Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Math. Anal. Appl. 301 (2005) 386–393

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

# Differential operators and entire functions with simple real zeros

D.A. Cardon *, S.A. de Gaston

*Department of Mathematics, Brigham Young University, Provo, UT 84602, USA*

Received 28 February 2004

Available online 11 September 2004

Submitted by B.C. Berndt

**Abstract**

Let $\phi$ and $f$ be functions in the Laguerre–Pólya class. Write $\phi(z) = e^{-\alpha z^2}\phi_1(z)$ and $f(z) = e^{-\beta z^2} f_1(z)$, where $\phi_1$ and $f_1$ have genus 0 or 1 and $\alpha, \beta \geqslant 0$. If $\alpha\beta < 1/4$ and $\phi$ has infinitely many zeros, then $\phi(D) f(z)$ has only simple real zeros, where $D$ denotes differentiation.


*Keywords:* Differential operators; Zeros of entire functions; Laguerre–Pólya class; Simple zeros

## 1. Introduction

In this paper we answer a question of Craven and Csordas stated in [1] regarding the simplicity of the zeros of $\phi(D) f(z)$, where both $\phi$ and $f$ are in the Laguerre–Pólya class and $D$ denotes differentiation. The Laguerre–Pólya class, denoted $\mathcal{LP}$, consists of the entire functions having only real zeros with Weierstrass products of the form

$$cz^m e^{\alpha z-\beta z^2} \prod_k \left(1 - \frac{z}{\alpha_k}\right) e^{z/\alpha_k},$$

* Corresponding author.
*E-mail address:* cardon@math.byu.edu (D.A. Cardon).

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2004.07.032

where $c, \alpha, \beta, \alpha_k$ are real, $\beta \geqslant 0$, $\alpha_k \neq 0$, $m$ is a nonnegative integer, and $\sum_{k=1}^{\infty} 1/\alpha_k^2 < \infty$. An entire function belongs to $\mathcal{LP}$ if and only if it is the uniform limit on compact sets of a sequence of real polynomials having only real zeros [2, Theorem 3, p. 331].

One of the reasons for studying the Laguerre–Pólya class is its relationship to the Riemann zeta function. Let

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

where $\zeta(s)$ is the Riemann zeta function. Then $\xi(1/2+iz)$ is an even entire function of genus 1 that is real for real $z$. The Riemann hypothesis, which predicts that the zeros of $\xi(s)$ have real part 1/2, can be stated as $\xi(1/2+iz) \in \mathcal{LP}$. Furthermore, evidence suggests that most, if not all, of the zeros of $\xi(s)$ are simple. Hence, functions in $\mathcal{LP}$ with simple zeros are especially interesting.

For $\phi(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{LP}$ and $f \in \mathcal{LP}$ we consider the differential operator $\phi(D)$ defined by

$$\phi(D)f(z) = \sum_{k=0}^{\infty} a_k f^{(k)}(z).$$

With suitable hypotheses $\phi(D)f(z) \in \mathcal{LP}$ (see Lemma 2 below). There are several cases in which the zeros of $\phi(D)f(z)$ are known to be simple. Craven and Csordas proved that if $\phi$ and $f$ have orders less than 2, if $\phi$ has infinitely many zeros, and if there is a bound on the multiplicity of the zeros of $f$, then $\phi(D)f(z)$ has only simple real zeros [1, Theorem 4.6]. They also showed that if $\phi$ and $f$ have orders less than 2, if $\phi$ has infinitely many zeros, and if the canonical product representation of $\phi$ has genus zero, then $\phi(D)f(z)$ has only simple real zeros [1, Theorem 4.7]. In the same paper they state the open problem of whether $\phi(D)f(z)$ has simple zeros without the extra hypothesis bounding the order of zeros of $f$ or the hypothesis that $\phi$ has genus zero [1, p. 819].

In this paper we answer that question in the affirmative with the following theorem.

**Theorem 1.** *Let $\phi$ and $f$ be in $\mathcal{LP}$. Write $\phi(z) = e^{-\alpha z^2}\phi_1(z)$ and $f(z) = e^{-\beta z^2} f_1(z)$, where $\phi_1$ and $f_1$ have genus* 0 *or* 1 *and $\alpha, \beta \geqslant 0$. If $\alpha\beta < 1/4$ and $\phi$ has infinitely many zeros, then $\phi(D)f(z)$ has only simple real zeros.*

This theorem is proved in Section 3.

We remark that the hypothesis $\alpha\beta < 1/4$ in Theorem 1 is necessary. The term $1/4$ arises in proving the convergence of the series defining $\phi(D)f(z)$ as in Lemma 3.1 in [1, p. 806] or Theorem 8 in [2, p. 360]. On the other hand, if the Weierstrass product for $\phi$ contains the genus two factor $e^{-\alpha z^2}$ and if $f$ has order less than two, the assumption that $\phi$ has infinitely many zeros is not necessary. Theorem 3.10 in [1] states that if $\alpha > 0$ and if $g$ is a function in $\mathcal{LP}$ of order less than 2, then the zeros of $e^{-\alpha D^2}g(z)$ are simple and real. Consequently, if $\phi(z) = e^{-\alpha z^2}\phi_1(z)$, where $\alpha > 0$ and $\phi_1(z)$ has genus less than two, then $\phi(D)f(z) = e^{-\alpha D^2}(\phi_1(D)f(z))$ has only simple zeros even if $\phi_1(z)$ has finitely many zeros. If $\phi$ lacks the genus two factor $e^{-\alpha z^2}$ and has finitely many zeros, the conclusion of the theorem does not hold.

## 2. Preliminaries

For $\phi(z)=\sum_{k=0}^{\infty}a_kz^k\in\mathcal{LP}$ and $f\in\mathcal{LP}$ it is important to know when the expression

$$\phi(D)f(z)=\sum_{k=0}^{\infty}a_kf^{(k)}(z)$$

makes sense. For our purposes, the following well-known result will suffice.

**Lemma 2.** *Write $\phi(z)=e^{-\alpha z^2}\phi_1(z)$ and $f(z)=e^{-\beta z^2}f_1(z)$, where $\phi_1(z)$ and $f_1(z)$ have genus 0 or 1 and $\alpha,\beta\geqslant 0$. If $\alpha\beta<1/4$, then $\phi(D)f(z)\in\mathcal{LP}$.*

**Proof.** See Levin [2, Theorem. 8, p. 360]. □

Lemma 2 shows that under the assumptions of Theorem 1 the expression $\phi(D)f(z)$ represents a function in the Laguerre–Pólya class. Thus, $\phi(D)f(z)$ has only real zeros. A natural question to ask is whether the zeros are also simple. As the convergence of the sum defining $\phi(D)f(z)$ is not in question, the proof of Theorem 1 in the following section focuses solely on the question of simplicity.

## 3. Proof of Theorem 1

In this section we will prove Theorem 1. The proof builds upon results from the paper of Craven and Csordas [1] and upon well-known facts about entire function as in Levin [2].

The basic outline of the proof of Theorem 1 is as follows: We begin by studying the effect of individual factors in the Weierstrass product for $\phi(D)$ on $f(z)$. Thus, in Lemmas 3–5, we consider the expression $h=f-\alpha^{-1}f'$. We show that if $h$ has a zero of order $m\geqslant 2$ at $x_0$, then $f$ has a zero of order at least $m+1$ at $x_0$. This fact will be used to prove Lemma 6 which says that in a fixed interval the expression $\prod_{k=1}^{n}(1-D/\alpha_k)f(z)$ has only simple zeros for sufficiently large $n$. This result is extended in Lemmas 7–10 to show that if $\phi(z)=\prod_{k=1}^{\infty}(1-z/\alpha_k)$ is of genus 0, then $\phi(D)f(z)$ has only simple real zeros. Finally, in Lemma 11 the result is extended to the more general case, stated in the hypotheses of Theorem 1, to show that $\phi(D)f(z)$ has only simple real zeros. This proves Theorem 1. We will now proceed with the proof.

**Lemma 3.** *Let $f\in\mathcal{LP}$ and let $\alpha\neq 0$ be real. Then*

(1) $f'\in\mathcal{LP}$, *and*
(2) $h=(I-\alpha^{-1}D)f=f-\alpha^{-1}f'\in\mathcal{LP}$.

**Proof.** Although this is a special case of Lemma 2, we recall the elementary argument. Since $f$ is the uniform limit of a sequence of real polynomials $\{f_n\}$ having only real zeros, $f'$ is the uniform limit of the sequence $\{f_n'\}$. Because each $f_n$ has only real zeros, each $f_n'$ also has only real zeros. Hence, the zeros of $f'$ are also real, and $f'\in\mathcal{LP}$. Then

$$h(z) = -\alpha^{-1} e^{\alpha z} D\big(e^{-\alpha z} f(z)\big).$$

So, $h$ is also in $\mathcal{LP}$. $\square$

**Lemma 4** (Laguerre inequalities). *Let $f \in \mathcal{LP}$. Then*

$$\big(f^{(n)}(z)\big)^2 - f^{(n-1)}(z) f^{(n+1)}(z) \geqslant 0, \quad -\infty < z < \infty,\ n \geqslant 1.$$

*Equality holds if and only if $f^{(n-1)}(z)$ is of the form $ce^{\alpha z}$ or if $z$ is a multiple root of $f^{(n-1)}(z)$.*

**Proof.** We follow the explanation in [3, p. 69]. If $f(z)$ is of the form $f(z) = ce^{\alpha z}$, then $[f'(z)]^2 - f(z)f''(z) = 0$ for all $z$. Otherwise, we express $f(z)$ as a Weierstrass product

$$f(z) = cz^m e^{\alpha z - \beta z^2} \prod_k (1 - z/\alpha_k) e^{z/\alpha_k}.$$

The logarithmic derivative of $f(z)$ is

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + \alpha - 2\beta z + \sum_{k=0}^{\infty} \left( \frac{1}{z - \alpha_k} + \frac{1}{\alpha_k} \right).$$

Hence,

$$\frac{d}{dz}\left(\frac{f'(z)}{f(z)}\right) = \frac{f''(z)f(z) - (f'(z))^2}{(f(z))^2} = -\frac{m}{z^2} - 2\beta - \sum_{k=1}^{\infty} \frac{1}{(z - \alpha_k)^2} < 0.$$

This shows that if $f(z)$ is not of the form $ce^{\alpha z}$ and if $z$ is real but not a root of $f$, then

$$\big(f'(z)\big)^2 - f(z)f''(z) > 0. \tag{1}$$

By continuity

$$\big(f'(z)\big)^2 - f(z)f''(z) \geqslant 0 \tag{2}$$

for all real $z$ with equality if and only if $f(z)$ is of the form $ce^{\alpha z}$ or $z$ is a multiple root of $f$. Since the derivative of a function in $\mathcal{LP}$ is also in $\mathcal{LP}$, inequalities (1) and (2) apply to the derivatives of $f$. $\square$

**Lemma 5** [1, Lemma 4.2]. *Let $f \in \mathcal{LP}$ and let $h(z) = f(z) - \alpha^{-1} f'(z)$, where $\alpha \neq 0$ is real. If $h(z)$ has a zero of order $m \geqslant 2$ at $x_0$, then $f(z)$ has a zero of order at least $m+1$ at $x_0$. Consequently, if the zeros of $f$ are simple, then the zeros of $h$ are also simple.*

**Proof.** Since $h(z)$ has a zero of order $m$ at $x_0$,

$$0 = h^{(k)}(x_0) = f^{(k)}(x_0) - \alpha^{-1} f^{(k+1)}(x_0)$$

for $0 \leqslant k \leqslant m-1$. This implies that

$$f^{(k)}(x_0) = \alpha^k f(x_0)$$

for $0 \leqslant k \leqslant m$. Then for $1 \leqslant k \leqslant m-1$,

$$\begin{aligned}&\big(f^{(k)}(x_0)\big)^2 - f^{(k-1)}(x_0)f^{(k+1)}(x_0)\\&\quad = \big(\alpha^k f(x_0)\big)^2 - \big(\alpha^{k-1}f(x_0)\big)\big(\alpha^{k+1}f(x_0)\big) = 0.\end{aligned}$$

Since $f, f', \dots, f^{(m-1)}$ are not exponential functions (otherwise $h$ could not have a zero of order $m$), the Laguerre inequalities (Lemma 4) imply that

$$f^{(k)}(x_0) = 0$$

for $0 \leqslant k \leqslant m$. In other words, $f$ has a zero of order at least $m+1$ at $x_0$. $\square$

**Lemma 6.** *Let $\phi_n(z) = \prod_{k=1}^{n}(1 - z/\alpha_k)$, where $\alpha_1, \alpha_2, \alpha_3, \dots$ are real and nonzero, and let $f \in \mathcal{LP}$. Given $A > 0$ there exists $N$ such that if $n \geqslant N$, then $\phi_n(D)f(z)$ has only simple zeros in the interval $(-A, A)$.*

**Proof.** Assume, to the contrary, that for some $A > 0$ there is a sequence $0 < n_1 < n_2 < n_3 < \cdots$ such that $\phi_{n_j}(D)f(z)$ has a zero $x_j$ of multiplicity at least two in the interval $(-A, A)$. By Lemma 5, $x_j$ is a zero of $f(z)$ of order at least $n_j + 2$. Since the sequence $n_j + 2$ is unbounded, $f(z)$ has zeros of arbitrarily large order in the finite interval $(-A, A)$. This is impossible since $f(z)$ is entire. $\square$

We will extend the previous lemma to show that if $\phi \in \mathcal{LP}$ and if $\phi$ has genus zero, then $\phi(D)f(z)$ has simple zeros. This is shown in Lemma 10. Lemmas 7–9 provide several technical results needed for the proof of Lemma 10.

**Lemma 7.** *Assume $f \in \mathcal{LP}$ is of the form*

$$f(z) = cz^m e^{\alpha z - \sigma z^2} \prod_{k=1}^{\infty}\left(1 - \frac{z}{\beta_k}\right)e^{z/\beta_k}$$

*and assume $\epsilon > 0$. Then*

$$\big|f^{(n)}(z)\big| \leqslant n! A_\epsilon \left(\frac{2e(\sigma+\epsilon)}{n}\right)^{n/2}$$

*for $|z| \leqslant R = \sqrt{\frac{n}{2(\sigma+\epsilon)}}$, where $A_\epsilon$ is a constant depending on $\epsilon$.*

**Proof.** As explained in [2, p. 13], the product

$$z^m e^{\alpha z} \prod_{k=1}^{\infty}\left(1 - \frac{z}{\beta_k}\right)e^{z/\beta_k}$$

(which lacks the term $e^{-\sigma z^2}$) is of order at most 2 and of minimal type. Thus $f(z)$ is of order 2 and normal type $\sigma$. Therefore, given $\epsilon > 0$ there exists $A_\epsilon$ such that

$$M_f(R) = \max_{|z| \leqslant R}\big|f(z)\big| < A_\epsilon \exp\big((\sigma+\epsilon)R^2\big)$$

for all $R$. By Cauchy's inequality, for $|z| \leqslant R$,

$$\big|f^{(n)}(z)\big| \leqslant \frac{n! M_f(R)}{R^n} \leqslant \frac{n! A_\epsilon \exp((\sigma+\epsilon)R^2)}{R^n}.$$

The last expression is minimized when $R = \sqrt{\frac{n}{2(\sigma+\epsilon)}}$. $\square$

**Lemma 8.** *For each $n$ let*

$$\psi_n(z) = \prod_{k=n+1}^{\infty} \left(1 - \frac{z}{\alpha_k}\right),$$

*where $\sum_{k=1}^{\infty} |\alpha_k|^{-1} < \infty$. Then*

$$\left|\psi_n^{(k)}(0)\right| \leqslant k!\left(\frac{eB_n}{k}\right)^k,$$

*where $B_n = \sum_{j=n+1}^{\infty} |\alpha_j|^{-1}$.*

**Proof.** Let $M(R, \psi_n) = \max_{|z| \leqslant R} |\psi_n(z)|$. Taking the logarithm of the Weierstrass product for $\psi_n$ gives

$$\log M(R, \psi_n) \leqslant \sum_{k=n+1}^{\infty} \log\bigl(1 + |R/\alpha_k|\bigr) \leqslant \sum_{k=n+1}^{\infty} |R/\alpha_k| = B_n R.$$

By Cauchy's inequality we obtain, for $|z| \leqslant R$,

$$\left|\psi_n^{(k)}(z)\right| \leqslant \frac{k! M(R, \psi_n)}{R^k} \leqslant \frac{k! \exp(B_n R)}{R^k}.$$

The last expression is minimized if $R = k/B_n$. $\square$

**Lemma 9.** *Let $\psi_n$ be as in the previous lemma and let $f \in \mathcal{LP}$. Then $\psi_n(D) f(z)$ converges to $f(z)$ uniformly on compact sets.*

**Proof.** Let $K$ be any compact subset of $\mathbb{C}$ and let $|z| < R$ for all $z \in K$. Then

$$\psi_n(D) f(z) = \sum_{k=0}^{\infty} \frac{\psi_n^{(k)}(0)}{k!} f^{(k)}(z).$$

Now let $\epsilon > 0$ as in Lemma 7. Then

$$\begin{aligned} &\left|\psi_n(D) f(z) - f(z)\right| \\ &\quad \leqslant \sum_{1 \leqslant k \leqslant 2(\sigma+\epsilon)R^2} \frac{|\psi_n^{(k)}(0)|}{k!} \left|f^{(k)}(z)\right| + \sum_{k > 2(\sigma+\epsilon)R^2} \frac{|\psi_n^{(k)}(0)|}{k!} \left|f^{(k)}(z)\right|. \end{aligned}$$

The reason for splitting the sum is that when $k > 2(\sigma + \epsilon)R^2$ the bound from Lemma 7 applies. Applying the bounds in Lemmas 7 and 8 gives

$$\begin{aligned} &\left|\psi_n(D) f(z) - f(z)\right| \\ &\quad \leqslant \sum_{1 \leqslant k \leqslant 2(\sigma+\epsilon)R^2} \left(\frac{eB_n}{k}\right)^k \frac{k! M(R, f)}{R^k} + \sum_{k > 2(\sigma+\epsilon)R^2} \left(\frac{eB_n}{k}\right)^k k! A_\epsilon \left(\frac{2e(\sigma + \epsilon)}{k}\right)^{k/2}. \end{aligned}$$

The second summation converges by the root test from elementary calculus. Since $B_n \to 0$ as $n \to \infty$, the right-hand side of the inequality can be made arbitrarily small when $|z| < R$ by taking $n$ sufficiently large. This proves the uniform convergence. $\square$

**Lemma 10.** *Let* $\phi(z) = \prod_{k=1}^{\infty}(1 - z/\alpha_k) \in \mathcal{LP}$ *and let* $f$ *be any function in* $\mathcal{LP}$. *Then* $\phi(D)f(z)$ *has only simple real zeros.*

**Proof.** Let $A$ be any positive number. We will show that $\phi(D)f(z)$ has only simple zeros in the interval $(-A, A)$. We factor $\phi(z)$ as

$$\phi(z) = \phi_n(z)\theta_{n,m}(z)\psi_m(z),$$

where $1 \leqslant n < m$ and where

$$\phi_n(z) = \prod_{k=1}^{n}\left(1 - \frac{z}{\alpha_k}\right), \qquad \theta_{n,m}(z) = \prod_{k=n+1}^{m}\left(1 - \frac{z}{\alpha_k}\right),$$
$$\psi_m(z) = \prod_{k=m+1}^{\infty}\left(1 - \frac{z}{\alpha_k}\right).$$

Recalling that products in $\mathcal{LP}$ correspond to composition of differential operators we have

$$\phi(D)f(z) = \phi_n(D)\big[\theta_{n,m}(D)\big(\psi_m(D)f(z)\big)\big].$$

As the composition of these differential operators is commutative, the terms $\phi_n(D)$, $\theta_{n,m}(D)$, and $\psi_m(D)$ can be written in any order. According to Lemma 6, there is an $N$ such that $\phi_N(D)f(z)$ has only simple zeros in the interval $(-A, A)$. According to Lemma 9, $\psi_m(D)(\phi_N(D)f(z))$ converges uniformly on compact sets to $\phi_N(D)f(z)$. By Hurwitz's theorem the simple zeros of $\phi_N(D)f(z)$ are limit points of the zeros of $\psi_m(D)(\phi_N(D)f(z))$. Thus, there exists an $M > N$ such that $\psi_M(D)(\phi_N(D)f(z))$ has only simple zeros in the interval $(-A, A)$. Then by Lemma 5,

$$\theta_{N,M}(D)\big[\psi_M(D)\big(\phi_N(D)f(z)\big)\big] = \phi(D)f(z)$$

has only simple zeros in the interval $(-A, A)$. Since $A$ is arbitrary this proves the lemma. $\square$

**Lemma 11.** *Let* $\phi$ *and* $f$ *be in* $\mathcal{LP}$. *Write* $\phi(z) = e^{-\alpha z^2}\phi_1(z)$ *and* $f(z) = e^{-\beta z^2}f_1(z)$, *where* $\phi_1$ *and* $f_1$ *have genus* 0 *or* 1 *and* $\alpha, \beta \geqslant 0$. *If* $\alpha\beta < 1/4$ *and* $\phi$ *has infinitely many zeros, then* $\phi(D)f(z)$ *has only simple real zeros.*

**Proof.** Since $\phi$ has infinitely many zeros, there is a subsequence $\{\alpha_k\}$ of zeros of $\phi$ such that $\sum_{k=1}^{\infty}|\alpha_k|^{-1} < \infty$. Write $\phi$ as

$$\phi(z) = \phi_0(z)\phi_2(z),$$

where $\phi_0(z) = \prod_{k=1}^{\infty}(1 - z/\alpha_k)$. Note that $\phi_0$ has genus 0 and $\phi_2$ has genus $\leqslant 2$. By Lemma 2, $\phi_2(D)f(z)$ is in $\mathcal{LP}$. Then by Lemma 10,

$$\phi(D)f(z) = \phi_0(D)\big[\phi_2(D)f(z)\big]$$

is in $\mathcal{LP}$ and has only simple zeros. $\square$

This completes the proof of Theorem 1.

## Acknowledgment

We thank Craven and Csordas for stating the interesting open problem [1, p. 819] that led to the writing of this paper.

## References

[1] T. Craven, G. Csordas, Differential operators of infinite order and the distribution of zeros of entire functions, J. Math. Anal. Appl. 186 (1994) 799–820.
[2] B.Ja. Levin, Distribution of Zeros of Entire Functions, American Mathematical Society, Providence, RI, 1980.
[3] H. Skovgaard, On inequalities of the Turán type, Math. Scand. 2 (1954) 65–73.