

ASYMPTOTIC VALUES, PREPOLES, AND PERIODIC POINTS

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ABSTRACT. Curves of parameter values for which an asymptotic value of

$$F_{\alpha,\beta}(z) = \frac{1}{\alpha + \beta e^{-z}}$$

lies on a pre-pole of arbitrary order are found through an iterative method. Some of these curves are shown to be real analytic. Regions bounded by certain pairs of consecutive curves of parameter values are found where periodic cycles exist for $F_{\alpha,\beta}$. Numerically some of these periodic cycles are shown to be attracting and the corresponding Julia sets are depicted.

1. INTRODUCTION

In a family of meromorphic maps without fixed points, the interaction between asymptotic values and pre-poles plays a key role in determining the topology and bifurcations of the Julia set. This paper focuses on the interaction of asymptotic values with pre-poles and their effect on the Julia set for the family of meromorphic maps:

$$(1) \quad F_{\alpha,\beta}(z) = \frac{1}{\alpha + \beta e^{-z}} \quad \alpha, \beta \in \mathbb{R} \text{ and } \beta > 0.$$

An analysis of how the asymptotic values and pre-poles vary is performed for this family, finding maps in this family where an asymptotic value is a pre-pole of arbitrary order. It is known that the Julia set contains a Cantor Bouquet when an asymptotic value is a prepole (see [4] and [5]). In addition, maps in this family which have periodic cycles are found by the same analysis. When the periodic cycle is attracting, it is known that at least one asymptotic value of the map must lie in the immediate basin of attraction of this cycle, in which case the Julia set is a nowhere dense subset of the plane (see [3], [4] and [5]).

Constructing the bifurcation diagram for $F_{\alpha,\beta}$ begins with its fixed points. It is well known that $F_{\alpha,\beta}$ can have no attracting fixed points (cycles) off of \mathbb{R} . Previous work has focused on parameter values where fixed points on the real line appear. Parameter values for which $F_{\alpha,\beta}$ has an elliptic fixed point lie on curves in the parameter plane. These curves, shown in Figure 1, define the region \mathcal{T} where no fixed points occur on \mathbb{R} . Outside of \mathcal{T} the bifurcations of the Julia set of $F_{\alpha,\beta}$ are well understood (see [4] and [8] for details).

The pre-poles of $F_{\alpha,\beta}$ are found through a sequence of iteratively defined functions that act from the parameter plane to the dynamical plane. Using these iteratively defined functions, parameter values are found where one of the asymptotic values, $a_0 = 0$ or $a_1 = 1/\alpha$, is a pre-pole of a specific order. These parameter values lie on ‘curves’ in the region \mathcal{T} , as shown in Figure 2. One of these curves, generically denoted by $\beta_i^k(\alpha)$, describes parameter values where a_i lies on a pre-pole of order

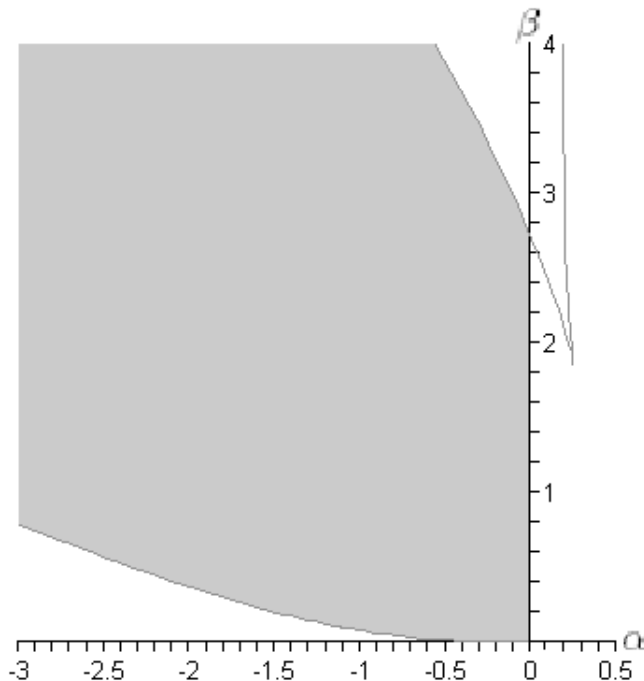


FIGURE 1. The region of interest \mathcal{T} in the parameter plane is shaded gray.

k . For each k there are two curves $\beta_i^{k\pm}(\alpha)$ which lie respectively above or below all the other $\beta_i^k(\alpha)$. These two curves $\beta_i^{k\pm}(\alpha)$ are called the bounding solutions. See Figure 2 for a numerical depiction of some of the bounding solutions. It will be shown that for each $k \in \mathbb{N}$ the bounding solutions are real analytic (see Theorem 2.2).

These bounding solutions are not the only curves where one of the asymptotic values is a pre-pole of order m . In fact, for $m \geq 3$ there are several other solutions which are located between some of the β_i^k for $k < m$. It will be shown that, if there are $p, q \in \mathbb{N}$ so that $m = k(p+1)+1 = n(q+1)-1$ (in addition to other conditions), then there are parameter values lying between the two curves β_0^{k-1} and β_1^{n-1} (or β_0^{n-1} and β_1^{k-1}) where one of the asymptotic values is a pre-pole of order m (see Theorem 2.3). See Figure 3 for a numerical depiction of these additional solution curves for $m = 3$.

Even with this complicated behavior each curve $\beta_0^k(\alpha)$ has a corresponding ‘consecutive’ solution curve $\beta_1^k(\alpha)$ which lies directly below. It is shown that no other solution curve lies between any two consecutive solutions (see Lemma 2.2). Parameter values that do lie between two consecutive curves produce a periodic cycle of period $k+2$ for $F_{\alpha,\beta}$ (see Theorem 3.1). Numerically these periodic cycles appear attracting, and hence the Julia set is a nowhere dense subset of the plane. Numerical depictions of the Julia set of $F_{\alpha,\beta}^3$ and $F_{\alpha,\beta}^6$ are given when the attracting cycles

are present. See Figures 6 and 7 for numerical depictions of their Julia sets, each with a zoomed-in part.

2. ASYMPTOTIC VALUES AS PRE-POLES

To find a description for all the pre-poles of order k a map $g_k(\alpha, \beta)$ is defined which acts on the space of parameter values (α, β) and finds all the pre-poles of order k . Poles (pre-poles of order $k = 0$) are found by setting the denominator of $F_{\alpha, \beta}(z)$ equal to zero and solving for z . These poles are denoted by:

$$g_0(\alpha, \beta) = \log\left(\frac{\beta}{-\alpha}\right),$$

where the log function is restricted to the principal branch. Suppose that the function $g_k(\alpha, \beta)$ describes the pre-poles of order k . (To simplify notation $g_k(\alpha, \beta)$ will be represented by g_k .) It follows that if g_{k+1} is a pre-pole of order $k + 1$, then $F_{\alpha, \beta}(g_{k+1}) = g_k$ and hence

$$(2) \quad g_{k+1} = \log\left(\frac{\beta g_k}{1 - \alpha g_k}\right).$$

To analyze each g_k , $\alpha = \alpha_0$ is fixed and g_k is viewed as a function of β . Only $\alpha_0 < 0$ is considered as this describes parameter values lying in \mathcal{T} .

The following is a consequence of the monotonicity and location of the singularities for the logarithm function.

Lemma 2.1. *For each $k \in \mathbb{N}$, $g_k(\alpha_0, \beta)$ restricted to \mathbb{R} is an increasing function of β in intervals where g_k is continuous and is discontinuous only for parameter values (α_0, β) where $g_{k-1}(\alpha_0, \beta) = a_i$.*

Parameter values which correspond to discontinuities of the function $g_{k+1}(\alpha, \beta)$ are of particular interest. These parameter values are solutions to the equation $g_k(\alpha, \beta) = a_i$ and are denoted by $\beta_i^k(\alpha)$. The solutions of $g_0 = a_i$ are:

$$\beta_0^0(\alpha) = -\alpha, \quad \beta_1^0(\alpha) = -\alpha e^{1/\alpha}.$$

By iteration, solutions are found to the equation $g_k = a_i$ for larger values of k , i.e., parameter values where an asymptotic value is a pre-pole of order k .

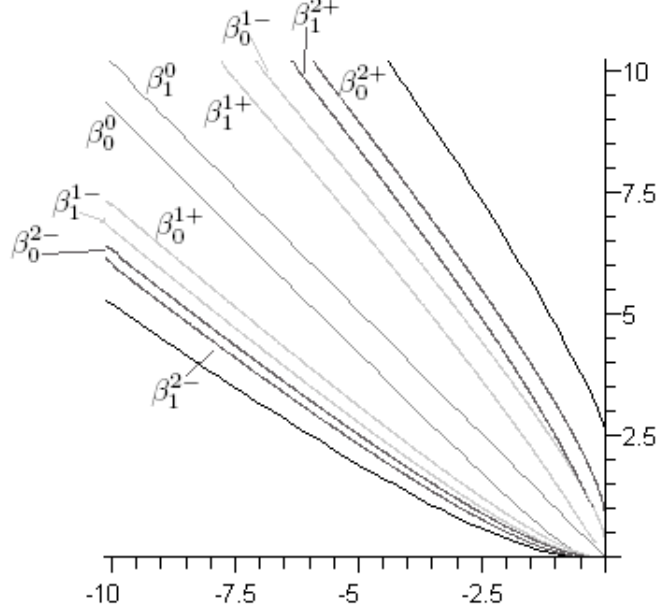
2.1. Bounding Solutions. Let $\beta_i^{k+}(\alpha_0)$ (referred to as β_i^{k+}) be the solution of $g_k(\alpha_0, \beta) = a_i$ which lies furthest above $\beta_0^0(\alpha_0)$ and $\beta_i^{k-}(\alpha_0)$ be the solution which lies furthest below $\beta_1^0(\alpha_0)$. These solutions appear in a related fashion as shown in Theorem 2.1 (see Figure 2). These bounding solutions are also shown to be real analytic curves in the parameter plane (see Theorem 2.2).

Theorem 2.1. *For each $k \geq 1$ the bounding solutions $\beta_i^{k\pm}$ exist and satisfy*

$$\beta_1^{k-}, \beta_0^{k-} \in \left(0, \beta_1^{(k-1)-}\right) \quad \text{and} \quad \beta_1^{k+}, \beta_0^{k+} \in \left(\beta_0^{(k-1)+}, +\infty\right).$$

Proof. Let $k = 1$. The function $g_1(\alpha_0, \beta)$ is continuous on the intervals $(0, \beta_1^0(\alpha_0))$ and $(\beta_0^0(\alpha_0), \infty)$. The following limits for g_1 hold:

$$\begin{aligned} \lim_{\beta \rightarrow 0} g_1(\alpha_0, \beta) &= -\infty, & \lim_{\beta \rightarrow \beta_1^0} g_1(\alpha_0, \beta) &= \infty, \\ \lim_{\beta \rightarrow \beta_0^0} g_1(\alpha_0, \beta) &= -\infty, & \lim_{\beta \rightarrow +\infty} g_1(\alpha_0, \beta) &= \infty. \end{aligned}$$

FIGURE 2. The bounding solutions for $k \leq 2$.

The Intermediate Value Theorem yields the two solutions β_0^1 and β_1^1 .

Now suppose for $k > 1$ the following limits hold for g_k :

$$(3) \quad \lim_{\beta \rightarrow 0} g_k(\alpha_0, \beta) = -\infty, \quad \lim_{\beta \rightarrow \beta_1^{(k-1)-}} g_k(\alpha_0, \beta) = \infty,$$

$$(4) \quad \lim_{\beta \rightarrow \beta_1^{(k-1)+}} g_k(\alpha_0, \beta) = -\infty, \quad \lim_{\beta \rightarrow +\infty} g_k(\alpha_0, \beta) = \infty,$$

in addition $\beta_1^{k-}, \beta_0^{k-} \in (0, \beta_1^{(k-1)-})$ and $\beta_1^{k+}, \beta_0^{k+} \in (\beta_0^{(k-1)+}, +\infty)$.

The following result immediately:

$$\begin{aligned} \lim_{\beta \rightarrow 0} g_{k+1}(\alpha_0, \beta) &= -\infty, & \lim_{\beta \rightarrow \beta_1^{k-}} g_{k+1}(\alpha_0, \beta) &= \infty, \\ \lim_{\beta \rightarrow \beta_0^{k+}} g_{k+1}(\alpha_0, \beta) &= -\infty, & \lim_{\beta \rightarrow +\infty} g_{k+1}(\alpha_0, \beta) &= \infty. \end{aligned}$$

Once again the Intermediate Value Theorem yields two β values one for each of these intervals of continuity as solutions to $g_{k+1}(\alpha_0, \beta) = a_i$ for each i . \square

Theorem 2.2. *The bounding solutions $\beta_i^{k\pm}(\alpha)$ are real analytic curves in \mathcal{T} .*

Proof. This will be shown using the Implicit Function Theorem. To this end the following are needed:

$$\frac{\partial}{\partial \beta} (g_k(\alpha, \beta)) \Big|_{\beta \geq \beta_1^{k+}(\alpha)} > 0 \quad \text{and} \quad \frac{\partial}{\partial \beta} (g_k(\alpha, \beta)) \Big|_{\beta \leq \beta_0^{k+}(\alpha)} > 0.$$

The first case is shown, and the other is similar.

Note that since each g_k is increasing with respect to β , the derivative with respect to β must be nonnegative. Hence it is only necessary to show that the derivative does not equal zero.

For $k = 0$,

$$\frac{\partial}{\partial \beta} (g_0(\alpha, \beta)) = \frac{1}{\beta} > 0.$$

Now suppose that

$$\frac{\partial}{\partial \beta} (g_k(\alpha, \beta)) \Big|_{\beta \geq \beta_1^{k+}(\alpha)} > 0.$$

If the derivative of g_{k+1} with respect to β (evaluated at the point (α, β) where $\beta \geq \beta_1^{(k+1)+}$) equals zero, then a direct application of the chain rule yields

$$\begin{aligned} \frac{1}{\beta} + \left(\frac{\partial g_k}{\partial \beta} \right) \left(\frac{1}{g_k - \alpha g_k^2} \right) &= 0 \\ \Rightarrow \left(\frac{\partial g_k}{\partial \beta} \right) \left(\frac{1}{g_k - \alpha g_k^2} \right) &< 0. \end{aligned}$$

Note from the proof of Theorem 2.1 that g_k is increasing with respect to β and $g_k(\alpha, \beta_0^{k+}(\alpha)) = 0$, so $g_k(\alpha, \beta) \geq 0$ at $\beta \geq \beta_1^{(k+1)+} > \beta_0^{k+} > \beta_1^{k+}$. However the previous assumption indicates that for g_k evaluated at $\beta \geq \beta_1^{(k+1)+} > \beta_0^{k+}$, $g_k - \alpha g_k^2 < 0$ and hence $g_k < \alpha g_k^2 < 0$. This contradiction shows that

$$\frac{\partial}{\partial \beta} (g_{k+1}(\alpha, \beta)) \Big|_{\beta \geq \beta_1^{k+1}} > 0.$$

An application of the Implicit Function Theorem yields the desired result. \square

2.2. Additional Solutions. The bounding solutions do not describe all solutions to the equation $g_m = a_i$. Other solutions in addition to the bounding solutions exist for this equation as shown in Theorem 2.3, but the bounding solutions do exhibit some properties which apply to all the solutions. The curves $\beta_i^{m\pm}(\alpha)$ occur in pairs as $\beta_0^{m\pm}(\alpha)$ and $\beta_1^{m\pm}(\alpha)$. This is true for all solutions, that is, if $g_m = a_1$ for a fixed α then $g_m = a_0$ for a nearby value of β . These solutions β_1^m and β_0^m are called ‘consecutive’ solutions. This concept of consecutive solutions applies to all the solution curves, bounding and otherwise.

Lemma 2.2. *If β_1^k and β_0^k are consecutive solutions and $\beta \in (\beta_1^k, \beta_0^k)$ then $g_m(\alpha_0, \beta) \in \mathbb{C} \setminus \mathbb{R}$ for all $m > k$.*

Proof. Note that for these β values, $1/\alpha_0 < g_k < 0$. It follows that the inner part of the logarithm of $g_{k+1}(\alpha_0, \beta)$ is less than zero and hence g_{k+1} lies off the real line. The rest of the statement follows from properties of the log function and the iterative definition of the g_m . \square

This restricts the appearance of solutions to the equation $g_m = a_i$. It also brings up the question of how g_m behaves as β approaches a previous solution β_i^k ($k < m$) from outside (β_1^k, β_0^k) . Using the iterative definition of the g_m ’s, the following are

derived from (3) and (4)

$$(5) \quad \lim_{\beta \rightarrow \beta_0^{m-2}} g_m(\alpha_0, \beta) = \lim_{g_{m-1} \rightarrow -\infty} g_m(\alpha_0, \beta) = \log \left(\frac{\beta}{-\alpha_0} \right),$$

$$(6) \quad \lim_{\beta \rightarrow \beta_1^{m-2}} g_m(\alpha_0, \beta) = \lim_{g_{m-1} \rightarrow +\infty} g_m(\alpha_0, \beta) = \log \left(\frac{\beta}{-\alpha_0} \right).$$

The following determines the behavior of each of the g_m near the curves $\beta_i^0(\alpha)$.

Lemma 2.3. *Let $m \in \mathbb{N}$.*

(1) *If m is even then*

$$(7) \quad \lim_{\beta \rightarrow \beta_i^0} g_m(\alpha_0, \beta) = a_i.$$

(2) *If m is odd then*

$$(8) \quad \lim_{\beta \rightarrow \beta_0^0} g_m(\alpha_0, \beta) = -\infty \text{ and } \lim_{\beta \rightarrow \beta_1^0} g_m(\alpha_0, \beta) = +\infty.$$

Proof. Let m be even. It follows that

$$\lim_{\beta \rightarrow \beta_i^0} g_m(\alpha_0, \beta) = \log \left(\frac{\beta}{-\alpha_0} \right) = g_0 = a_i.$$

In addition it can be seen from the definition of g_m (see equation 2) that

$$\lim_{\beta \rightarrow \beta_0^0} g_{m+1}(\alpha_0, \beta) = -\infty \text{ and } \lim_{\beta \rightarrow \beta_1^0} g_{m+1}(\alpha_0, \beta) = +\infty.$$

□

This relationship between the curves $\beta_i^0(\alpha)$ and the functions g_m has an analog for additional solution curves. The following Lemma states some conditions which determine the value that g_m approaches near one of the solution curves $\beta_i^k(\alpha)$ for some values of $k < m$.

Lemma 2.4. *If $k \neq 1$, $n \in \mathbb{N}$ and $m = k(p+1) + p - 1$ for some $p \in \mathbb{N} \cup \{0\}$,*

$$(9) \quad \lim_{\beta \rightarrow \beta_i^{k-1}} g_m(\alpha_0, \beta) = a_i.$$

Moreover if for some $l \neq 0$ and $r \in \mathbb{N} \cup \{0\}$, $m = l(r+1) + r$,

$$(10) \quad \lim_{\beta \rightarrow \beta_1^{l-1}} g_m(\alpha_0, \beta) = +\infty \text{ and } \lim_{\beta \rightarrow \beta_0^{l-1}} g_m(\alpha_0, \beta) = -\infty.$$

Before this Lemma can be proved, there are some properties of the g_k that need to be noted. It can be seen from (2) that

$$g_1(\alpha, \beta) = \log \left(\frac{\beta g_0}{1 - \alpha g_0} \right).$$

Thus, g_1 can be written as a function of α , β and of g_0 , i.e., $g_1 = g_1(\alpha, \beta, g_0)$. Using a similar approach, g_{k+1} can be defined as a function of g_0 as well, or g_{k+1} can be written in terms of g_1 :

$$g_{k+1} = g_1(\alpha, \beta, g_k) = \log \left(\frac{\beta g_k}{1 - \alpha g_k} \right).$$

Implementing this concept to all values $m < k$, the following relationship is found:

$$(11) \quad g_{m+k} = g_m(\alpha, \beta, g_k).$$

Proof. This will be shown by Induction on p for the first case (the second follows just as the odd case in Lemma 2.3). Consider $m = k(0 + 1) + 0 - 1 = k - 1$. It follows from the definition of the β_i^{k-1} that

$$\lim_{\beta \rightarrow \beta_i^{k-1}} g_m(\alpha_0, \beta) = a_i.$$

Thus the statement holds for $p = 0$. Now suppose that for $p \in \mathbb{N}$ and $m = k(p + 1) + p - 1$,

$$\lim_{\beta \rightarrow \beta_i^{k-1}} g_m(\alpha_0, \beta) = a_i.$$

Applying g_2 to this limit produces

$$\begin{aligned} \lim_{\beta \rightarrow \beta_i^{k-1}} g_2(\alpha_0, \beta, g_m) &= \lim_{\beta \rightarrow \beta_i^{k-1}} g_2(\alpha_0, \beta, a_i) = \lim_{g_0 \rightarrow \pm\infty} g_1(\alpha_0, \beta, g_0) = \log\left(\frac{\beta}{-\alpha_0}\right) \\ &= g_0(\alpha_0, \beta). \end{aligned}$$

This can also be rewritten as

$$\lim_{\beta \rightarrow \beta_i^{k-1}} g_{m+2}(\alpha_0, \beta) = \lim_{\beta \rightarrow \beta_i^{k-1}} g_0(\alpha_0, \beta).$$

Applying g_{k-1} will result in the following,

$$\begin{aligned} \lim_{\beta \rightarrow \beta_i^{k-1}} g_{k-1}(\alpha_0, \beta, g_{m+2}) &= \lim_{\beta \rightarrow \beta_i^{k-1}} g_{k-1}(\alpha_0, \beta, g_0) \\ \Rightarrow \lim_{\beta \rightarrow \beta_i^{k-1}} g_{m+k+1}(\alpha_0, \beta) &= a_i. \end{aligned}$$

The case for $p+1$ follows from these limits because $m+k+1 = k(p+1)+p-1+k+1 = k((p+1)+1) + (p+1) - 1$. \square

Before proceeding it is valuable to note that for parameter values lying above the curve $\beta_0^0(\alpha) = -\alpha$, $\log(\beta/\alpha) > 0$, and for parameter values lying below the curve $\beta_1^0(\alpha) = \exp(1/\alpha)$, $\log(\beta/\alpha) < 1/\alpha$. This indicates where the bounds found in (5) and (6) lie in relation to the asymptotic values. This combined with the previous two Lemmata culminates in the following Theorem.

Theorem 2.3. *Let $m \geq 3$, $\alpha_0 < 0$, $k \geq 0$, and $n < m$. There are solutions to $g_m = a_i$ lying in $(\beta_0^{n-1}, \beta_1^{k-1})$ and $(\beta_0^{k-1}, \beta_1^{n-1})$ if the following conditions hold.*

- (1) *For all $s < m$ there are no solutions β_i^s lying in the interval $(\beta_0^{n-1}, \beta_1^{k-1})$ or in the interval $(\beta_0^{k-1}, \beta_1^{n-1})$.*
- (2) *$\beta_0^{n-1} > \beta_0^0$, $\beta_1^{k-1} > \beta_0^0$, $\beta_0^{k-1} < \beta_1^0$, and $\beta_1^{n-1} < \beta_1^0$.*
- (3) *$\beta_0^{n-1} < \beta_1^{k-1}$ and $\beta_0^{k-1} < \beta_1^{n-1}$.*
- (4) *There exist $p, q \in \mathbb{N}$ so that $m = k(p+1) + p + 1 = n(q+1) + q$.*

These additional solutions for $g_3 = a_i$ are illustrated in Figure 3, note that the bounding curves $\beta_i^{3\pm}$ still appear and two additional sets of solutions are found between $\beta^{0\pm}$ and $\beta^{1\pm}$. This follows from Theorem 2.3 for $m = 3$, $n = 1$, $k = 2$ and $p = 0$, $q = 1$.

Proof. Let m, k and n exist for the conditions given above. It follows from Lemma 2.4 that

$$\lim_{\beta \rightarrow \beta_1^{n-1}} g_m(\alpha_0, \beta) = +\infty \text{ and } \lim_{\beta \rightarrow \beta_0^{n-1}} g_m(\alpha_0, \beta) = -\infty.$$

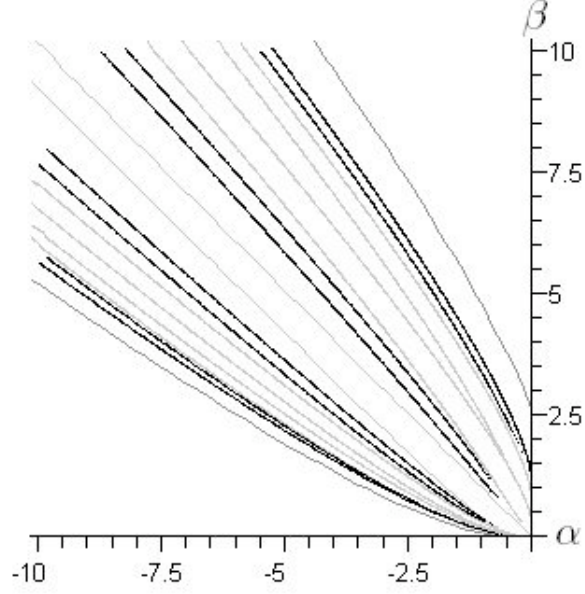


FIGURE 3. Two sets of additional solution curves for $m = 3$ are shown in black.

In addition

$$\lim_{\beta \rightarrow \beta_i^{k-1}} g_{m-2}(\alpha_0, \beta) = a_i.$$

It follows that

$$\lim_{\beta \rightarrow \beta_i^{k-1}} g_m(\alpha_0, \beta) = \log \left(\frac{\beta}{-\alpha_0} \right)$$

where condition 2 above guarantees that

$$\lim_{\beta \rightarrow \beta_0^{n-1}} g_m(\alpha_0, \beta) < 1/\alpha_0$$

and

$$\lim_{\beta \rightarrow \beta_1^{n-1}} g_m(\alpha_0, \beta) > 0.$$

This implies that on the interval $(\beta_0^{k-1}, \beta_1^{n-1})$, the function $g_m(\alpha_0, \beta)$ increases from below $1/\alpha_0$ to $+\infty$ and on the interval $(\beta_0^{n-1}, \beta_1^{k-1})$, the function $g_m(\alpha_0, \beta)$ increases from $-\infty$ to above 0. It follows from the Intermediate Value Theorem that there are solutions to the equation $g_m = a_i$ in both of these intervals. \square

3. APPEARANCE OF PERIODIC CYCLES

The solution curves of $g_k = a_i$ that have been found, have important consequences on the dynamics of $F_{\alpha, \beta}$. The interactions between the pre-poles and asymptotic values when β is between two consecutive solution curves β_1^k and β_0^k ,

force $F_{\alpha,\beta}^{k+2}|_{\mathbb{R}}$ to cross the line $y = x$, producing a periodic point of period $k+2$ (not necessarily of minimal period). It is observed that numerically these periodic orbits appear to be attracting. The topology of the Julia set is illustrated numerically when this occurs for $k = 1$ and $k = 4$.

Before proceeding it is important to note that the image of an asymptotic value is an asymptotic value for higher powers of the map. It follows that there are $2k$ asymptotic values for $F_{\alpha,\beta}^k(z)$.

Lemma 3.1. *If $\beta \in (\beta_1^k(\alpha), \beta_0^k(\alpha))$ then $F_{\alpha,\beta}^{k+1}(a_0) < 0$ and $F_{\alpha,\beta}^{k+1}(a_1) > 0$.*

Proof. Note that the equation $g_k(\alpha, \beta) = a_i$ is equivalent to

$$F_{\alpha,\beta}^k(a_i) = g_0(\alpha, \beta) = \log\left(\frac{\beta}{-\alpha}\right), \text{ and}$$

$$F_{\alpha,\beta}^{k+1}(a_i) = \frac{1}{\alpha + \beta \exp\left(-F_{\alpha,\beta}^k(a_i)\right)}.$$

For $\beta_1^k(\alpha) < \beta < \beta_0^k(\alpha)$ it follows that $F_{\alpha,\beta}^k(a_0) < g_0(\alpha, \beta) < F_{\alpha,\beta}^k(a_1)$. The monotonicity of the exponential yields

$$\beta \exp\left(-F_{\alpha,\beta}^k(a_0)\right) < \beta \exp\left(-g_0(\alpha, \beta)\right) = -\alpha.$$

Thus

$$F_{\alpha,\beta}^{k+1}(a_0) = \frac{1}{\alpha + \beta \exp\left(F_{\alpha,\beta}^k(a_0)\right)} < 0.$$

Similarly,

$$\beta \exp\left(-F_{\alpha,\beta}^k(a_1)\right) > \beta \exp\left(-g_0(\alpha, \beta)\right) = -\alpha.$$

And thus

$$F_{\alpha,\beta}^{k+1}(a_1) = \frac{1}{\alpha + \beta \exp\left(F_{\alpha,\beta}^k(a_1)\right)} > 0.$$

□

Before the influence of the asymptotics on pre-poles is discussed, the number of pre-poles on the real line is determined.

Lemma 3.2. *There are exactly $k+1$ pre-poles $\{p_k\}$ lying on the real line for $F_{\alpha,\beta}$, where the order of p_i is i .*

Proof. Let $k \in \mathbb{N}$. Note that the appearance of a unique pole on the real line at $\log(-\beta/\alpha)$ is guaranteed since $\alpha < 0$ and $\beta > 0$. Suppose that for $n-1 < k+1$ there are $n-1$ pre-poles $\{p_0, p_1, \dots, p_{n-2}\}$ where the order of p_i is i . It follows that the set

$$\{p_0\} \cup \left(\bigcup_{i=0}^{n-2} \mathbb{R} \cap \{F_{\alpha,\beta}^{-1}(p_i)\} \right)$$

describes all the pre-poles up to order $n-1$ on the real line. Since $F_{\alpha,\beta}^{-1}(x)|_{\mathbb{R}}$ is well-defined it follows that there are n pre-poles for each $n \leq k+1$. Thus we are guaranteed the existence of $k+1$ pre-poles one each of order $0, 1, \dots, k$.

Recall from Lemma 2.2 that $g_{k+1}(\alpha, \beta) \in \mathbb{C} \setminus \mathbb{R}$ in this region. From the definition of g_{k+1} it follows that there are no pre-poles on the real line of order $n > k$. □

The location of each pre-pole is determined in relation to asymptotic values of powers of $F_{\alpha,\beta}$.

Lemma 3.3. *Let $\beta \in (\beta_1^k, \beta_0^k)$ and p_n denote the pre-pole of order n . Then $p_n \in (F_{\alpha,\beta}^{k-n}(a_1), F_{\alpha,\beta}^{k-n}(a_0))$.*

Proof. Consider $n = k$. Since $g_k(\alpha, \beta)$ is monotonic with respect to β then applying g_k to $\beta_1^k < \beta < \beta_0^k$ yields $p_k = g_k(\alpha, \beta) \in (a_1, a_0)$.

Now suppose that for $0 < n < k$, $p_n \in (F_{\alpha,\beta}^{k-n}(a_1), F_{\alpha,\beta}^{k-n}(a_0))$. It follows that since $F_{\alpha,\beta}$ is continuous and monotonic on this interval, $F_{\alpha,\beta}(p_n) = p_{n-1} \in (F_{\alpha,\beta}^{k-n+1}(a_1), F_{\alpha,\beta}^{k-n+1}(a_0))$. \square

Now that the location of these pre-poles is determined, their impact on the dynamics of $F_{\alpha,\beta}^{k+2}$ is observed.

Lemma 3.4. *Let $\{p_0, p_1, \dots, p_k\}$ denote the pre-poles with corresponding order $\{0, 1, \dots, k\}$. Then for $0 \leq n \leq k$, on the real line*

$$\lim_{x \rightarrow p_n^+} F_{\alpha,\beta}^{k+2}(x) = F_{\alpha,\beta}^{k-n}(a_0) \quad \text{and} \quad \lim_{x \rightarrow p_n^-} F_{\alpha,\beta}^{k+2}(x) = F_{\alpha,\beta}^{k-n}(a_1).$$

Proof. Recall that

$$(12) \quad \lim_{x \rightarrow -\infty} F_{\alpha,\beta}(x) = a_0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} F_{\alpha,\beta}(x) = a_1, \quad \text{and}$$

$$\lim_{x \rightarrow p_0^+} F_{\alpha,\beta}(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow p_0^-} F_{\alpha,\beta}(x) = +\infty.$$

It follows from the monotonicity of $F_{\alpha,\beta}$ that

$$\lim_{x \rightarrow p_n^+} F_{\alpha,\beta}^{n+1}(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow p_n^-} F_{\alpha,\beta}^{n+1}(x) = +\infty.$$

Using (12) and applying $F_{\alpha,\beta}$ in the limit yields

$$\lim_{x \rightarrow p_n^+} F_{\alpha,\beta}^{n+2}(x) = a_0 \quad \text{and} \quad \lim_{x \rightarrow p_n^-} F_{\alpha,\beta}^{n+2}(x) = a_1.$$

When $F_{\alpha,\beta}^{k-n}$ is applied to these limits, the desired result is achieved. \square

A consequence of Lemmas 3.4 and 3.1 is that on the real line, the range of $F_{\alpha,\beta}^{k+2}(x)$ is restricted to $k+2$ intervals, as shown in the proof of the following Theorem (see Figure 4).

Theorem 3.1. *If $\beta \in (\beta_1^k(\alpha), \beta_0^k(\alpha))$, then $F_{\alpha,\beta}(x)|_{\mathbb{R}}$ has a periodic point of period $k+2$.*

Proof. It is sufficient to show that $F_{\alpha,\beta}^{k+2}(x)$ has $k+2$ fixed points, i.e., that the graph of $F_{\alpha,\beta}^{k+2}(x)$ crosses the line $y = x$ on every interval in its range (see Figures 4 and 5).

Consider the graph of $F_{\alpha,\beta}^{k+2}|_{\mathbb{R}}$ restricted to the sub-interval $(-\infty, p_k)$. From Lemma 3.4 (Equation (12)) the image of this interval under the map $F_{\alpha,\beta}^{k+2}$, is $(F_{\alpha,\beta}^{k+1}(a_0), a_1)$. Thus the left end of the graph of $F_{\alpha,\beta}^{k+2}|_{(-\infty, p_k)}$ is the point $\{-\infty, F_{\alpha,\beta}^{k+1}(a_0)\}$ which clearly lies above the line $y = x$. Similarly the right end-point is $\{p_k, a_1\}$. Because $p_k > a_1$ (see Lemma 3.3 with $n = k$) then the right

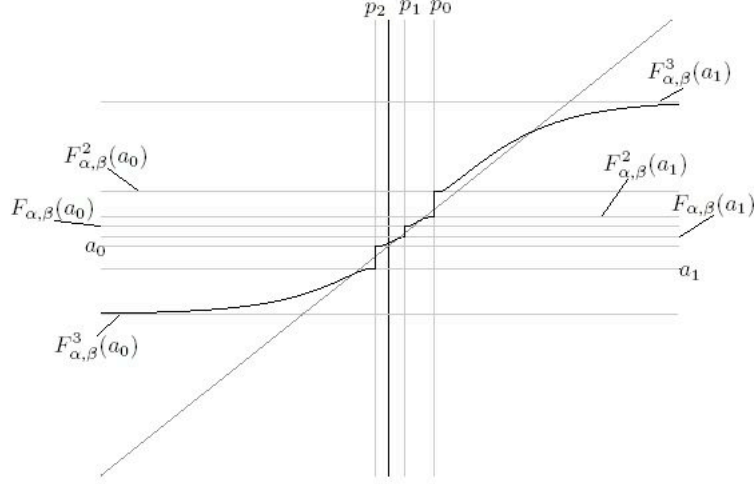


FIGURE 4. The graph of $F_{\alpha, \beta}^4$, its pre-poles and asymptotic values for $\alpha = -2$ and $\beta = 4.35$.

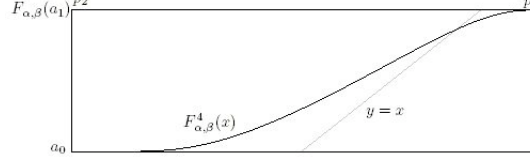


FIGURE 5. The graph of $F_{\alpha, \beta}^4$ for $\alpha = -2$ and $\beta = 4.35$ restricted to the interval (p_2, p_1) .

endpoint lies below the line $y = x$. It follows since $F_{\alpha, \beta}^{k+2}$ is continuous and increasing on this interval, that there is a point $x_1 \in (-\infty, p_k)$ so that $F_{\alpha, \beta}^{k+2}(x_1) = x_1$.

Now consider the graph of $F_{\alpha, \beta}^{k+2}|_{\mathbb{R}}$ restricted to the sub-interval (p_n, p_{n-1}) for any $1 \leq n \leq k$ (see Figure 5). It follows from Lemma 3.4 that:

$$\lim_{x \rightarrow p_n^+} F_{\alpha, \beta}^{k+2}(x) = F_{\alpha, \beta}^{k-n}(a_0) \text{ and } \lim_{x \rightarrow p_{n-1}^-} F_{\alpha, \beta}^{k+2}(x) = F_{\alpha, \beta}^{k-n+1}(a_1).$$

It follows that $\{p_n, F_{\alpha, \beta}^{k-n}(a_0)\}$, the left endpoint of the graph of $F_{\alpha, \beta}^{k+2}|_{(p_n, p_{n-1})}$, lies above the line $y = x$. The right endpoint, $\{p_{n-1}, F_{\alpha, \beta}^{k-n+1}(a_1)\}$ lies below the line $y = x$. Because $F_{\alpha, \beta}^{k+2}$ is continuous and increasing on this interval, there is an additional fixed point lying in (p_n, p_{n-1}) for each $1 \leq n \leq k$. Thus there are k additional fixed points x_i for $F_{\alpha, \beta}^{k+2}(x)$ lying in the intervals (p_i, p_{i-1}) on the real line.

There is an additional fixed point contained in the interval $(p_0, +\infty)$ that can be found in a similar manner to that found for the interval $(-\infty, p_k)$. This produces $k + 2$ fixed points for $F_{\alpha,\beta}^{k+2}(x)$. \square

As shown in Figures 4 and 5, numerically these periodic cycles appear attracting in which case the Julia set of $F_{\alpha,\beta}$ is a nowhere dense subset of the plane. To determine what the Julia set of $F_{\alpha,\beta}$ is, the dynamics of $F_{\alpha,\beta}^{k+2}$ are considered.

It is shown in [2] that the presence of a pole surrounded by two attracting fixed points produces a Jordan curve as the Julia set of the function. A similar system is found when $\beta \in (\beta_1^0, \beta_0^0)$ in which case the Julia set of $F_{\alpha,\beta}^2$ is homeomorphic to a Jordan curve.

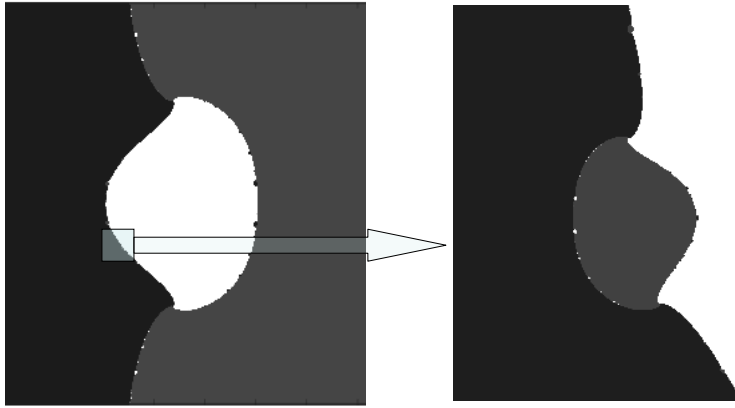


FIGURE 6. A part of the Julia set of $F_{\alpha,\beta}^3(z)$ for $\alpha = -3$ and $\beta = 1.9$ is the boundary between any two shades of gray. Also, a zoomed depiction.

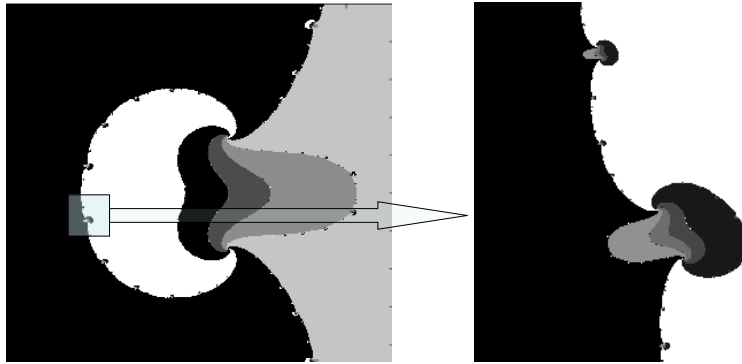


FIGURE 7. A part of the Julia set of $F_{\alpha,\beta}^6(z)$ for $\alpha = -1$ and $\beta = 3.75$, along with a zoomed-in depiction.

For higher values of k the Julia set is more complicated. Some numerically generated images of the Julia set of $F_{\alpha,\beta}^{k+2}$ when $\beta \in (\beta_1^k, \beta_0^k)$ are illustrated in

Figures 6 and 7. In this case the basins of attraction for the $k + 2$ fixed points of $F_{\alpha,\beta}^{k+2}$ are each given a different shade of gray. The Julia set will be the boundary between any two shades of gray, or between any two basins of attraction.

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