Math 521 Lecture #38
§4.5.2: The Inverse Problem

A variational principle exists for a given physical system if there is a Lagrangian L such that the Euler-Lagrange equations

$$L_{y_i} - \frac{d}{dx} L_{\dot{y}_i} = 0,\ i = 1, \ldots, n,$$

are the governing equations for the physical system.

When the physical system is conservative, Hamilton's principle states that the Lagrangian is the kinetic energy minus the potential energy of the system.
How can we determine the Lagrangian for other systems?

Finding the Lagrangian is known as the inverse problem in the calculus of variations.

We formulate the problem for $n = 1$, where we start with the scalar second-order ODE

$$\ddot{y} = F(t, y, \dot{y}).$$

We then seek for a Lagrangian L such that Euler-Lagrange equation

$$L_y - \frac{d}{dt} L_{\dot{y}} = 0$$

is the second-order ODE $\ddot{y} = F(t, y, \dot{y})$.

Expanding the time derivative in the Euler-Lagrange equation gives

$$0 = L_y - L_{\dot{y}t} - L_{\dot{y}y}\dot{y} - L_{\dot{y}\dot{y}}\ddot{y} = 0.$$

Since we want $\ddot{y} = F(t, y, \dot{y})$, we obtain

$$0 = L_y - L_{\dot{y}t} - L_{\dot{y}y}\dot{y} - L_{\dot{y}\dot{y}}F(t, y, \dot{y}) = 0$$

a partial differential equation that the Lagrangian L must satisfy.

There are typically many correct choices of the Lagrangian L that satisfy this PDE.

Rather than learn the method of solving this PDE, we instead resort to matching terms.

Example 4.27. Consider the damped unforced mass-spring system

$$m\ddot{y} + a\dot{y} + ky = 0,$$

where y is the displacement of the object of mass m from its equilibrium, a is the damping coefficient, and k is the spring constant.

This is not a conservative system, because the presence of damping implies every solution tends to 0, a state which has 0 energy.

We cannot apply Hamilton's Principle to find the Lagrangian.

To find L we multiply the second-order ODE through by a undetermined nonzero function or integrating factor $f(t)$:

$$mf(t)\ddot{y} + af(t)\dot{y} + kf(t)y = 0.$$

We will match terms of this with the terms of the Euler-Lagrange equation

$$-L_y + L_{\dot{y}t} + L_{\dot{y}y}\dot{y} + L_{\dot{y}\dot{y}}\ddot{y} = 0.$$

An obvious choice of matching is the terms with $\ddot{y}$:

$$L_{\dot{y}\dot{y}} = mf(t).$$

Integrating this with respect to $\dot{y}$ gives

$$L_{\dot{y}} = mf(t)\dot{y} + M(t, y)$$

for an arbitrary C^1 function $M(t, y)$.

Integrating one more time with respect to $\dot{y}$ gives

$$L = mf(t)\frac{\dot{y}^2}{2} + M(t, y)\dot{y} + N(t, y)$$

for an arbitrary C^1 function $N(t, y)$.

Using this guess for L we have that

$$\begin{aligned} af(t)\dot{y} + kf(t)y &= -L_y + L_{\dot{y}t} + L_{\dot{y}y}\dot{y} \\ &= -(M_y\dot{y} + N_y) + mf'(t)\dot{y} + M_t + M_y\dot{y} \\ &= -N_y + mf'(t)\dot{y} + M_t. \end{aligned}$$

Upon matching the $\dot{y}$ terms we get

$$af(t) = mf'(t) + M_y.$$

This is not a differential equation in $f(t)$ unless $M_y = 0$.

So we assume that $M_y = 0$, and by solving the differential equation we get

$$f(t) = C\exp(at/m)$$

for an arbitrary constant C.
We need only one Lagrangian, so we choose $C = 1$.

The remaining terms must match, that is

$$kf(t)y = -N_y + M_t.$$

Since we have assumed that $M_y = 0$, then M_t is at most a function of t.

The function N satisfies the PDE

$$N_y = -kf(t)y + M_t,$$

so that

$$N(t,y) = -\frac{kf(t)y^2}{2} + \int M_t \, dy.$$

Needing only one Lagrangian, we assume that $M_t = 0$, so that M is a constant (take it to be zero) and

$$N(t,y) = \frac{ky^2 e^{at/m}}{2}.$$

Thus the Lagrangian we have found is the time dependent

$$L(t,y,\dot{y}) = \left(\frac{m\dot{y}^2}{2} - \frac{ky^2}{2}\right) \exp\left(\frac{at}{m}\right).$$

We can check that the Euler-Lagrangian equation for this Lagrangian does indeed give the damped unforced mass-spring system.

With

$$L_y = -ky \exp(at/m), \;\; L_{\dot{y}} = m\dot{y} \exp(at/m),$$

we have

$$0 = L_y - \frac{d}{dt} L_{\dot{y}} = -ky \exp(at/m) - m\ddot{y} \exp(at/m) - a\dot{y} \exp(at/m).$$

Cancellation of the common exponent factors gives $m\ddot{y} + a\dot{y} + ky = 0$.

We can convert this second-order ODE into a Hamiltonian system of 2 first-order ODEs.

The canonical momentum is

$$p = L_{\dot{y}} = m\dot{y} \exp(at/m).$$

This inverts to give

$$\dot{y} = \frac{p}{m} \exp(-at/m).$$

As a function of y and p, the Hamiltonian is the time dependent

$$\begin{aligned}
H &= -L + \dot{y} L_{\dot{y}} \\
&= \left(\frac{ky^2}{2} - \frac{m\dot{y}^2}{2}\right) \exp(at/m) + \dot{y}(m\dot{y}) \exp(at/m) \\
&= \left(\frac{m\dot{y}^2}{2} + \frac{ky^2}{2}\right) \exp(at/m) \\
&= \left(\frac{p^2}{2m} \exp(-2at/m) + \frac{ky^2}{2}\right) \exp(at/m) \\
&= \frac{p^2}{2m} \exp(-at/m) + \frac{ky^2}{2} \exp(at/m).
\end{aligned}$$

Hamilton's equations are

$$\dot{y} = \frac{\partial H}{\partial p} = \frac{p}{m}\exp(-at/m),\ \ \dot{p} = -\frac{\partial H}{\partial y} = -ky\exp(at/m).$$

To check these we compute

$$\begin{aligned}\ddot{y} &= \frac{\dot{p}}{m}\exp(-at/m) - \frac{ap}{m^2}\exp(-at/m) \\ &= -\frac{ky}{m} - \frac{a\dot{y}}{m},\end{aligned}$$

where we replaced $\dot{p}$ with $-\partial H/\partial y$ and replaced p with an expression involving $\dot{y}$ via $\dot{y} = \partial H/\partial p$.