Journal of Number Theory 130 (2010) 27–39

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Matrices related to Dirichlet series

David A. Cardon

Department of Mathematics, Brigham Young University, Provo, UT 84602, USA

ARTICLE INFO

Article history:
Received 30 August 2008
Revised 5 May 2009
Available online 16 September 2009
Communicated by David Goss

ABSTRACT

We attach a certain $n \times n$ matrix A_n to the Dirichlet series $L(s) = \sum_{k=1}^{\infty} a_k k^{-s}$. We study the determinant, characteristic polynomial, eigenvalues, and eigenvectors of these matrices. The determinant of A_n can be understood as a weighted sum of the first n coefficients of the Dirichlet series $L(s)^{-1}$. We give an interpretation of the partial sum of a Dirichlet series as a product of eigenvalues. In a special case, the determinant of A_n is the sum of the Möbius function. We disprove a conjecture of Barrett and Jarvis regarding the eigenvalues of A_n.

1. Introduction

To the Dirichlet series

$$L(s) = \sum_{k=1}^{\infty} \frac{a_k}{k^s},$$

we attach the $n \times n$ matrix

$$D_n = \sum_{k=1}^{\infty} a_k E_n(k),$$

where $E_n(k)$ is the $n \times n$ matrix whose ijth entry is 1 if $j = ki$ and 0 otherwise. For example,

E-mail address: cardon@math.byu.edu.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2009.05.013

$$D_6 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ & a_1 & & a_2 & & a_3 \\ & & a_1 & & & a_2 \\ & & & a_1 & & \\ & & & & a_1 & \\ & & & & & a_1 \end{pmatrix}.$$

Since

$$E_n(k_1)E_n(k_2) = E_n(k_1k_2)$$

for every $k_1, k_2 \in \mathbb{N}$, formally manipulating linear combinations of $E_n(k)$ is very similar to formally manipulating Dirichlet series. However, because $E_n(k)$ is the zero matrix whenever $k > n$, the sum defining D_n is guaranteed to converge. Of course, the $n \times n$ matrix contains less information than the Dirichlet series. Letting n tend to infinity produces semi-infinite matrices, the formal manipulation of which is exactly equivalent to formally manipulating Dirichlet series.

Let W_n be the matrix whose first column is the weight vector $(0, w_2, w_3, \ldots, w_n)^T$ and whose other entries are zeros. Define the $n \times n$ matrix A_n (and the special cases B_n and C_n) by

$$\begin{aligned} A_n &= W_n + D_n, \\ B_n &= W_n + D_n \quad \text{when } a_k = 1 \text{ for all } k, \\ C_n &= W_n + D_n \quad \text{when } a_k = 1 \text{ and } w_k = 1 \text{ for all } k. \end{aligned} \tag{1}$$

For example, A_6, B_6, and C_6 are the following three matrices:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ w_2 & a_1 & & a_2 & & a_3 \\ w_3 & & a_1 & & & a_2 \\ w_4 & & & a_1 & & \\ w_5 & & & & a_1 & \\ w_6 & & & & & a_1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ w_2 & 1 & & 1 & & 1 \\ w_3 & & 1 & & & 1 \\ w_4 & & & 1 & & \\ w_5 & & & & 1 & \\ w_6 & & & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & & 1 & & 1 \\ 1 & & 1 & & & 1 \\ 1 & & & 1 & & \\ 1 & & & & 1 & \\ 1 & & & & & 1 \end{pmatrix}.$$

We will always assume that $a_1 = 1$ since this ensures that the Dirichlet series $\sum a_k k^{-s}$ has a formal inverse and since this is true for many Dirichlet series that arise in number theory. For notational convenience, we set $w_1 = 1$, and occasionally we will write $a(i)$ instead of a_i. Several authors have studied the matrices B_n and C_n. In [4], it was observed that

$$\det B_n = \sum_{k=1}^{n} w_k \mu(k), \tag{2}$$

where μ is the Möbius μ-function. This is a special case of the slightly more general fact (see Theorem 2.1 below) that

$$\det A_n = \sum_{k=1}^{n} w_k b_k, \tag{3}$$

where the numbers b_k are the coefficients of the formal series

$$L(s)^{-1} = \sum_{k=1}^{\infty} \frac{b_k}{k^s}.$$

Thus, $\det A_n$ is a weighted sum of the coefficients of $L(s)^{-1}$.

To obtain (2) from (3), choose the Dirichlet series to be the Riemann zeta function $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ so that $a_k = 1$ for all k. This corresponds to the case of the matrix B_n. Since $\zeta(s)^{-1} = \sum_{k=1}^{\infty} \mu(k)k^{-s}$, where μ is the Möbius μ-function, it follows that $b_k = \mu(k)$. One particularly intriguing choice for w_k is $w_k = k^{-s}$. Then (3) results in the truncated Dirichlet series

$$\det A_n = \sum_{k=1}^{n} \frac{b_k}{k^s}.$$

As the asymptotic growth of sums of the type in Eq. (3) is important to analytic number theory, representing those sums in terms of determinants becomes very interesting.

Recall that the Riemann hypothesis is equivalent to the statement

$$\sum_{k=1}^{n} \mu(k) = O\left(n^{1/2+\epsilon}\right),$$

for every positive ϵ. Thus, the Riemann hypothesis is equivalent to

$$\det C_n = O\left(n^{1/2+\epsilon}\right),$$

for every positive ϵ.

In [1], Barrett, Forcade, and Pollington expressed the characteristic polynomial of C_n as

$$p_n(x) = (x-1)^{n-r-1}\left((x-1)^{r+1} - \sum_{k=1}^{r} v(n,k)(x-1)^{r-k}\right), \tag{4}$$

where $r = \lfloor \log_2 n \rfloor$ and where the coefficients $v(n,k)$ were described in terms of directed graphs. We will refer to the eigenvalue 1, whose multiplicity is $n-r-1$, as the *trivial* eigenvalue. The eigenvalues $\lambda \neq 1$ will be called *nontrivial* eigenvalues. In Theorem 3.2 we extend this result by determining the characteristic polynomial of the more general matrix A_n. In [1], it was shown that the spectral radius $\rho(C_n)$ of C_n is asymptotic to $\sqrt{n}$.

Barrett and Robinson [5] determined that the sizes of the Jordan blocks of B_n corresponding to the trivial eigenvalue 1 are

$$\left\lfloor \log_2(n/3) \right\rfloor + 1, \left\lfloor \log_2(n/5) \right\rfloor + 1, \ldots, \left\lfloor \log_2\left(n/\{n\}\right) \right\rfloor + 1,$$

where $\{n\}$ denotes the greatest odd integer $\leqslant n$. Theorem 4.1 of this paper shows that each nontrivial eigenvalue of A_n is simple and expresses a basis for the one-dimensional eigenspace in terms of a recursion involving the coefficients of $p_m(x)$ for $m < n$, enhancing our understanding of the Jordan form of A_n. Theorem 4.2 gives a similar result for the transpose of A_n.

The coefficients of the characteristic polynomial of C_n are related to the Riemann zeta function as follows: If $(\zeta(s)-1)^k$ is expressed as a Dirichlet series $\sum_{m=1}^{\infty} \frac{c(m,k)}{m^s}$ so that

$$\frac{1}{1+(\zeta(s)-1)} = \sum_{k=0}^{\infty}(-1)^k\left(\zeta(s)-1\right)^k = \sum_{k=0}^{\infty}(-1)^k\left(\sum_{m=1}^{\infty}\frac{c(m,k)}{m^s}\right),$$

then

$$v(n,k) = \sum_{j \leqslant n} c(j,k).$$

Evaluating $p_n(x)$ at $x=0$ gives the fundamental relationship

$$\det C_n = \sum_{i=1}^{n} \mu(i) = \prod_{\lambda \text{ nontrivial}} \lambda = \sum_{k=0}^{\lfloor \log_2 n \rfloor} (-1)^k v(n,k),$$

where $v(n,0)$ is defined to equal 1.

Barrett and Jarvis [2] showed that C_n has two large real eigenvalues $\lambda_\pm$ satisfying

$$\lambda_\pm = \pm\sqrt{n} + \log\sqrt{n} + \gamma - 1/2 + O\left(\frac{\log^2 n}{\sqrt{n}}\right), \tag{5}$$

and that the remaining $\lfloor \log_2 n \rfloor - 1$ small nontrivial eigenvalues satisfy

$$|\lambda| < \log_{2-\epsilon} n$$

for any small positive ϵ and sufficiently large n. Based on numerical evidence for various values of n as large as $n = 10^6$, they also made the following two-part conjecture:

Conjecture 1.1. *(See Barrett and Jarvis [2].) The small nontrivial eigenvalues λ of C_n satisfy*

(i) $|\lambda| < 1$, *and*
(ii) $\mathrm{Re}(\lambda) < 1$.

The statement $\mathrm{Re}(\lambda) < 1$ is, of course, weaker than the statement $|\lambda| < 1$.

Vaughan [6] refined the asymptotic formula (5) for the two large eigenvalues and showed, unconditionally, that the small eigenvalues satisfy

$$|\lambda| \ll (\log n)^{2/5},$$

and, upon the Riemann hypothesis, that the small eigenvalues satisfy

$$|\lambda| \ll \log\log(2+n). \tag{6}$$

He later showed [7] that C_n has nontrivial eigenvalues arbitrarily close to 1 for sufficiently large n, suggesting that a proof of Conjecture 1.1 would likely be quite subtle.

Investigations of the Redheffer matrix have been extended to group theory by Humphries [3] and to partially ordered sets by Wilf [8].

In Section 5, we resolve Conjecture 1.1 by showing that both parts are false. There exist values of n for which a small eigenvalue λ satisfies both $|\lambda| > 1$ and $\mathrm{Re}(\lambda) > 1$. To accomplish this we computed the characteristic polynomials for A_n for values of n as large as $n = 2^{36}$.

2. The determinant of A_n

We now find the determinant of A_n.

Theorem 2.1. *Let D_n be the Dirichlet matrix associated with the formal Dirichlet series $L(s) = \sum_{k=1}^\infty a_k k^{-s}$ where $a_1 = 1$, and write $L(s)^{-1} = \sum_{k=1}^\infty b_k k^{-s}$. Let W_n be the matrix whose first column is $(0, w_2, \ldots, w_n)^T$ and whose other entries are zeros. Let $A_n = W_n + D_n$ as in (1). Also, let $\tilde{A}_n = W_n + D_n^{-1}$. Then*

$$\det(A_n) = \sum_{k=1}^{n} w_k b_k \quad \text{and} \quad \det(\tilde{A}_n) = \sum_{k=1}^{n} w_k a_k. \tag{7}$$

Corollary 2.2. *The choice $w_k = 1$ produces partial sums of coefficients of Dirichlet series*:

$$\det A_n = \sum_{k=1}^{n} b_k \quad \textit{and} \quad \det \tilde{A}_n = \sum_{k=1}^{n} a_k. \tag{8}$$

Corollary 2.3. *The choice $w_k = k^{-s}$ gives truncations of the Dirichlet series $L(s)^{-1}$ and $L(s)$*:

$$\det A_n = \sum_{k=1}^{n} \frac{b_k}{k^s} \quad \textit{and} \quad \det \tilde{A}_n = \sum_{k=1}^{n} \frac{a_k}{k^s}. \tag{9}$$

If s is a complex number at which $L(s)$ and $L(s)^{-1}$ converge,

$$\lim_{n\to\infty} \det A_n = L(s)^{-1} \quad \text{and} \quad \lim_{n\to\infty} \det \tilde{A}_n = L(s).$$

So, Corollary 2.3 says that we may interpret $\det A_n$ and $\det \tilde{A}_n$ as approximating values of Dirichlet series. Since the determinant is the product of the eigenvalues, this relates values of Dirichlet series with eigenvalues of matrices.

Proof of Theorem 2.1. This is essentially the same argument as the one given in Redheffer's note [4] where he found the determinant of B_n. Since D_n is upper triangular with diagonal entry 1, $\det D_n = \det D_n^{-1} = 1$. Then

$$\det A_n = \det D_n^{-1} \det A_n = \det D_n^{-1} \det(W_n + D_n) = \det\bigl(D_n^{-1} W_n + I_n\bigr).$$

The matrix $D_n^{-1} W_n$ has zeros in columns 2 through n, and its $(1,1)$-entry is $\sum_{k=2}^{n} w_k b_k$. Thus, $\det A_n$ equals the $(1,1)$-entry of $D_n^{-1} W_n + I_n$ which is $\sum_{k=1}^{n} w_k b_k$. Replacing D_n with D_n^{-1} in the argument gives $\det \tilde{A}_n = \sum_{k=1}^{n} w_k a_k$. $\square$

3. The characteristic polynomial of A_n

The characteristic polynomial $p_n(x) = \det(I_n x - A_n)$ plays a significant role. Previously, $p_n(x)$ was obtained for the special case C_n in [1] and [6]. In this section, we will determine the characteristic polynomial of the more general matrix A_n. The following definition will be instrumental in describing both the characteristic polynomial of A_n and its eigenvectors.

Definition. For integers $n \geqslant 1$ and $k \geqslant 0$, we define $d(n,k)$ to be the Dirichlet series coefficients of $(L(s) - 1)^k$. That is,

$$\bigl(L(s) - 1\bigr)^k = \left(\sum_{k=2}^{\infty} \frac{a_n}{n^s}\right)^k = \sum_{n=1}^{\infty} \frac{d(n,k)}{n^s}. \tag{10}$$

We define $v(n,k)$ and $v_\ell(n,k)$ to be the weighted sums:

$$v(n,k) = \sum_{j \leqslant n} w(j) d(j,k), \quad \text{and}$$

$$v_\ell(n,k) = \sum_{j \leqslant n} w(j\ell) d(j,k). \tag{11}$$

Several cases of this definition are important to keep in mind: $d(1,0)=1$ and $d(n,0)=0$ for $n>1$; also, both $d(n,k)$ and $\nu(n,k)$ are zero if $n<2^k$ since a number smaller than 2^k cannot be written as a product of k nontrivial factors.

From the definition of $d(n,k)$,

$$\sum_{n=1}^{\infty}\frac{d(n,k)}{n^s}=\left(\sum_{k=2}^{\infty}\frac{a_n}{n^s}\right)\left(\sum_{k=2}^{\infty}\frac{a_n}{n^s}\right)^{k-1}=\left(\sum_{k=2}^{\infty}\frac{a_n}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{d(n,k-1)}{n^s}\right),$$

which immediately gives the elementary recurrence relation:

Lemma 3.1. *If $k\geqslant 1$, then*

$$d(n,k)=\sum_{\substack{i|n\\1<i}}a(i)d(n/i,k-1)=\sum_{\substack{j|n\\j<n}}a(n/j)d(j,k-1). \tag{12}$$

Theorem 3.2. *The characteristic polynomial $p_n(x)=\det(xI_n-A_n)$ is*

$$p_n(x)=(x-1)^{n-r-1}\left((x-1)^{r+1}-\sum_{k=1}^{r}\nu(n,k)(x-1)^{r-k}\right), \tag{13}$$

where $r=\lfloor\log_2 n\rfloor$. Consequently, if $\nu(n,r)\neq 0$, the algebraic multiplicity of the trivial eigenvalue $\lambda=1$ is $n-r-1$.

Proof. We will use the cofactor expansion to calculate the characteristic polynomial $p_n(x)=\det(xI_n-A_n)$. Write $M_n=xI_n-A_n$ and let

$$M_n(i_1,\ldots,i_s\mid j_1,\ldots,j_t)$$

denoted the matrix obtained by removing the rows indexed by $i_1,\ldots,i_s$ and the columns indexed by $j_1,\ldots,j_t$ from M_n. The cofactor expansion of the determinant along the first column is

$$p_n(x)=(x-1)^n+\sum_{k=2}^{n}(-1)^k w_k\det M_n(k\mid 1).$$

The matrix $M_n(k\mid 1)$ is a block matrix whose upper left $(k-1)\times(k-1)$ block is $M_k(k\mid 1)$, whose lower left $(n-k)\times(k-1)$ block is zero, and whose lower right $(n-k)\times(n-k)$ block is upper triangular with diagonal entries $x-1$. Thus

$$\det M_n(k\mid 1)=(x-1)^{n-k}\det M_k(k\mid 1),$$

where we understand $\det M_1(1\mid 1)$ to be 1, and

$$p_n(x)=(x-1)^n+\sum_{k=2}^{n}(-1)^k w_k(x-1)^{n-k}\det M_k(k\mid 1). \tag{14}$$

The ℓth entry in the last column of $M_k(k \mid 1)$ is $-a_{n/\ell}$ if ℓ divides k; otherwise, it is zero. Then the cofactor expansion of $\det M_k(k \mid 1)$ along the last column is

$$\det M_k(k \mid 1) = \sum_{\substack{\ell \mid k \\ \ell < k}} (-1)^{k+\ell} a_{k/\ell} \det M_k(\ell, k \mid 1, k).$$

The matrix $M_k(\ell, k \mid 1, k)$ is also a block matrix. Since the upper left $(\ell - 1) \times (\ell - 1)$ block is $M_\ell(\ell \mid 1)$, the lower left $(k - \ell - 1) \times (\ell - 1)$ block is zero, and the lower right $(k - \ell - 1) \times (k - \ell - 1)$ block is upper triangular with diagonal entries $x - 1$,

$$\det M_k(\ell, k \mid 1, k) = (x - 1)^{k-\ell-1} \det M_\ell(\ell \mid 1).$$

This shows that

$$\det M_k(k \mid 1) = \sum_{\substack{\ell \mid k \\ \ell < k}} (-1)^{k+\ell} a_{k/\ell} (x - 1)^{k-\ell-1} \det M_\ell(\ell \mid 1). \tag{15}$$

In other words, the quantity $q_k(x) = (-1)^{k-1} M_k(k \mid 1)$ satisfies the recurrence relation:

$$\begin{aligned} q_1(x) &= 1, \\ q_k(x) &= \sum_{\substack{\ell \mid k \\ \ell < k}} a_{k/\ell} (x - 1)^{k-\ell-1} q_\ell(x) \quad \text{for } k > 1. \end{aligned} \tag{16}$$

On the other hand, consider the polynomial $t_\ell(x)$ defined by

$$t_\ell(x) = \sum_{j \geqslant 0} d(\ell, j)(x - 1)^{\ell-j-1}. \tag{17}$$

Then $t_1(x) = 1$. For $\ell > 1$, the term in the sum corresponding to $j = 0$ is zero since $d(\ell, 0) = 0$ in that case. For $k > 1$, calculating the right-hand side of (16) with $t_\ell(x)$ in place of $q_\ell(x)$ and applying Lemma 3.1 gives

$$\begin{aligned} \sum_{\substack{\ell \mid k \\ \ell < k}} a_{k/\ell} (x - 1)^{k-\ell-1} t_\ell(x) &= \sum_{\substack{\ell \mid k \\ \ell < k}} \sum_{j \geqslant 0} a_{k/\ell} d(\ell, j)(x - 1)^{k-j-2} \\ &= \sum_{j \geqslant 0} d(k, j + 1)(x - 1)^{k-j-2} \\ &= \sum_{j \geqslant 1} d(k, j)(x - 1)^{k-j-1} \\ &= t_k(x). \end{aligned}$$

Since $t_k(x)$ and $q_k(x)$ both satisfy the same recurrence relations, they are equal. This shows that

$$(-1)^{k-1} M_k(k \mid 1) = q_k(x) = t_k(x) = \sum_{j \geqslant 0} d(k, j)(x - 1)^{k-j-1}.$$

Substituting the last expression into (14) gives

$$p_n(x) = (x-1)^n - \sum_{k=2}^{n} \sum_{j \geqslant 1} w_k d(k,j)(x-1)^{n-j-1}$$
$$= (x-1)^n - \sum_{j \geqslant 1} v(n,j)(x-1)^{n-j-1}.$$

Since $v(n,j) = 0$ for $j > r = \lfloor \log_2(n) \rfloor$, this is

$$p_n(x) = (x-1)^n - \sum_{j=1}^{r} v(n,j)(x-1)^{n-j-1}$$
$$= (x-1)^{n-r-1}\left((x-1)^{r+1} - \sum_{j=1}^{r} v(n,j)(x-1)^{r-j}\right),$$

which proves the theorem. □

4. The eigenvectors of A_n

Theorem 4.1. *Let $\lambda \neq 1$ be a nontrivial eigenvalue of A_n. Then λ is a simple eigenvalue, and a basis for the one-dimensional eigenspace of A_n associated with λ is the vector*

$$u = \left[\lambda - 1, X_2(\lfloor n/2 \rfloor), X_3(\lfloor n/3 \rfloor), X_4(\lfloor n/4 \rfloor), \ldots, X_n(\lfloor n/n \rfloor)\right]^T$$

where

$$X_j(q) = \sum_{k \geqslant 0} \frac{v_j(q,k)}{(\lambda-1)^k} = 1 + \frac{v_j(q,1)}{\lambda - 1} + \frac{v_j(q,2)}{(\lambda-1)^2} + \frac{v_j(q,3)}{(\lambda-1)^3} + \cdots.$$

Proof. For $i \geqslant 2$, the ith entry of $A_n u$ is

$$(A_n u)_i = w_i(\lambda - 1) + \sum_{1 \leqslant \ell \leqslant n/i} a_\ell u_{\ell i}$$
$$= w_i(\lambda - 1) + \sum_{1 \leqslant \ell \leqslant n/i} a_\ell X_{\ell i}\left(\left\lfloor n/(\ell i) \right\rfloor\right)$$
$$= w_i(\lambda - 1) + \sum_{1 \leqslant \ell \leqslant n/i} a_\ell \sum_{\substack{k \geqslant 0 \\ 1 \leqslant m \leqslant n/i}} w(i\ell m) d(m,k)(\lambda - 1)^{-k}$$
$$= w_i(\lambda - 1) + \sum_{k \geqslant 0}\left(\sum_{\substack{1 \leqslant \ell \leqslant n/i \\ 1 \leqslant m \leqslant n/(\ell i)}} a(\ell) w(i\ell m) d(m,k)\right)(\lambda - 1)^{-k}$$
$$= w_i(\lambda - 1) + \sum_{k \geqslant 0}\left(\sum_{1 \leqslant t \leqslant n/i} w(it) \sum_{s \mid t} a(t/s) d(s,k)\right)(\lambda - 1)^{-k} \quad [\text{set } t = i\ell]$$
$$= w_i(\lambda - 1) + \sum_{k \geqslant 0} \sum_{1 \leqslant t \leqslant n/i} w(it)\big(d(t,k) + d(t,k+1)\big)(\lambda - 1)^{-k} \quad \big[\text{by (12)}\big]$$

$$
\begin{aligned}
&= \sum_{k\geqslant 0} v_i\big(\lfloor n/i\rfloor, k\big)(\lambda-1)^{-k} + w_i(\lambda-1) + \sum_{k\geqslant 1} v_i\big(\lfloor n/i\rfloor, k\big)(\lambda-1)^{-k+1} \\
&= \sum_{k\geqslant 0} v_i\big(\lfloor n/i\rfloor, k\big)(\lambda-1)^{-k} + (\lambda-1)\sum_{k\geqslant 0} v_i\big(\lfloor n/i\rfloor, k\big)(\lambda-1)^{-k} \\
&= \lambda \sum_{k\geqslant 0} v_i\big(\lfloor n/i\rfloor, k\big)(\lambda-1)^{-k} \\
&= \lambda X_i\big(\lfloor n/i\rfloor\big) \\
&= \lambda u_i.
\end{aligned}
$$

In the calculation for $(A_n u)_i$ with $i \geqslant 2$, the term $a_i u_{\ell i}$ when $\ell = 1$ was equal to $a_i X_i(\lfloor n/i\rfloor)$, but this term should be omitted from the case $i = 1$. Taking this into account and going to the second to last step of the previous calculation gives

$$
\begin{aligned}
(A_n u)_1 &= \lambda X_1(n) - X_1(n) \\
&= (\lambda-1)\left(1 + \sum_{k\geqslant 1} v(n,k)(\lambda-1)^{-k}\right) \\
&= (\lambda-1)\big[1 + (\lambda-1)\big] \quad \text{by Theorem 3.2} \\
&= \lambda(\lambda-1) \\
&= \lambda u_1.
\end{aligned}
$$

This shows that the vector u is a nonzero eigenvector for λ.

To see why the eigenspace of λ is one-dimensional, consider the submatrix of $A_n - \lambda I$ obtained by deleting the first row and column. This $(n-1)\times(n-1)$ matrix is upper triangular with nonzero entries on the diagonal. Hence, it is invertible implying that the rank of $A_n - \lambda I_n$ is $\geqslant n-1$. Since we found a nontrivial eigenvector, the nullity is $\geqslant 1$. So, the nullity of $A_n - \lambda I$ must be exactly one. This completes the proof. □

Theorem 4.2. *Let $\lambda \neq 1$ be a nontrivial eigenvalue of A_n. A basis for the one-dimensional eigenspace of A_n^T associated with λ is the vector*

$$
v = \big[1, Y_\lambda(2), Y_\lambda(3), \ldots, Y_\lambda(n)\big]^T, \tag{18}
$$

where

$$
Y_\lambda(q) = \sum_{k\geqslant 0} \frac{d(q,k)}{(\lambda-1)^k} = d(q,0) + \frac{d(q,1)}{\lambda-1} + \frac{d(q,2)}{(\lambda-1)^2} + \cdots.
$$

Interestingly, the algebraic expression for v does not explicitly rely on the symbols $w_2, \ldots, w_n$ in the first column of A_n. However, altering $w_2, \ldots, w_n$ changes the possible numeric values of λ.

Proof of Theorem 4.2. For $i \geqslant 2$, the ith entry of $A_n^T v$ is

$$
\begin{aligned}
\big(A_n^T v\big)_i &= \sum_{\ell \mid i} a(i/\ell) Y(\ell) \\
&= \sum_{k\geqslant 0}\sum_{\ell \mid i} a(i/\ell) d(\ell,k)(\lambda-1)^{-k}
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{k\geqslant 0}\big[d(i,k)+d(i,k+1)\big](\lambda-1)^{-k} \quad \text{by (12)}\\
&= Y(i) + (\lambda-1)\sum_{k\geqslant 1} d(i,k)(\lambda-1)^{-k}\\
&= Y(i) + (\lambda-1)Y(i) \quad \big[\text{since } d(i,0)=0\big]\\
&= \lambda Y(i).
\end{aligned}$$

The first entry of $A_n^T v$ is

$$\begin{aligned}
\big(A_n^T\big)_i &= \sum_{1\leqslant j\leqslant n} w_j Y(j)\\
&= \sum_{k\geqslant 0}\sum_{1\leqslant j\leqslant n} w_j d(j,k)(\lambda-1)^{-k}\\
&= \sum_{k\geqslant 0} v(n,k)(\lambda-1)^{-k}\\
&= 1 + \sum_{k\geqslant 1} v(n,k)(\lambda-1)^{-k}\\
&= 1 + (\lambda-1) \quad [\text{by Theorem 3.2}]\\
&= \lambda v_1.
\end{aligned}$$

This shows that $v = [Y(1),\dots,Y(n)]^T$ is a nonzero eigenvector of A_n^T. The dimension of the eigenspace is one, as explained in the proof of Theorem 4.1. □

5. Computing eigenvalues of C_n for large n

Theorem 3.2 expresses the characteristic polynomial of the matrix A_n in terms of the numbers $v(n,k)$. In this section, we will explain how to explicitly calculate the characteristic polynomial $p_n(x)$ for large values of n for the special case C_n in which $w_i = a_i = 1$ for all i. The method given below in Theorem 5.2 was used to find $p_n(x)$ for n as large as $n = 2^{36}$ in a few hours on a desktop computer. To accomplish this, it is necessary to use a more efficient algorithm for finding the coefficients than a brute force approach based directly on the definition of matrix C_n. Even with Theorem 3.2 we need a better method for computing $v(n,k)$ than the direct application of the definition of $v(n,k)$ in (11).

Lemma 5.1. *Suppose $a_\ell = w_\ell = 1$ for all ℓ. If $1 \leqslant 2^k \leqslant n$, then*

$$v(n,k) = \sum_{i>1} v\left(\left\lfloor \frac{n}{i}\right\rfloor, k-1\right) = \sum_{j<n}\left(\left\lfloor \frac{n}{j}\right\rfloor - \left\lfloor \frac{n}{j+1}\right\rfloor\right) v(j,k-1). \tag{19}$$

If both $a_k = 1$ and $w_k = 1$ for all k, then $v(n,k)$ represents the number of ways to form products of k nontrivial factors whose product is $\leqslant n$ and where order matters. In this case, $v(n,k)$ represents a count of lattices points in k-dimensional space:

$$v(n,k) = \left|\left\{(\ell_1,\dots,\ell_k)\in\mathbb{Z}^k\colon\ \ell_1\ell_2\cdots\ell_k \leqslant n \text{ and } \ell_i \geqslant 2 \text{ for all } i\right\}\right|. \tag{20}$$

Proof. The first equality in (19) is evident from (20) by letting one component of $(\ell_1, \dots, \ell_k)$, say ℓ_k, be the index of summation i. The second equality in (19) is obtained by re-indexing the sum over the distinct values of $j = \lfloor n/i \rfloor$. For a given positive integer j,

$$j = \left\lfloor \frac{n}{i} \right\rfloor \quad \Leftrightarrow \quad j \leqslant \frac{n}{i} < j+1 \quad \Leftrightarrow \quad \frac{n}{j+1} < i \leqslant \frac{n}{j}.$$

Thus, the number of distinct i for which $\lfloor n/i \rfloor = j$ is $\lfloor \frac{n}{j} \rfloor - \lfloor \frac{n}{j+1} \rfloor$. $\square$

The first recursion formula in (19) is computationally inefficient since there can be many distinct values of i_1 and i_2 such that $\lfloor n/i_1 \rfloor = \lfloor n/i_2 \rfloor$. The second is inefficient since there can be many values of j such that $\lfloor n/j \rfloor - \lfloor n/(j+1) \rfloor$ is zero. The next theorem helps to remove this redundancy by rewriting the summation to have significantly fewer terms.

Theorem 5.2. *Assume $a_\ell = w_\ell = 1$ for all ℓ. Suppose $1 \leqslant 2^k \leqslant n$ and $k \geqslant 1$. Then*

$$v(n,k) = \sum_{i=2}^{s} v\left(\left\lfloor \frac{n}{i} \right\rfloor, k-1\right) + \sum_{j=2^{k-1}}^{\lfloor \sqrt{n} \rfloor} \left(\left\lfloor \frac{n}{j} \right\rfloor - \left\lfloor \frac{n}{j+1} \right\rfloor\right) v(j, k-1), \tag{21}$$

where $s = \lfloor \frac{n}{\lfloor \sqrt{n} \rfloor + 1} \rfloor$.

Proof. This argument applies the hyperbola method from analytic number theory. Rewrite (19) as

$$v(n,k) = \sum_{\lfloor n/i \rfloor \geqslant \lfloor \sqrt{n} \rfloor + 1} v\left(\left\lfloor \frac{n}{i} \right\rfloor, k-1\right) + \sum_{\lfloor n/i \rfloor \leqslant \lfloor \sqrt{n} \rfloor} v\left(\left\lfloor \frac{n}{i} \right\rfloor, k-1\right), \tag{22}$$

where the index i in each summation satisfies $2 \leqslant i \leqslant \lfloor n/2^{k-1} \rfloor$. In the first summation, since both i and $\lfloor \sqrt{n} \rfloor + 1$ are integers,

$$\left\lfloor \frac{n}{i} \right\rfloor \geqslant \lfloor \sqrt{n} \rfloor + 1 \quad \Leftrightarrow \quad \frac{n}{i} \geqslant \lfloor \sqrt{n} \rfloor + 1 \quad \Leftrightarrow \quad i \leqslant \frac{n}{\lfloor \sqrt{n} \rfloor + 1} \quad \Leftrightarrow \quad i \leqslant \left\lfloor \frac{n}{\lfloor \sqrt{n} \rfloor + 1} \right\rfloor.$$

This gives the value $s = \lfloor n/(\lfloor \sqrt{n} \rfloor + 1) \rfloor$ in the first summation in Eq. (21). In the second summation in (22), we re-index the sum over the distinct values of $j = \lfloor n/i \rfloor \leqslant \lfloor \sqrt{n} \rfloor$. For a given positive integer j,

$$j = \left\lfloor \frac{n}{i} \right\rfloor \quad \Leftrightarrow \quad j \leqslant \frac{n}{i} < j+1 \quad \Leftrightarrow \quad \frac{n}{j+1} < i \leqslant \frac{n}{j}.$$

Thus, the number of distinct i for which $\lfloor n/i \rfloor = j$ is $\lfloor \frac{n}{j} \rfloor - \lfloor \frac{n}{j+1} \rfloor$, allowing the second summation in (22) to be written as

$$\sum_{j=2^{k-1}}^{\lfloor \sqrt{n} \rfloor} \left(\left\lfloor \frac{n}{j} \right\rfloor - \left\lfloor \frac{n}{j+1} \right\rfloor\right) v(j, k-1).$$

This proves (21). $\square$

Table 1
Values of $v(n,k)$ for $n=10^6$, $n=2^{28}$, and $n=2^{36}$.

k	$v(10^6,k)$	$v(2^{28},k)$	$v(2^{36},k)$
1	999,999	268,435,455	68,719,476,735
2	11,970,035	4,714,411,991	1,587,951,104,025
3	67,120,491	39,550,266,080	17,712,699,735,807
4	233,959,922	210,866,000,001	127,006,997,038,631
5	567,345,854	801,946,179,797	657,738,684,402,616
6	1,015,020,739	2,314,766,752,399	2,620,541,404,211,325
7	1,386,286,166	5,267,935,378,357	8,354,699,452,581,663
8	1,475,169,888	9,693,670,870,002	21,888,970,237,054,221
9	1,237,295,133	14,675,212,443,928	48,028,484,118,248,949
10	822,451,796	18,500,845,515,388	89,496,511,738,284,187
11	433,656,192	19,585,798,031,078	143,118,705,146,069,804
12	180,821,164	17,506,983,509,953	197,979,547,265,239,162
13	59,146,673	13,254,336,924,806	238,336,089,820,847,725
14	14,935,574	8,508,754,910,066	250,812,663,743,567,239
15	2,829,114	4,628,591,443,629	231,467,885,026,020,936
16	383,693	2,128,656,115,076	187,727,209,728,498,411
17	34,630	824,357,770,148	133,949,812,310,943,213
18	1672	267,263,904,116	84,103,735,312,636,462
19	20	71,941,723,387	46,433,832,280,215,021
20		15,889,930,335	22,505,741,596,654,059
21		2,830,811,858	9,551,600,816,612,963
22		396,537,923	3,536,981,261,202,340
23		42,162,106	1,137,490,727,898,326
24		3,284,753	315,879,734,318,303
25		177,731	75,228,001,661,856
26		4707	15,244,074,212,812
27		55	2,604,780,031,507
28		1	371,154,513,760
29			43,388,420,848
30			4,049,932,603
31			290,175,811
32			15,487,073
33			582,143
34			9555
35			71
36			1

It is interesting to note that s in Theorem 5.2 is equal to either $\lfloor\sqrt{n}\rfloor$ or $\lfloor\sqrt{n}\rfloor-1$ according to

$$s=\left\lfloor \frac{n}{\lfloor\sqrt{n}\rfloor+1}\right\rfloor=\begin{cases}\lfloor\sqrt{n}\rfloor & \text{if } n-\lfloor\sqrt{n}\rfloor^2\geqslant\lfloor\sqrt{n}\rfloor,\\ \lfloor\sqrt{n}\rfloor-1 & \text{if } n-\lfloor\sqrt{n}\rfloor^2<\lfloor\sqrt{n}\rfloor.\end{cases}$$

By implementing the recursive formula in Theorem 5.2, we were able to calculate the characteristic polynomial for the special case C_n for relatively large values of n, such as $n=2^{36}$, within a few hours on a desktop computer. Recall that the characteristic polynomial $p_n(x)$ of the general matrix A_n was given in Theorem 3.2. The author implemented the algorithm in Theorem 5.2, using exact integer arithmetic, with the Mathematica programming language. To add confidence to the validity of the calculation, Rodney Forcade independently verified the author's calculations of various characteristic polynomials for large values of n with a program written in the Maple programming language. A sample of the coefficients $v(n,k)$ of $p_n(x)$ for $n=10^6$, $n=2^{28}$, and $n=2^{36}$ is given in Table 1. Once the characteristic polynomials were obtained, roots were computed numerically to a high degree of precision.

A table listing the maximum absolute value and real part of small nontrivial eigenvalues of C_n for $n=10^6$ and $n=2^r$ with $28\leqslant r\leqslant 36$ is given below:

n	$\max\{\lvert\lambda\rvert\}$	$\max\{\mathrm{Re}(\lambda)\}$
$10^6 = 1{,}000{,}000$	0.983108	0.983108
$2^{28} = 268{,}435{,}456$	0.998885	0.998739
$2^{29} = 536{,}870{,}912$	0.999120	0.998989
$2^{30} = 1{,}073{,}741{,}824$	0.999324	0.999206
$2^{31} = 2{,}147{,}483{,}648$	0.999501	0.999395
$2^{32} = 4{,}294{,}967{,}296$	0.999676	0.999560
$2^{33} = 8{,}589{,}934{,}592$	1.002646	0.999704
$2^{34} = 17{,}179{,}869{,}184$	1.005213	0.999829
$2^{35} = 34{,}359{,}738{,}368$	1.007423	0.999939
$2^{36} = 68{,}719{,}476{,}736$	1.031192	1.000036

The example with $n = 2^{36}$ provides a counter-example to both parts of Conjecture 1.1; that is, there exist values of n for which a small eigenvalue λ of C_n satisfies both $|\lambda| > 1$ and $\mathrm{Re}(\lambda) > 1$.

Acknowledgments

The author thanks Wayne Barrett who introduced him to the problem discussed in this paper and for extensive conversations about several of the results described here. Also the author thanks Rodney Forcade for several helpful suggestions and for independently verifying the author's numerical computation of characteristic polynomials.

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