

ASSIGNMENT B: INTEGRATION BY PARTS

(Due Thursday, January 31)

In this assignment you will study an application of the technique of integration by parts to find polynomials that approximate a given function.

Exercise 1. Assume that f is a differentiable function and that f' is continuous. Let x and a be real numbers. By the Fundamental Theorem of Calculus, we know that

$$f(x) = f(a) + \int_a^x f'(t) dt.$$

Use integration by parts to show that

$$f(x) = f(a) + f'(a)(x - a) - \int_a^x f''(t)(t - x) dt.$$

Hint: Let $u = f'(t)$ and $dv = dt$. Do **not** use the antiderivative $v = t$. Use a different antiderivative. It is very important to note that the dummy variable is t rather than x .

Exercise 2. Assume that f is at least twice differentiable and that f'' is continuous. Use integration by part and the previous exercise to show that

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{1}{2!} \int_a^x f^{(3)}(t)(t - x)^2 dt.$$

Note: Recall that $f^{(k)}(t)$ means the k th derivative of $f(t)$ with respect to t .

Exercise 3. Assume that f is at least three times differentiable and that $f^{(3)}(t)$ is continuous. Use integration by part and the previous exercise to show that

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f^{(3)}(a)}{3!}(x - a)^3 - \frac{1}{3!} \int_a^x f^{(4)}(t)(t - x)^3 dt.$$

Exercise 4. Assume f is at least $n + 1$ times differentiable and that $f^{(n+1)}(t)$ is continuous. Use the method of proof by induction, to show that

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x - a)^k + \frac{(-1)^n}{n!} \int_a^x f^{(n+1)}(t)(t - x)^n dt.$$

Note: In Exercise 1 you already verified this formula when $n = 1$. To complete the argument, assume that the formula holds for $n = j$ where $1 \leq j < n$ and show that this implies that the formula also is true when $n = j + 1$. Please use summation notation!

Remark: We regard the formula in the previous exercise as being the sum of two main pieces $T_n(x)$ and $R_n(x)$ defined by

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x - a)^k$$

and

$$R_n(x) = \frac{(-1)^n}{n!} \int_a^x f^{(n+1)}(t)(t - x)^n dt = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x - t)^n dt.$$

The term $T_n(x)$ is called the n th Taylor polynomial of $f(x)$ centered at $x = a$, and the term $R_n(x)$ is called the n th remainder. Often $T_n(x)$ is thought of as an approximation of the function $f(x)$ near $x = a$ while the term $R_n(x)$ is thought of as the error in the approximation.

A slightly different method of obtaining the same polynomials is explained on pages 253-254 at the end of Section 3.10 in volume 1 of the text.

Exercise 5. Calculate the following:

- (a) $T_3(x)$ if $f(x) = x^5$ and $a = 2$. Leave the answer in terms of powers of $x - 2$.
- (b) $T_n(x)$ if $f(x) = e^{3x}$ and $a = 7$. Express the answer in summation notation.
- (c) $T_n(x)$ if $f(x) = \cos(x)$ and $a = 0$. Express the answer in summation notation.
- (d) $T_n(x)$ if $f(x) = \cos(x)$ and $a = \pi/4$. In part (d), you are not required to express the solution in summation notation.

Exercise 6.

- (a) Find the general formula for the Taylor polynomial $T_n(x)$ for $f(x) = 1/x$ centered at $x = 1$. Use summation notation to express the answer.
- (b) Use a computer to plot $T_n(x)$ from part (a) and $1/x$ together in the viewing rectangle $[-1, 3]$ by $[-10, 10]$ for $n = 4, 5, 6, 7$. (Make separate plots for each value of n .) Comment on what is going on near $x = 0$ and comment on the difference between n even and n odd. Include the printout of the plots.