Math 521 Lecture #35
§4.4.3: Natural Boundary Conditions

To this point we have considered the fixed endpoint problem of the Calculus of Variations, that is, the functions y in the admissible set A satisfies boundary conditions $y(a) = y_a$ and $y(b) = y_b$.

For some problems, we may not have a condition at one or both endpoints.

Example. A river has parallel straight banks b units apart.

Let the x-axis be perpendicular to the parallel straight banks, with $x = 0$ being the left bank and $x = b$ being the right bank.

Let the y-axis be the left bank of the river with the positive direction being in the direction that the water is moving.

Suppose that the water moves with continuous velocity $\mathbf{v}(x, y) = (0, v(x))$, $0 \leq x \leq b$.

A boat is launched from the left river bank at the origin.

In still water the boat moves with constant velocity c.

We assume that $c > \|v\|_M$.

What route should the boat take to reach the right river bank in the shortest possible time?

The route the boat takes is the graph of a function $y(x)$ with $y(0) = 0$ and $y(b)$ unspecified or free.

The time to traverse the river is the functional

$$J(y) = \int_0^b \frac{\sqrt{c^2(1 + [y']^2) - v^2} - vy'}{c^2 - v^2}\, dx$$

over the set of admissible functions

$$A = \{y \in C^2[0, b] : y(0) = 0,\ \ y(b) \text{ free}\}.$$

This is a called a **free endpoint problem**.

If y is an extremal of J on A, are there any conditions that $y(b)$ should satisfy?

Yes, there are, and they are called **natural boundary conditions**.

To get to the natural boundary conditions, we fix a free endpoint problem,

$$J(y) = \int_a^b L(x, y, y')\, dx$$

over

$$A = \{y \in C^2[a, b] : y(a) = y_a,\ \ y(b) \text{ free}\}.$$

Suppose that $y_0 \in A$ is a local minimizer.

An admissible variation is $h \in C^2[0, 1]$ with $h(0) = 0$.

The first variation of J at y_0 in the direction h is

$$\begin{aligned}
\delta J(y_0, h) &= \frac{d}{d\epsilon} \int_a^b L(x, y_0 + \epsilon h, y_0' + \epsilon h')\ dx \bigg|_{\epsilon=0} \\
&= \int_a^b (L_y(x, y_0, y_0')h + L_{y'}(x, y_0, y_0')h')\ dx.
\end{aligned}$$

Using integrating by parts gives

$$\delta J(y_0, h) = \int_a^b \left(L_y - \frac{d}{dx} L_{y'} \right) h\ dx + L_{y'}(x, y_0, y_0') \bigg|_{x=a}^{x=b}.$$

Since $h(a) = 0$, a necessary condition for y_0 to be a local minimizer is that for all $h \in C^2[a, b]$ with $h(0) = 0$ there holds

$$\int_a^b \left(L_y - \frac{d}{dx} L_{y'} \right) h\ dx + L_{y'}(b, y_0(b), y_0'(b))h(b) = 0.$$

In particular this holds when $h(b) = 0$, we have

$$\int_a^b \left(L_y - \frac{d}{dx} L_{y'} \right) h\ dx = 0,$$

so that the Fundamental Lemma we obtain the Euler equation

$$L_y - \frac{d}{dx} L_{y'} = 0$$

that the local minimizer y_0 must satisfy.

This implies that for all $h \in C^2[a, b]$ with $h(0) = 0$ that

$$0 = \int_a^b \left(L_y - \frac{d}{dx} L_{y'} \right) h\ dx + L_{y'}(b, y_0(b), y_0'(b))h(b) = L_{y'}(b, y_0(b), y_0'(b))h(b).$$

Since $h(b)$ is unspecified or free, it follows that

$$L_{y'}(b, y_0(b), y_0'(b)) = 0$$

which is called a **natural boundary condition** on the minimizer y_0 at $x = b$.

The preceding argument shows that when y_0 is an extremal of $J(y)$, then y_0 satisfies the Euler equations and the natural boundary condition.

Similarly, if y_0 is an extremal of $J(y)$ with $y(a)$ free and $y(b) = y_b$, then y_0 satisfies the Euler equation and the natural boundary condition

$$L_{y'}(a, y_0(a), y_0'(b)) = 0.$$

Again, by a similar argument, if y_0 is an extremal of $J(y)$ with $y(a)$ free and $y(b)$ free, then y_0 satisfies the Euler equation and both natural boundary conditions.

Example (Continued). We return to the free endpoint problem for the route of the boat across the river in the shortest time possible.

The Lagrangian for variational problem is

$$L(x,y,y') = \frac{\sqrt{c^2(1+[y']^2)-v^2} - vy'}{c^2-v^2}.$$

From this we have

$$L_{y'} = \frac{1}{c^2-v^2}\left(\frac{c^2 y'}{(c^2(1+[y']^2)-v^2)^{1/2}} - v\right).$$

From the natural boundary condition $L_{y'}(b, y_0(b), y_0'(b)) = 0$ we obtain

$$\frac{c^2 y_0'(b)}{(c^2(1+[y_0'(b)]^2)-[v(b)]^2)^{1/2}} - v(b) = 0.$$

Moving the $v(b)$ to the right-hand side and squaring both sides gives

$$\frac{c^4[y_0'(b)]^2}{c^2(1+[y_0'(b)]^2)-[v(b)]^2} = [v(b)]^2.$$

This simplifies to

$$c^4[y_0'(b)]^2 = [v(b)]^2\big(c^2(1+[y_0'(b)]^2)-[v(b)]^2\big).$$

Collecting the $[y_0'(b)]^2$ terms to the left-hand side we have

$$c^4[y_0'(b)]^2 - c^2[v(b)]^2[y_0'(b)]^2 = c^2[v(b)]^2 - [v(b)]^4.$$

Factoring out the $[y_0'(b)]^2$ terms gives

$$(c^4 - c^2[v(b)]^2)[y_0'(b)]^2 = c^2[v(b)]^2 - [v(b)]^4.$$

Isolating $[y_0'(b)]^2$ gives

$$[y_0'(b)]^2 = \frac{[v(b)]^2(c^2-[v(b)]^2)}{c^2(c^2-[v(b)]^2)} = \frac{[v(b)]^2}{c^2}.$$

Thus the natural boundary condition for an extremal y_0 of $J(y)$ is

$$y_0'(b) = \frac{v(b)}{c}.$$

(We took the positive root because physically the boat will move in the positive y direction.)

The boat arrives at the right bank of the river with a slope that is the ratio of the water speed at right bank to the boat velocity in still water.

Example. Find the differential equations and natural boundary condition for the free endpoint variational problem

$$J(y) = \int_0^1 \left(p(x)(y')^2 - q(x)y^2\right)\, dx,$$
$$y(0) = 0,\ \ y(1) \text{ free},$$

where $p(x)$ and $q(x)$ are positive functions on $[0,1]$.

With the Lagrangian being

$$L(x, y, y') = p(x)(y')^2 - q(x)y^2,$$

the Euler equation is

$$0 = L_y - \frac{d}{dx} L_{y'} = -2q(x)y - \frac{d}{dx}\left(p(x)y'\right),$$

or the Sturm-Liouville type ODE

$$\frac{d}{dx}\left(p(x)y'\right) + q(x)y = 0,\ \ 0 < x < 1.$$

The natural boundary condition at $x = 1$ is

$$2p(1)y'(1) = 0.$$

Since $p(1) > 0$, we obtain the condition

$$y'(1) = 0.$$