Math 521 Lecture #32
§4.3.1: The Euler Equation

Recall that last time we derive a second-order differential equation that a local minimizer must satisfy for the arc length functional.

The derivation can be carried out for functionals of the form

$$J(y) = \int_a^b L(x, y, y')\ dx$$

where the Lagrangian $L(x, y, y')$ is a twice continuously differentiable function defined on $[a, b] \times \mathbb{R} \times \mathbb{R}$.

A key step is the following result.

Lemma 4.13. If $f(x)$ is continuous on $[a, b]$ and if

$$\int_a^b f(x)h(x)\ dx = 0$$

for every $h \in C^2[a, b]$ with $h(a) = 0$ and $h(b) = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

Proof. Assume, by way of contradiction, that there is $x_0 \in [a, b]$ such that $f(x_0) \neq 0$.

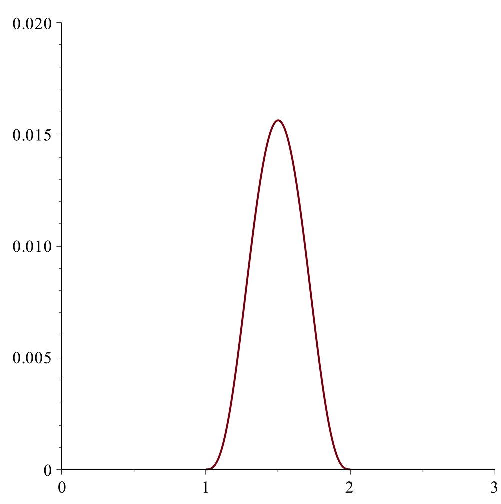
With loss of generality, we may assume that $f(x_0) > 0$.

Because f is continuous, there is an interval $[x_1, x_2]$ with $a \leq x_1 < x_2 \leq b$ such that $x_0 \in [x_1, x_2]$.

For a function $h(x) \in C^2[a, b]$ with $h(a) = 0$ and $h(b) = 0$, we choose

$$h(x) = \begin{cases} (x - x_1)^3(x_2 - x)^3 & \text{if } x_1 \leq x \leq x_2, \\ 0 & \text{otherwise.} \end{cases}$$

Here is the graph of this function $h(x)$.



The reason for the cubic powers is to ensure that the first and second derivatives of $h(x)$ at x_1 and x_2 are all zero, thus ensuring that $h \in C^2[a, b]$.

With this choice of $h(x)$ we reach the contradiction,

$$\int_a^b f(x)h(x)\ dx = \int_{x_1}^{x_2} f(x)(x - x_1)^3(x_2 - x)^3\ dx > 0.$$

This shows that $f(x) = 0$ for all $x \in [a, b]$. □

We can now state the necessary condition of a differential equation that a local minimizer must satisfy.

The differential equations obtained do not depend on the choice of a norm on the function space which contains the set of admissible functions.

Theorem 4.14. If $y_0 \in A = \{y \in C^2[a, b] : y(a) = y_a,\ y(b) = y_b\}$ is a local minimizer of the functional

$$J(y) = \int_a^b L(x, y, y')\ dx,$$

then y_0 must satisfy the **Euler equation** (or **Euler-Lagrange equation**),

$$L_y(x, y, y') - \frac{d}{dx} L_{y'}(x, y, y') = 0.$$

Proof. For $h \in C^2[a, b]$ with $h(a) = 0$ and $h(b) = 0$, the variation $y_0 + \epsilon h$ is admissible for small enough ϵ.

Then

$$J(y_0 + \epsilon h) = \int_a^b L(x, y + \epsilon h, y' + \epsilon h')\ dx,$$

and so

$$\begin{aligned}\frac{d}{d\epsilon} J(y_0 + \epsilon h) &= \int_a^b \frac{\partial}{\partial \epsilon} L(x, y_0 + \epsilon h, y_0' + \epsilon h')\ dx \\ &= \int_a^b \left[L_y(x, y_0 + \epsilon h, y_0' + \epsilon h')h + L_{y'}(x, y_0 + \epsilon h, y_0' + \epsilon h')h'\right]\ dx,\end{aligned}$$

where $L_y = \partial L/\partial y$ and $L_{y'} = \partial L/\partial y'$.

Thus

$$\left.\frac{d}{d\epsilon} J(y_0 + \epsilon h)\right|_{\epsilon=0} = \int_a^b \left\{L_y(x, y_0, y_0')h + L_{y'}(x, y_0, y_0')h'\right\} dx.$$

Since y_0 is a local minimizer, we know that

$$\delta J(y_0, h) = \frac{d}{d\epsilon} J(y_0 + \epsilon h) = 0$$

for all $h \in C^2[a, b]$ with $h(a) = 0$ and $h(b) = 0$.

This implies that

$$\int_a^b \{L_y(x, y_0, y_0')h + L_{y'}(x, y_0, y_0')h'\}dx = 0$$

holds for all $h \in C^2[a, b]$ with $h(a) = 0$ and $h(b) = 0$.

We perform integration by parts on the second term in the integral: with $u = L_{y'}(x, y_0, y_0')$ and $dv = h'dx$, we get

$$\int_a^b \left(L_y(x, y_0, y_0') - \frac{d}{dx} L_{y'}(x, y_0, y_0') \right) h \; dx + L_{y'}(x, y_0, y_0')h \Big|_{x=a}^{x=b} = 0.$$

Because $h(a) = 0$ and $h(b) = 0$, the evaluations vanishes, and we obtain

$$\int_a^b \left(L_y(x, y_0, y_0') - \frac{d}{dx} L_{y'}(x, y_0, y_0') \right) h \; dx = 0$$

which holds for all $h \in C^2[a, b]$ with $h(a) = 0$ and $h(b) = 0$.

We apply Lemma 4.13 to obtain

$$L_y(x, y_0, y_0') - \frac{d}{dx} L_{y'}(x, y_0 . y_0') = 0,$$

which is the Euler equation. □

This necessary condition was derived from the assumption of a local minimizer.

We do NOT know if any solution y of the Euler equation will be a local minimizer of $J(y)$.

However, any solution of the Euler equation y will satisfy $\delta J(y, h) = 0$ for all h (we get this by working some of the steps of the proof of the Theorem backwards).

So, in general, each solution of the Euler equation is a local extremum of J.

Carrying out the derivative with respect to x in the Euler equation gives

$$L_y(x, y, y') + L_{y'x}(x, y, y') + L_{y'y}(x, y, y')y' + L_{y'y'}(x, y, y')y'' = 0$$

and so the Euler equation is a second order differential equation when $L_{y'y'} \neq 0$.

Example. The Euler equation for

$$J(y) = \int_0^1 \left(\frac{m(y')^2}{2} - V(y) \right) \; dx$$

is

$$\begin{aligned} 0 &= L_y(x, y, y') - \frac{d}{dx} L_{y'}(x, y, y') \\ &= -V'(y) - my''. \end{aligned}$$

The extremals of J are precisely the solutions of the mechanical system $my'' = -V'(y)$.