Math 521 Lecture #39
§4.6: Isoperimetric Problem

We review the Lagrange multiplier rule from multivariable calculus.

Theorem 4.28. Suppose $f(x, y)$ and $g(x, y)$ are differentiable functions on open sets containing (x_0, y_0) such that $\nabla g(x_0, y_0) \neq 0$. If f has an extreme value at (x_0, y_0) subject to the constraint $g(x, y) = c$ (i.e., $g(x_0, y_0) = c$), then there exists λ such that the function $f^* = f + \lambda g$ satisfies $\nabla f^*(x_0, y_0) = 0$.

Typically the Lagrange multiplier can be eliminated from the systems of equations $\nabla f^*(x_0, y_0) = 0$ and $g(x_0, y_0) = c$, thereby enabling the finding of candidate points (x_0, y_0) at which the constrained optimization problem may have a solution.

We will show how to apply the Lagrange multiplier rule to a functional subject to a certain kind of constraint.

For C^2 functions $L(x, y, y')$ and $G(x, y, y')$, the variational problem of minimizing the functional

$$J(y) = \int_a^b L(x, y, y')\, dx$$

subject to the integral constraint

$$W(y) = \int_a^b G(x, y, y')\, dx = k,$$

for a constant k, where $y \in C^2[a, b]$ and $y(a) = y_a$ and $y(b) = y_b$, is known as **isoperimetric problem**.

The equation $W(y) = k$ is known as an **isoperimetric constraint**.

We suppose that y_0 is a local minimum of the functional $J(y)$ that satisfies the isoperimetric constraint.

For the variation of y_0, the one-parameter family $z = y_0 + \epsilon h$ may not satisfy the isoperimetric constraint.

We can resolve this issue by using a two-parameter family of variations

$$z = y_0 + \epsilon_1 h_1 + \epsilon h_2$$

for $h_1, h_2 \in C^2[a, b]$ with $h_i(a) = h_i(b) = 0$ for $i = 1, 2$.

To make this work, we assume that W does not have an extremum at y_0.

Then for any choice of h_1 and h_2 there will be choices of ϵ_1 and ϵ_2 near $(0, 0)$ such that

$$W(z) = k.$$

Evaluation of J and W on the two-parameter family of variations z gives functions

$$\mathcal{J}(\epsilon_1, \epsilon_2) = \int_a^b L(x, z, z')\, dt$$

and

$$\mathcal{W}(\epsilon_1, \epsilon_2) = \int_a^b G(x, z, z')\ dt = k.$$

Because y_0 is a local minimum of $J(y)$ subject to the isoperimetric constraint, the point $(\epsilon_1, \epsilon_2) = (0, 0)$ is where $\mathcal{J}(\epsilon_1, \epsilon_2)$ has a local minimum subject to the constraint $\mathcal{W}(\epsilon_1, \epsilon_2) = k$.

We can thus apply the Lagrange multiplier rule to $\mathcal{J}$ subject to $\mathcal{W} = k$.

There is a Lagrange multiplier λ for which the function $\mathcal{J}^* = \mathcal{J} + \lambda\mathcal{W}$ satisfies

$$\nabla\mathcal{J}^*(0, 0) = (0, 0).$$

Now

$$\mathcal{J}^* = \int_a^b L(x, z, z')\ dx + \lambda \int_a^b G(x, z, z')\ dt = \int_a^b \big(L(x, z, z') + \lambda G(x, z, z')\big)\ dt.$$

We define the auxiliary function to be the integrand:

$$L^*(x, z, z') = L(x, z, z') + \lambda G(x, z, z').$$

The partial derivatives of $\mathcal{J}^*$ at $(\epsilon_1, \epsilon_2) = (0, 0)$ are

$$\frac{\partial \mathcal{J}^*}{\partial \epsilon_i}(0, 0) = \int_a^b \big(L^*_y(x, y_0, y_0')h_i + L^*_{y'}(x, y_0, y_0')h_i'\big)\ dx,\ \ i = 1, 2.$$

Integration by parts on the second term in the integral and the values of h_i are the endpoints gives

$$\frac{\partial \mathcal{J}^*}{\partial \epsilon_i}(0, 0) = \int_a^b \left(L^*_y(x, y_0, y_0') - \frac{d}{dx}L^*_{y'}(x, y_0, y_0')\right) h_i\ dx,\ \ i = 1, 2.$$

Because of the arbitrariness of h_1 and h_2, the Fundamental Lemma gives the Euler-Lagrange equation

$$L^*_y(x, y, y') - \frac{d}{dx}L^*_{y'}(x, y, y') = 0$$

which the local minimizer y_0 must satisfy.

The solutions of the Euler-Lagrange equations will involve two arbitrary constants and the Lagrange multiplier λ.

These may be determined from the conditions $y(a) = y_a$ and $y(b) = y_b$, and the substitution of y_0 into the isoperimetric constraint (thereby bringing in the value k).

Example 4.29. What is the shape of a rope of length l and constant linear density ρ that is suspended between two fixed points (a, y_a) and (b, y_b)?

We can answer this with an isoperimetric problem.

Let $y(x)$ be any shape of the rope with the assumption that $y(x) > 0$.

A small segment of the rope has length ds and a mass of ρds.

The potential energy acting on the small segment of the rope is that determined by gravity acting, and is $\rho gyds$ relative to $y = 0$.

The total potential energy of the rope is the functional

$$J(y) = \int_0^1 \rho gy \; ds = \int_a^b \rho gy\sqrt{1 + [y']^2} \; dx.$$

There is no kinetic energy because we are considering the rope in an equilibrium position of hanging between two fixed points.

The shape of the curve of the hanging rope is the function $y(x)$ that minimizes the potential energy.

The isoperimetric constraint is the constant length l of the rope, and is the integral

$$W(y) = \int_a^b \sqrt{1 + [y']^2} \; dx = l.$$

For the Lagrange multiplier λ, the auxiliary function here is

$$L^* = L + \lambda G = \rho gy\sqrt{1 + [y']^2} + \lambda\sqrt{1 + [y']^2}.$$

The Euler-Lagrange equations are

$$0 = L_y - \frac{d}{dx}L_{y'} = \rho g\sqrt{1 + [y']^2} - \frac{d}{dx}\left(\frac{\rho gyy'}{\sqrt{1 + [y']^2}} + \frac{\lambda y'}{\sqrt{1 + [y']^2}}\right).$$

Rather than solving this nonlinear second-order ODE, we recognize that the auxiliary function L^* does not depend on x, so there is a first integral,

$$\begin{aligned}
C &= L^* - y'L^*_{y'} \\
&= \rho gy\sqrt{1 + [y']^2} + \lambda\sqrt{1 + [y']^2} - y'\left(\frac{\rho gyy'}{\sqrt{1 + [y']^2}} + \frac{\lambda y'}{\sqrt{1 + [y']^2}}\right) \\
&= (\rho gy + \lambda)\sqrt{1 + [y']^2} - (\rho gy + \lambda)\left(\frac{[y']^2}{\sqrt{1 + [y']^2}}\right) \\
&= (\rho gy + \lambda)\left(\sqrt{1 + [y']^2} - \frac{[y']^2}{\sqrt{1 + [y']^2}}\right) \\
&= (\rho gy + \lambda)\left(\frac{1}{\sqrt{1 + [y']^2}}\right).
\end{aligned}$$

Solving for y' gives

$$C^2[y']^2 = (\rho gy + \lambda)^2 - C^2.$$

Separation of variables gives

$$\frac{dy}{\sqrt{(\rho g y + \lambda)^2 - C^2}} = \frac{dx}{C}.$$

The substitution $u = \rho g y + \lambda$ gives

$$\frac{du}{\rho g \sqrt{u^2 - C^2}} = \frac{dx}{C}.$$

Integration of this gives

$$\frac{1}{\rho g} \cosh^{-1}\left(\frac{u}{C}\right) = \frac{x}{C} + C_2.$$

Undoing the substitution we get the solutions

$$y = -\frac{\lambda}{\rho g} + \frac{C}{\rho g} \cosh\left(\frac{\rho g x}{C} + C_2\right).$$

This necessary condition for the shape of the hanging rope is known as a **catenary**.