Math 521 Lecture #37
§4.5: Hamilton's Equations

The Euler equation for the variational problem

$$J(y) = \int_{t_0}^{t_1} L(t, y, \dot{y})\ dt$$

is the second-order ODE

$$L_y - \frac{d}{dx} L_{\dot{y}} = 0.$$

We introduce a **canonical way** to convert this second-order ODE into a system to first-order ODEs.

We keep y as one variable, and define a new variable, called the **canonical momentum**, by

$$p = L_{\dot{y}}(t, y, \dot{y}).$$

The change of variables from $(y, \dot{y})$ to (y, p) is called the **Legendre transformation**.

We seek a system of first order ODEs in the variables y and p.

From the definition of p, the Euler equation becomes

$$\frac{dp}{dx} = L_y.$$

When $L_{\dot{y}\dot{y}} \neq 0$, the Implicit Function Theorem guarantees that we can solve $p = L_{\dot{y}}(t, y, \dot{y})$ for $\dot{y}$ to get

$$\dot{y} = \phi(t, y, p).$$

The Hamiltonian $H = -L + \dot{y} L_{\dot{y}}$, which is a function of y and $\dot{y}$, can now be expressed in terms of y and p:

$$H(t, y, p) = -L(t, y, \phi(t, y, p)) + \phi(t, y, p)p.$$

Then

$$\frac{\partial H}{\partial y} = -\frac{\partial L}{\partial y} - \frac{\partial L}{\partial \dot{y}}\frac{\partial \phi}{\partial y} + p\frac{\partial \phi}{\partial y} = -\frac{\partial L}{\partial y} - p\frac{\partial \phi}{\partial y} + p\frac{\partial \phi}{\partial y} = -\frac{\partial L}{\partial y} = -L_y = -\dot{p},$$

and

$$\frac{\partial H}{\partial p} = -\frac{\partial L}{\partial \dot{y}}\frac{\partial \phi}{\partial p} + p\frac{\partial \phi}{\partial p} + \phi = -p\frac{\partial \phi}{\partial p} - p\frac{\partial \phi}{\partial p} + \phi = \phi = \dot{y}.$$

Rewriting these gives a system of first-order ODEs in the variables y and p:

$$\dot{y} = \frac{\partial H}{\partial p},$$
$$\dot{p} = -\frac{\partial H}{\partial y}.$$

These are called **Hamilton's equations**.

Example 4.26. Recall that the Lagrangian of the harmonic oscillator is

$$L(t, y, \dot{y}) = \frac{m\dot{y}^2}{2} - \frac{ky^2}{2}.$$

The canonical momentum is defined by

$$p = L_{\dot{y}} = m\dot{y}.$$

This inverts to give

$$\dot{y} = \phi(t, y, p) = \frac{p}{m}.$$

The total energy, or Hamiltonian, of the system in the variables y and p is

$$\begin{aligned} H &= -L + \dot{y}L_{\dot{y}} \\ &= -\frac{m\dot{y}^2}{2} + \frac{ky^2}{2} + \dot{y}(m\dot{y}) \\ &= \frac{m\dot{y}^2}{2} + \frac{ky^2}{2} \\ &= \frac{m}{2}\left(\frac{p}{m}\right)^2 + \frac{ky^2}{2} \\ &= \frac{p^2}{2m} + \frac{ky^2}{2}. \end{aligned}$$

Since

$$\frac{\partial H}{\partial y} = ky, \quad \frac{\partial H}{\partial p} = \frac{p}{m},$$

Hamilton's equations are

$$\dot{y} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial y} = -ky.$$

We can solve this system of first-order ODEs by eliminating the t variable to obtain a single first-order ODE,

$$\frac{dp}{dy} = \frac{dp/dt}{dy/dt} = \frac{-ky}{p/m} = -\frac{kmy}{p}.$$

Separation of variables then gives

$$pdp = -kmydy$$

so that

$$\frac{p^2}{2} = -\frac{kmy^2}{2} + C.$$

Dividing through by m gives the Hamiltonian

$$\frac{C}{m} = \frac{p^2}{m} + \frac{ky^2}{2} = H,$$

whose level curves are ellipses in the yp-plane.

For the general action integral

$$J(y_1,\ldots,y_n)=\int_{t_0}^{t_1} L(t,y_1,\ldots,y_n,\dot y_1,\ldots,\dot y_n)\;dt,$$

the canonical momenta are defined by

$$p_i = L_{\dot y_i}(t,y_1,\ldots,y_n,\dot y_1,\ldots,\dot y_n),\ i=1,\ldots,n.$$

This determines the Legendre transformation from the variables $(y_1,\ldots,y_n,\dot y_1,\ldots,\dot y_n)$ to the variables $(y_1,\ldots,y_n,p_1,\ldots,p_n)$.

Assuming that the $n\times n$ matrix with entries $L_{\dot y_i \dot y_j}$ is invertible, we can solve $p_i = L_{\dot y_i}(t,y_1,\ldots,y_n,\dot y_1,\ldots,\dot y_n)$ for $\dot y_1,\ldots,y_n$ (via the Implicit Function Theorem) to obtain

$$\dot y_i = \phi_i(t,y_1,\ldots,y_n,p_1,\ldots,p_n),\ i=1,\ldots,n.$$

The Hamilton in terms of the variables $y_1,\ldots,y_n$ and $p_1,\ldots,p_n$ is then

$$\begin{aligned} H &= -L+\sum_{i=1}^n \dot y_i L_{\dot y_i} \\ &= -L(t,y_1,\ldots,y_n,\phi_1,\ldots,\phi_n)+\sum_{i=1}^n \phi_i(t,y_1,\ldots,y_n,p_1,\ldots,p_n)p_i, \end{aligned}$$

where we have suppressed the dependence of ϕ_i on $(t,y_1,\ldots,y_n,p_1,\ldots,p_n)$.

We can convert the Euler-Lagrange equations (a system of n second-order ODEs),

$$0 = L_{y_j}-\frac{d}{dt}L_{\dot y_j} = L_{y_j}-\frac{dp_j}{dt},\ j=1,\ldots,n,$$

into a system of $2n$ first-order ODEs (Hamilton's equations):

$$\begin{aligned} \frac{\partial H}{\partial y_j} &= -L_{y_j}-\sum_{i=1}^n \frac{\partial L}{\partial \dot y_i}\frac{\partial \phi_i}{\partial y_j}+\sum_{i=1}^n \frac{\partial \phi_i}{\partial y_j}p_i \\ &= -L_{y_j}-\sum_{i=1}^n \frac{\partial L}{\partial \dot y_i}\frac{\partial \phi_i}{\partial y_j}+\sum_{i=1}^n \frac{\partial \phi_i}{\partial y_j}\frac{\partial L}{\partial \dot y_i} \\ &= -L_{y_j} = -\dot p_j \end{aligned}$$

and

$$\begin{aligned} \frac{\partial H}{\partial p_j} &= -\sum_{i=1}^n \frac{\partial L}{\partial \dot y_i}\frac{\partial \phi_i}{\partial p_j}+\sum_{i=1}^n \frac{\partial \phi_i}{\partial p_i}p_j+\phi_j \\ &= -\sum_{i=1}^n \frac{\partial L}{\partial \dot y_i}\frac{\partial \phi_i}{\partial p_j}+\sum_{i=1}^n \frac{\partial \phi_i}{\partial p_i}\frac{\partial L}{\partial \dot y_i}+\phi_j \\ &= \phi_j = \dot y_j \end{aligned}$$

Thus we obtain the Hamiltonian system of first-order ODEs,

$$\dot y_j = \frac{\partial H}{\partial p_j},\ \dot p_j = -\frac{\partial H}{\partial y_j},\ j=1,\ldots,n.$$