Math 521 Lecture #33
§4.3.2: Solved Examples; 4.3.3: First Integrals

Recall that the Euler equation $L_y-(d/dx)L_{y'}=0$ is a necessary condition that is satisfied by a minimizer (or extremum) y_0 of a functional

$$J(y)=\int_a^b L(x,y,y')\ dt,\ \ y\in C^2[a,b],\ \ y(a)=y_a,\ \ y(b)=y_b.$$

Showing that a solution of the Euler equation is a minimizer requires analysis beyond the scope of this course.

However, the search for a minimizer is reduced through the Euler equation to solutions of a differential equation.

Example 4.15. Find the extremals of the functional

$$J(y)=\int_0^1 \big((y')^2+3y+2x\big)\ dx,\ \ y(0)=0,\ \ y(1)=1.$$

We tacitly assume that $y\in C^2[0,1]$ so that we can obtain the Euler equation.

With

$$L(x,y,y')=(y')^2+3y+2x,$$

the Euler equation for J is

$$3+\frac{d}{dx}\big(2y'\big)=0.$$

This is the second-order linear non homogeneous equation

$$y''=\frac{3}{2}.$$

We solve this by integrating twice to obtain

$$y=C_1x+C_2+\frac{3x^2}{4}.$$

From the boundary conditions $y(0)=0$ and $y(1)=1$ we obtain the linear system of equations,

$$\begin{aligned} C_2&=0\\ C_1+C_2&=1-\frac{3}{4}. \end{aligned}$$

We have a unique solution for C_1 and C_2, and hence a unique solution of Euler equation subject to the boundary conditions, namely,

$$y_0=\frac{x}{4}+\frac{3x^2}{4}.$$

Unfortunately, we do not know if this is a minimizer or maximizer of $J(y)$, or if it is neither, because all we know is that $\delta J(y_0,h)=0$ for all admissible variations h.

Example 4.16. Recall that we found the Euler equation for the arc length functional

$$J(y) = \int_0^1 \sqrt{1 + [y'(x)]^2}\, dx,\ y(0) = 0,\ y(1) = 1.$$

We illustrated the proof of the Theorem for the Euler equation and got

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+[y']^2}}\right) = 0.$$

We can verify it this now using the Lagrangian

$$L(x, y, y') = \sqrt{1 + [y']^2}.$$

For then the Euler equation is

$$0 + \frac{d}{dx}\left(\frac{y'}{\sqrt{1+[y']^2}}\right) = 0.$$

Solving this gives for an arbitrary constant C that

$$\frac{y'}{\sqrt{1+[y']^2}} = C.$$

We can solve this for y':

$$\begin{aligned} y' &= C\sqrt{1+[y']^2} \\ [y']^2 &= C^2(1 + [y']^2) \\ [y']^2 - C^2[y'] &= C_2 \\ [y']^2 &= \frac{C_2}{1 - C_2}. \end{aligned}$$

This implies that y' is a constant, so that

$$y(x) = Kx + M.$$

From the boundary conditions $y(0) = 0$ and $y(1) = 0$ we get that $K = 1$ and $M = 0$, so that $y_0(x) = x$ is the only solution of the Euler equation subject to the boundary conditions.

On geometric grounds we can argue that $y_0(x) = x$ is a global minimizer of the arc length functional.

When the Lagrangian L does not explicitly depend on one of the three variables x, y, and y', we can make a simplification in the Euler equation.

When L does not depend on y', then the Euler equation is not a differential equation, but an algebraic equation,

$$L_y(x, y) = 0.$$

Generally (through the implicit function theorem), this implicitly (and sometimes explicitly) defines y as a function of x.

When L does not depend on y, then the Euler equation is

$$\frac{d}{dx}L_{y'}(x,y') = 0, \text{ or } L_{y'}(x,y') = C$$

for an arbitrary constant C (as it did for the arc length functional).

When L does not depend on x, then multiplying the Euler equation through by y' gives

$$\begin{aligned} 0 &= y'\left(L_y - \frac{d}{dx}L_{y'}\right) \\ &= y'L_y - y'\frac{d}{dx}L_{y'} + L_{y'}y'' - L_{y'}y'' \\ &= L_y y' + L_{y'}y'' - y'\frac{d}{dx}L_{y'} - y''L_{y'} \\ &= \frac{dL}{dx} - y'\frac{d}{dx}L_{y'} - y''L_{y'} \\ &= \frac{d}{dx}\left[L - y'L_{y'}\right]. \end{aligned}$$

This says that the quantity $L - y'L_{y'}$ is a conserved quantity or a first integral.

Formally, a **first integral** of a second-order ordinary differential equation $F(x,y,y',y'') = 0$ is a function $g(x,y,y')$ which is constant along each solution of $F(x,y,y',y'') = 0$.

In the Calculus of Variations, when L is independent of x, a first integral or **conservation law** of the Euler equations of the extremals is

$$g(y,y') = L(y,y') - y'L_{y'}(y,y').$$

We can use a first integral to determine the extremals of a functional.

Example 4.17. The functional for the brachistochrone problem is

$$J(y) = \int_0^a \frac{\sqrt{1+[y']^2}}{\sqrt{2g(b-y)}}\,dx, \; y(0) = b, \; y(1) = 0.$$

The Lagrangian for this functional is independent of x, so a first integral of the Euler equation is

$$C = L - y'L_{y'} = \frac{\sqrt{1+[y']^2}}{\sqrt{2g(b-y)}} - (y')^2\frac{(1+[y']^2)^{-1/2}}{\sqrt{2g(b-y)}}.$$

We can absorb the common number $\sqrt{2g}$ into C.

Multiplying the resulting equation through by $\sqrt{1+[y']^2}$ gives

$$C\sqrt{1+[y']^2} = \frac{1+(y')^2}{\sqrt{b-y}} - \frac{(y')^2}{\sqrt{b-y}} = \frac{1}{\sqrt{b-y}}.$$

Squaring both sides gives

$$C^2(1 + [y']^2) = \frac{1}{b-y}.$$

Simplifying this gives

$$(y')^2 = \frac{1}{C^2(b-y)} - 1 = \frac{1 - C^2(b-y)}{C^2(b-y)}.$$

From physical considerations (the bead is rolling downward), we know that $dy/dx < 0$.

Thus we have obtain the first order equation

$$\frac{dy}{dx} = -\sqrt{\frac{1 - C^2(b-y)}{C^2(b-y)}}$$

in addition to the second-order Euler equation that an extremal of the brachistochrone function must satisfy.

If we write $C_1 = C^{-2}$ and separate variables, the first-order equation becomes

$$dx = -\frac{\sqrt{b-y}}{\sqrt{C_1 - (b-y)}} dy.$$

Through the substitution

$$b - y = C_1 \sin^2(\phi/2), \ -dy = C_1 \sin(\phi/2)\cos(\phi/2) d\phi,$$

we obtain

$$\begin{aligned} dx &= \frac{\sqrt{C}_1 \sin\phi/2}{\sqrt{C_1(1 - \sin^2\phi/2)}} C_1 \sin(\phi/2)\cos(\phi/2)\ d\phi \\ &= C_1 \sin^2(\phi/2)\ d\phi \\ &= \frac{C_1}{2}\big(1 - \cos\phi\big)\ d\phi. \end{aligned}$$

Integration gives

$$x = \frac{C_1}{2}\big(\phi - \sin\phi\big) + C_2.$$

The functions x and y of ϕ are parametric equations for a cycloid.