Math 521 Lecture #36
§4.5: Hamilton's Principle

An approach to modeling the dynamics of a system is to assume that the state of the system at one moment determines the state of the system at all past and future moments.

To apply the classical approach, one identifies the relevant state variables and a system of differential equations in those variables, that describe the time evolution of the variables.

The principle of determinism is found in the existence and uniqueness theory of system of differential equations.

Another approach to modeling the dynamics of a system is to assume that the system evolves along a path of least resistance.

The variational principle is a means to describe a path of least resistance as a minimizer of some functional.

Hamilton's principle states that the time evolution of the state variables occurs in such a manner that the integral of the difference between the kinetic and potential energies is stationary.

Suppose a set of *generalized coordinates* $y_1, \ldots, y_n$ specifies the state of the system at any instant of time t.

We further assume that there are no relations among the generalized coordinates used.

The time derivatives $\dot{y}_1, \ldots, y_n$ are called the **generalized velocities**.

The **kinetic energy** T of the system is a *quadratic form* in the generalized velocities:

$$T = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(y_1, \ldots, y_n)\dot{y}_i\dot{y}_j,$$

where the a_{ij} are known functions of the generalized coordinates.

The potential energy V of the system is a scalar function of the form

$$V = V(t, y_1, \ldots, y_n).$$

The **Lagrangian** of the system is defined to be

$$L(t, y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n) = T - V.$$

Hamilton's Principle states that the evolution of the generalized coordinates of the system evolve in such a way that the functional

$$J(y_1, \ldots, y_n) = \int_{t_0}^{t_1} L(t, y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n)\, dt$$

is stationary.

Thinking of this another way, if we regard $(y_1(t), \ldots, y_n(t))$, $t \in [t_0, t_1]$, as a parameterized curve in **configuration space** $\mathbb{R}^n$ that joins the initial and final states,

$$S_0 = (y_1(t_0), \ldots, y_n(t_0)) \text{ and } S_1 = (y_1(t_1), \ldots, y_n(t_1)),$$

then among all paths in configuration space joining S_0 to S_1, that actual evolution of the generalized coordinates of the system takes place along a curve $(y_1, \ldots, y_n)$ for which

$$\delta \int_{t_0}^{t_1} L(t, y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n)\ dt = 0.$$

The functional $\int L dt$ is called the **action integral** for the system.

A curve $(y_1, \ldots, y_n)$ at which J is stationary satisfies the system of Euler equations

$$L_{y_i} - \frac{d}{dt} L_{\dot{y}_i} = 0,\ i = 1, \ldots, n.$$

This system of second-order ODES, or governing equations, is called **Lagrange's equations**.

We say that the governing equations follow from a variational principle if there exists a Lagrangian L such that $\delta \int L\ dt = 0$ gives the governing equations as necessary conditions for an extremum.

If L is independent of t, then the governing equations has a first integral or conservation law given by

$$L - \sum_{i=1}^{n} \dot{y}_i L_{\dot{y}_i} = C.$$

The expression

$$H(y_1, \ldots . y_n, \dot{y}_1, \ldots, \dot{y}_n) = -L + \sum_{i=1}^{n} \dot{y}_i L_{\dot{y}_i}$$

is called the **Hamiltonian** or **total energy** of the system, and is conserved when L is independent of t.

Example 4.22 (Harmonic Oscillator). Consider the undamped motion of an object of mass m attached to a spring obeying Hooke's law.

For the generalized coordinate y representing the positive displacement of the object from equilibrium, Hooke's law states that the restoring force of the spring is $F(x) = -ky$ for a positive constant k.

The kinetic energy of the object is

$$T(\dot{y}) = \frac{m\dot{y}^2}{2}.$$

The potential energy is the negative of the work done by the spring in moving the object from 0 to a displacement of y:

$$V(y) = \int_0^y kw\ dw = \frac{ky^2}{2}.$$

The Lagrangian for the undamped mass-spring system is

$$L(y, \dot{y}) = T - V = \frac{m\dot{y}^2}{2} - \frac{ky^2}{2}.$$

Hamilton's Principle states that the motion of the undamped mass-spring system occurs in such a way that the action integral

$$J(y) = \int_{t_0}^{t_1} L(y, \dot{y})\ dt = \int_{t_0}^{t_1} \left(\frac{m\dot{y}^2}{2} - \frac{ky^2}{2} \right)\ dt$$

is stationary.

Lagrange's equation for the action integral is

$$0 = L_y - \frac{d}{dt} L_{\dot{y}} = -ky - \frac{d}{dt}\big(m\dot{y}\big) = -ky - m\ddot{y}.$$

This is the familiar harmonic oscillator

$$m\ddot{y} + ky = 0.$$

Because the Lagrangian is independent of t, the Hamiltonian

$$H(y, \dot{y}) = -L + \dot{y} L_{\dot{y}} = -\frac{m\dot{y}^2}{2} + \frac{ky^2}{2} + \dot{y}\big(m\dot{y}\big) = \frac{m\dot{y}^2}{2} + \frac{ky^2}{2}$$

is conserved.

The level curves of H are ellipses, and so every non-equilibrium solution of the harmonic oscillator is a periodic solution.

Example 4.23. Consider a simple pendulum of length l and bob of mass m suspended from a frictionless hinge.

We use Cartesian coordinates (x, y) for the plane of motion of the bob where $(0, 0)$ is the center of the circle of radius l, $x > 0$ is the horizontal axis to the right, and $y > 0$ is the vertical axis upward.

The position of the bob is measured by the generalized coordinate of the angle θ that the pendulum makes with its downward vertical equilibrium position.

The vector position of the bob in Cartesian coordinates as a function of the θ is

$$\vec{r}(\theta) = (l \sin\theta, -l \cos\theta).$$

The velocity of the bob is then

$$\vec{r}\,'(\theta) = \big(l(\cos\theta)\dot{\theta}, l(\sin\theta)\dot{\theta}\big).$$

The kinetic energy is then

$$T = \frac{m}{2}\|\vec{r}\,'(\theta)\|^2 = \frac{ml^2\dot{\theta}^2}{2}.$$

The potential energy is the negative of the work done by the conservative force $\vec{F} = (0, -mg)$ in moving the bob from the equilibrium position to the angle θ (from $y = -l$ to $y = -l\cos\theta$):

$$V(\theta) = -\int_C \vec{F} \cdot d\vec{r} = -\int_{-l}^{-l\cos\theta} -mg\ dy = mgl(1 - \cos\theta).$$

By Hamilton's principle, the motion of the simple pendulum takes place so that

$$J(y) = \int_{t_0}^{t_1} \left(\frac{ml^2\dot{\theta}^2}{2} - mgl\big(1 - \cos\theta\big) \right) \, dt$$

is stationary.

The extrema of J satisfy the Lagrange equation

$$0 = L_\theta - \frac{d}{dt} L_{\dot{\theta}} = -mgl \sin\theta - \frac{d}{dt} ml^2\dot{\theta} = -mgl \sin\theta - ml^2\ddot{\theta}.$$

Simplifying the Lagrange equation gives

$$\ddot{\theta} + \frac{g}{l} \sin\theta = 0.$$

Since the Lagrangian is independent of t, there is a first integral

$$C = L - \dot{\theta} L_{\dot{\theta}} = \frac{ml^2\dot{\theta}^2}{2} - mgl(1 - \cos\theta) - ml^2\dot{\theta}^2 = -\frac{ml^2\dot{\theta}^2}{2} - mgl(1 - \cos\theta).$$

Abusing the arbitrary constant C, this first integral simplifies to

$$C = \frac{\dot{\theta}^2}{2} + \frac{g}{l}\big(1 - \cos\theta\big).$$

The Hamiltonian of the system is

$$-L + \dot{\theta} L_{\dot{\theta}} = -\frac{ml^2\dot{\theta}^2}{2} + mgl(1 - \cos\theta) + ml^2\dot{\theta}^2 = \frac{ml^2\dot{\theta}^2}{2} + mgl(1 - \cos\theta).$$