Math 521 Lecture #34
§4.4: Generalizations

In the simplest variational problem the Lagrangian L depended on three scalar quantities x, y, and y'.

Obvious generalizations of this are (1) to include higher-order derivatives of y in the Lagrangian, and to include more than one function y.

§4.2.1: Higher Derivatives. For a C^2 function L of four scalar variables, we consider the *second-order* variational problem given by the functional

$$J(y) = \int_a^b L(x, y, y', y'')\ dx$$

on the admissible set

$$A = \{y \in C^4[a, b] : y(a) = A_1, y'(a) = A_2, y(b) = B_1, y'(b) = B_2\}.$$

In seeking necessary conditions for $y_0 \in A$ to be a minimizer of J on A, we will see the need to have $y \in C^4[a, b]$ instead of $C^3[a, b]$.

Suppose that $y_0 \in A$ is a local minimizer of J with respect to some norm on $C^4[a, b]$.

An admissible variation if $h \in C^4[a, b]$ such that

$$h(a) = h'(a) = h(b) = h'(b) = 0.$$

Then $J(y_0 + \epsilon h)$ is defined for all small ϵ, and the first variation of J at y_0 in the direction of h is

$$\begin{aligned}
\delta J(y_0, h) &= \frac{d}{d\epsilon} J(y_0 + \epsilon h)\bigg|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} \int_a^b L\big(x, y_0 + \epsilon h, y_0' + \epsilon h', y_0'' + \epsilon h''\big)\ dx\bigg|_{\epsilon=0} \\
&= \int_a^b \frac{\partial}{\partial \epsilon} L\big(x, y_0 + \epsilon h, y_0' + \epsilon h', y_0'' + \epsilon h''\big)\ dx\bigg|_{\epsilon=0} \\
&= \int_a^b \big[L_y(x, y_0, y_0', y_0'')h + L_{y'}(x, y_0, y_0', y_0'')h' + L_{y''}(x, y_0, y_0', y_0'')h''\big]\ dx.
\end{aligned}$$

For the middle term of the integrand, we proceed as before with integration by parts and the conditions $h(a) = 0$ and $h(b) = 0$ to get

$$\begin{aligned}
\int_a^b L_{y'}(x, y_0, y_0', y_0'')h'\ dx &= L_{y'}(x, y_0, y_0', y_0'')h\bigg|_a^b - \int_a^b \frac{d}{dx} L_{y'}(x, y_0, y_0', y_0'')h\ dx \\
&= -\int_a^b \frac{d}{dx} L_{y'}(x, y_0, y_0', y_0'')h\ dx.
\end{aligned}$$

For the third term in the integrand, we use integration by parts twice along with the zero values of h and h' at $x = a$ and $x = b$.

Doing this gives

$$\begin{aligned}\int_a^b L_{y''}(x, y_0, y_0', y_0'')h'' \, dx &= L_{y''}(x, y_0, y_0', y_0'')h' \Big|_a^b - \int_a^b \frac{d}{dx} L_{y''}(x, y_0, y_0', y_0'')h' \, dx \\ &= -L_{y''}(x, y_0, y_0', y_0'')h \Big|_a^b + \int_a^b \frac{d^2}{dx^2} L_{y''}(x, y_0, y_0', y_0'')h \, dx \\ &= \int_a^b \frac{d^2}{dx^2} L_{y''}(x, y_0, y_0', y_0'')h \, dx.\end{aligned}$$

Thus we have that

$$\delta J(y_0, h) = \int_a^b \left[L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''} \right] h \, dx$$

where L_y, $L_{y'}$, and $L_{y''}$ are evaluated at (x, y_0, y_0', y_0'').

Because y_0 is a local minimizer, we have that $\delta J(y_0, h) = 0$ for all admissible variations $h \in C^4[a, b]$ satisfying the conditions $h(a) = 0$, $h(b) = 0$, $h'(a) = 0$, $h'(b) = 0$.

To get the Euler equation, we need another version of the *fundamental lemma*, the proof of which is similar to the earlier version.

Lemma 4.19. If f is a continuous function on $[a, b]$, and if

$$\int_a^b f(x)h(x) = 0$$

for all $h \in C^4[a, b]$ with $h(a) = 0$, $h(b) = 0$, $h'(0) = 0$, $h'(b) = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

By the Fundamental Lemma, we arrive at the *fourth-order* Euler equation,

$$L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''} = 0, \ a \leq x \leq b.$$

We assumed that $y \in C^4[a, b]$ because the third term in the Euler equation generally requires taking the fourth derivative of y.

The n^{th}-order variational problem is given by the functional

$$J(y) = \int_a^b L(x, y', y'', \ldots, y^{(n)}) dx$$

on the admissible set

$$A = \{y \in C^{2n}[a, b] : y^{(i)}(a) = A_i, y^{(i)}(b) = B_i, i = 0, \ldots, n-1\}.$$

The Euler equation for the n^{th} variational problem is

$$L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''} + \cdots + (-1)^n \frac{d^n}{dx^n} L_{y^{(n)}} = 0.$$

§:4.4.2: Several Functions. Another generalization of the simplest variational problem is when J depends on several functions $y_1, y_2, \ldots, y_n$.

When $n = 2$, the functional is

$$J(y_1, y_2) = \int_a^b L(x, y_1, y_2, y_1', y_2')\ dt$$

on the admissible set

$$A = \{(y_1, y_2) \in C^2[a,b] \times C^2[a,b] : y_i(a) = A_i, y_i'(b) = B_i, i = 1, 2\}.$$

Suppose that the pair $(\tilde{y}_1, \tilde{y}_2) \in A$ provides a local minimum (relative to a choice of norm on A).

We vary each of y_1 and y_2 independently with admissible variations $h_1, h_2 \in C^2[a,b]$ satisfying

$$h_1(a) = h_2(a) = h_1(b) = h_2(b) = 0.$$

Then

$$\mathcal{J}(\epsilon) = \int_a^b L(x, y_1 + \epsilon h_1, y' + \epsilon h_1', y_2 + \epsilon h_2, y_2' + \epsilon h_2')\ dx$$

has a local minimum at $\epsilon = 0$ so that

$$0 = \mathcal{J}'(0) = \int_a^b \big(L_{y_1} h_1 + L_{y_1'} h_1' + L_{y_2} h_2 + L_{y_2'} h_2'\big) dx.$$

Integration by parts on the terms with h_1' and h_2' gives

$$\int_a^b \left\{ \left(L_{y_1} - \frac{d}{dx} L_{y_1'} \right) h_1 + \left(L_{y_2} - \frac{d}{dx} L_{y_2'} \right) h_2 \right\} dx = 0$$

which holds for all admissible pairs (h_1, h_2).

By choosing $h_2 = 0$, we get

$$\int_a^b \left(L_{y_1} - \frac{d}{dx} L_{y_1'} \right) h_1\ dx = 0,$$

and so by the fundamental lemma we obtain

$$L_{y_1} - \frac{d}{dx} L_{y_1'} = 0.$$

Similarly, by choosing instead $h_1 = 0$ we obtain

$$L_{y_2} - \frac{d}{dx} L_{y_2'} = 0.$$

Thus the minimizing pair $(\tilde{y}_1, \tilde{y}_2)$ satisfies the system of (second-order) Euler equations given above.

The generalization to n functions $y_1, y_2, \ldots, y_n$ is straightforward.