THE BIG FUNDAMENTAL GROUP, BIG HAWAIIAN EARRINGS, AND THE BIG FREE GROUPS. November 26, 1998

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Abstract. In this second paper in a series of three we generalize the notions of fundamental group and Hawaiian earring . In the first paper [CC1] we generalized the notion of free group to that of a *big free group*. In the current article we generalize the notion of fundamental group by defining the *big fundamental group* of a topological space. We also describe *big Hawaiian earrings*, which are generalizations of the classical Hawaiian earring. We then prove that the big fundamental group of a big Hawaiian earring is a big free group.

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1. Introduction

This is the second in a series of three papers. The first paper in the series [CC1], gives a combinatorial description of the Hawaiian earring group, introduced the notion of *big free groups* as a generalization of both finite rank free groups and the Hawaiian earring group and studies their group theoretic properties.

The current paper has two goals. The first is to introduce the notion of the *big fundamental group* of a topological space. This group is similar to the standard fundamental group, but rather than being based on maps of the unit interval into a given space, the big fundamental group is based on maps of *big intervals* (compact totally ordered connected spaces). The second goal is motivate the ideas of both big fundamental groups and big free groups. We do this by defining the notion of a *big Hawaiian earring*, and proving that the big fundamental group of a big Hawaiian earring is a big free group.

The third paper in the series, [CC3], studies the fundamental groups of one–dimensional spaces. For example, we prove there that the fundamental group of a second countable, connected, locally path connected, one–dimensional metric space is free if and only if it is countable if and

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only if the space has a universal cover, and we prove that the fundamental group of a compact, one–dimensional, connected metric space embeds in an inverse limit of finitely generated free groups and thus is locally free, residually free, and residually finite.

The notion of the fundamental group of a topological space has been important in the development of many branches of topology. Indeed, it was first put to use in Poincaré's construction of a nontrivial homology 3-sphere. This classical invariant is central to the beautiful theory of covering spaces and to combinatorial and geometric group theories. It has also played an historically significant role in the study of manifolds.

The fundamental group measures a space's connectivity. However, "large" (non-second countable) spaces do not lent themselves to study via fundamental groups. Connected spaces which are not second countable are often too big to be path connected (the long line is just short enough to be path connected,) as is evident when one notices that an arc can only delve countably deep into a nested family of open sets.

There are natural generalizations of the notion of path connectivity. One can replace the notion of "arc" by something "longer", something akin to the one-point compactification of the long line. What properties should such a "big interval" satisfy?

As we show in Theorem 4.14, the unit interval is the unique compact totally ordered, connected topological space which contains a countable dense set. It seems natural that if one wishes to study large spaces one would could weaken or (as we will do) completely remove the separability condition. We shall call a compact, connected totally ordered topological space a *big interval* or *big arc*. We can construct big intervals of arbitrarily large cardinality: take any well-ordered set, introduce an honest-to-goodness interval between each point of the original set and its successor, topologize naturally and take the one point compactification. The resultant is a big interval. The resulting notion of *big path connectivity* is thus significantly weaker than that of normal path connectivity.

How should one define the *big fundamental group* of a space ? There is a subtle obstacle to making the obvious generalization. Namely, the set of homotopy classes of closed big paths based at a point is too large to be a set (let alone a group)– even for a space with only one point ! One, however, can also generalize the notion of a homotopy to that of a *big homotopy* by allowing the parameter space to be any big interval, and thus obtain a group of *big homotopy classes*. There are a number of technical details (such as how to homotop between big paths with differing domains, and how to concatenate homotopies with differing parameter spaces) which we deal with in Section 4.3.

The set of big homotopy classes of closed big paths based at the point x_0 of the topological space X form not only a set, but also a group, $\Pi(X, x_0)$, which we call the *big fundamental group of the space X based at x_0*. The big fundamental group and the classical fundamental group are not comparable (see Section 4.3.6), in the sense that either can be trivial for a particular space, while the other is not. One surprising fact (Theorem 4.26) is that these two groups coincide for separable metric spaces.

Having generalized the fundamental group by replacing arcs by big intervals, one sees many opportunities to generalize other standard topological notions. One could obtain a two parameter family of invariants based on the big fundamental group by restricting the cardinality of domains of big paths and restricting the cardinality of the homotopy parameter spaces. These restricted groups support various natural embeddings and projection maps among themselves. Also, it would certainly be desirable to have a theory of “big covering spaces” which would work with big fundamental groups analogously to the way covering spaces work with fundamental groups. The surprising fact is that there need be no generalization: big fundamental groups work well with covering spaces and many of the standard elementary results hold. As an example we give the following

Exercise 1.1. Show that if $p : (\widetilde{X}, \widetilde{x}_0) \longrightarrow (X, x_0)$ is a covering space and P_1 and P_2 are big paths emanating from x_0 which are big homotopic (rel endpoints) via a big homotopy H then

1. P_1, P_2 lift to big paths $\widetilde{P}_1, \widetilde{P}_2$ in $\widetilde{X}$ emanating from $\widetilde{x_0}$ so that $p \circ \widetilde{P}_i = P_i$ (path lifting).

2. There is a big homotopy $\widetilde{H}$ between $\widetilde{P}_1$ and $\widetilde{P}_2$ (rel endpoints) such that $p \circ \widetilde{H} = H$ (homotopy lifting).

Finally one could naturally define big variants of n-cell, higher homotopy, homology, cohomology, and contractibility and could hope for big versions of the Hurewicz, and Whitehead theorems and for theories of duality, fixed-point theorems, etc. We leave these.

We were motivated to define big fundamental groups by our study of the *Hawaiian earring group.* This group is the fundamental group of the *Hawaiian earring* space which is the countable union of planar circles tangent to the x-axis at the origin, one of radius $1/n$ for each n in $\mathbb{N}$. This group is well-known to topologists since it has the interesting properties of being uncountable and not free even though it is the fundamental group of a compact, one–dimensional metric space.

In [CC1] we showed that the fundamental group of the Hawaiian earring could be considered as a generalized free group – a group of *transfinite words*. Just as a word in a free group can be defined as a map from a finite totally ordered set in to a set of generators and their inverses (an alphabet), a *transfinite word* is defined to be a map from any ordered set into an alphabet satisfying one finiteness condition: no generator (or letter, if you will) can be used can be used infinitely often in any given transfinite word (i.e. the preimage of a letter is finite.) If one chooses to use a finite alphabet then the set of transfinite words give rise to the corresponding finite rank free group. As we show in [CC1], if one uses a countably infinite alphabet then the set of transfinite words (modulo a corresponding "free" cancellation condition) forms a group which is isomorphic to the fundamental group of Hawaiian earring. We call the group of transfinite words on the alphabet of cardinality c the big free group, $\mathrm{BF}(c)$. This group is described in more detail in Section 2.

David Wright and Mladen Bestvina each asked us if the big free groups of higher cardinality were also fundamental groups of *big Hawaiian earrings*; one–dimensional "Hawaiian earring–like" metric spaces. Our immediate answer was "no". It was clear to us that these groups were just too big to be the fundamental group of such a space. Because an arc can have only countably many disjoint open subarcs, a word of uncountable length could never represent a closed path. After some thought we found that it was possible to get around this difficulty by using closed big paths (big loops) to replace loops in the definition of fundamental group. When we asked our colleague Steve Humphries what the notion of "big square" should be, he gave us the idea of using (as we do in the definition of big homotopies) the product of two big arcs, but not requiring that the two arcs have the same cardinality.

There was an obvious candidate for big Hawaiian earrings once we realized the following facts:

1. The bouquet of n-circles is the one-point compactification of the disjoint union of n open intervals.
2. The Hawaiian earring is the one-point compactification of disjoint union of countably infinitely many open intervals.

In Section 3 we define the *big Hawaiian earring, $E(c)$,* where c is a cardinal number, to be the one-point compactification of a disjoint union of a set of c open intervals.

In the final section of this paper we show that the big fundamental group of a big Hawaiian earring is a big free group. More precisely we prove that $\Pi(E(c)) = \mathrm{BF}(c)$.

2. The Big free groups

The following results and definitions are discussed in detail in [CC1].

Definition 2.1. Let A be an alphabet of arbitrary cardinality c, and let A^{-1} denote a formal inverse set for A. A *transfinite word* over A is a function $\alpha : S \longrightarrow A \cup A^{-1}$ satisfying the following two conditions:

$\alpha 1$: S is totally ordered.
$\alpha 2$: $\alpha^{-1}(a)$ is finite for each $a \in A \cup A^{-1}$.

We completely identify two transfinite words $\alpha_i : S_i \longrightarrow A \cup A^{-1}$ for $i = 0, 1$ if there is an order-preserving bijection $\phi : S_0 \longrightarrow S_1$ such that the following diagram commutes.

$$\begin{array}{ccc} S_0 & \xrightarrow{\alpha_0} & A \cup A^{-1} \\ \phi \downarrow & & \equiv\downarrow \\ S_1 & \xrightarrow{\alpha_1} & A \cup A^{-1} \end{array}$$

Definition 2.2. If a and b are elements of the totally ordered set S then we define $[a, b]_S = \{s \in S \mid a \leq s \leq b\}$.

Definition 2.3. We say that a transfinite word $\alpha : S \longrightarrow A \cup A^{-1}$ admits a *cancellation* $*$ if there is a subset T of S and a pairing $* : T \longrightarrow T$ satisfying the following three conditions:

$1^\star$: $*$ is an involution of the set T.
$2^\star$: $*$ is *complete* in the sense that $[t, t^*]_S = [t, t^*]_T$ for every $t \in T$ and *noncrossing* in the sense that $[t, t^*]_T = ([t, t^*]_T)^*$ for every $t \in T$.
$3^\star$: $*$ is an *inverse pairing* in the sense that $\alpha(t^*) = \alpha(t)^{-1}$.

One might describe such a cancellation as a *complete noncrossing inverse pairing*. Note that $3^\star$ forces $*$ to be fixed–point–free.

We define $S/* = S \setminus T$ and $\alpha/* = \alpha|(S/*) : S/* \longrightarrow A \cup A^{-1}$. We say that $\alpha/*$ arises from α by cancellation, that α arises from $\alpha/*$ by expansion. We say that two transfinite words are equivalent if we can pass from one to the other by a finite number of expansions and cancellations.

Definition 2.4 (BF(c))**.** We can multiply transfinite words by concatenation. This multiplication is clearly consistent with equivalence. Recall that if c is the cardinality of the alphabet A, we define BF(c) to be the set of equivalence classes of transfinite words on the alphabet A with this multiplication.

Theorem 2.5. *The set BF*(c)*, with its multiplication induced by concatenation of transfinite words, is a group. We call it the* big free group

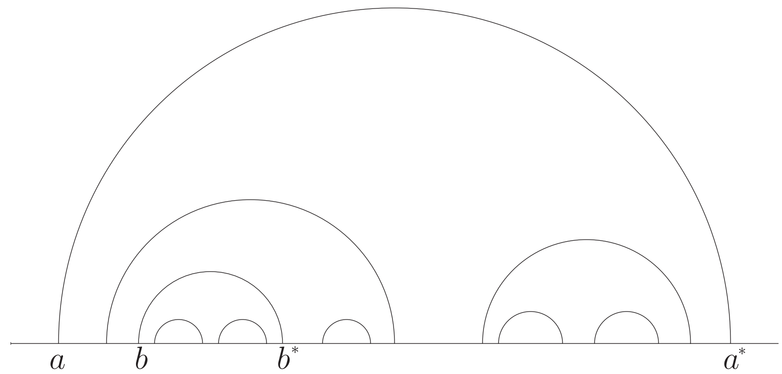

FIGURE 1. A complete noncrossing inverse pairing

on alphabet of cardinality c. *If A is countably infinite so that c is the countably infinite cardinal ω, then* $\mathrm{BF}(\omega)$ *and* $G = \pi_1(E)$ *are isomorphic, where E is the classical Hawaiian earring.*

We shall now recall a result which shows that all of the big free groups $\mathrm{BF}(c)$ act much like the classical free groups in the sense that each element is represented by a unique reduced word; we say that a word α is reduced if it admits no nonempty cancellations. Clearly, every word α admits a maximal cancellation $*$ by Zorn's Lemma, and $\alpha/*$ is reduced. However, maximal cancellations are not unique

Theorem 2.6. *Each equivalence class of transfinite words contains exactly one reduced word.*

For each finite subset $A' \subset A$ there is a natural projection

$$\pi(A') : \mathrm{BF}(c) \longrightarrow F(A')$$

onto the free group $F(A')$ with free basis A' which simply erases all letters other than those from $A' \cup A'^{-1}$. Thus there is a natural map from $\mathrm{BF}(c)$ into the inverse limit of these groups $F(A')$. The following lemma shows that the map is injective so that $\mathrm{BF}(c)$ is a subgroup of this inverse limit.

Theorem 2.7. *Suppose that* $\alpha : S \longrightarrow A \cup A^{-1}$ *has trivial projection* $\pi(A')([\alpha]) \in F(A')$ *for each finite subset A' of A. Then α is equivalent to the empty word.*

3. Big Hawaiian earrings

The fundamental group of the classical Hawaiian earring is interesting precisely because loops can travel around infinitely many of the circles. That is, circles get small as their indices get large. One's first instinct about a generalized Hawaiian earring is simply to use a larger index set, use more circles, and have circles get smaller with larger index. How is one to do that metrically? In any metric space any uncountable family of circles will have uncountably many with radius larger than some fixed real number. Our task of making sizes go to zero seems impossible.

The structure of the classical Hawaiian earring, as a subset of the plane, disguises its very homogeneous nature: any permutation of the circles can be realized by a homeomorphism of the earring. Can one then describe the earring in a more homogeneous way that does not depend upon the size of the circles? We note that the complement of the base point is the disjoint union of countably many open arcs. We then examine a neighborhood of the base point: it includes all but finitely many of the open arcs and omits only a compact set of the remaining. That is, the classical Hawaiian earring is precisely the one-point compactification of a countable disjoint union of open arcs. Hence the following definition solves the problem of creating larger Hawaiian earrings.

Definition 3.1. The (big, generalized) Hawaiian earring $E(c)$, where c is an arbitrary cardinal number, is the one-point compactification of the disjoint union of c copies of the open unit interval. Among all generalized earrings, the cardinal c is a complete topological invariant.

One can now examine the fundamental group $\pi_1(E(c))$. It will, for infinite c, be uncountable, locally free, and not free by the same arguments given for the Hawaiian earring group in [CC1]. However, no loop is long enough to traverse more than countably many of the circles of $E(c)$. It would be nice if a loop could traverse all of them, as is possible in the classical Hawaiian earring. We shall see in Section 4.3 how one can expand the notion of loop to make this possible.

For the moment, however, we will satisfy ourselves with understanding the construction of mappings and homotopies into a generalized Hawaiian earring.

Theorem 3.2. *Consider an arbitrary function $f : X \longrightarrow E(c)$. For each of the open intervals I_α from which $E(c)$ was formed, define X_α to be the inverse of I_α under f and define Y_α to be the closure of X_α in X. Then the function f is continuous if and only if each of the sets X_α is open and each of the functions $f_\alpha = f \mid Y_\alpha$ is continuous.*

Proof. In order that f be continuous it is clear that the stated conditions concerning X_α and f_α must be satisfied. Suppose conversely that they are satisfied. We shall then prove that f is continuous at each point x of X. To that end we fix a neighborhood N of $f(x)$ in $E(c)$.

Case 1. Suppose that $x \in X_\alpha$. Since f_α is continuous on Y_α, there is an open set U in X such that $f(U \cap Y_\alpha) \subset N$. Then $U \cap X_\alpha$ is a neighborhood of x in X that is mapped by f into N. Thus f is continuous at x.

Case 2. Suppose that $f(x)$ is the base point ∞ of $E(c)$, the compactification point. Then there are compact sets $C_1, \ldots, C_k$ in finitely many $I_1, \ldots, I_k$ of the constituent open intervals such that $N = E(c) \setminus (C_1 \cup \cdots \cup C_k)$. The set $f_j^{-1}(C_j)$ is closed in the closed set Y_j; hence its complement U_j is open in X and contains x. The intersection $U = U_1 \cap \cdots \cap U_k$ is open in X, contains x, and maps into the complement of $(C_1 \cup \cdots \cup C_k)$, that is, into N. We conclude that f is continuous at x.

□

Theorem 3.2 will be used in Section 5.

4. The big fundamental group

Our goal is to generalize the fundamental group of a space X in such a way as to allow transfinitely long loops and transfinitely big disks. The precise construction is carried out in section Section 4.3. The resulting group is called the big fundamental group and is denoted by $\Pi_1(X)$. Applications to generalized Hawaiian earrings and big free groups appear in section Section 5 where we prove that $\Pi_1(E(c)) = \mathrm{BF}(c)$. But before we can carry out the construction, we need to review the fundamental properties of compact, connected, linearly-ordered spaces.

4.1. **Compact, connected, linearly-ordered spaces.** Let X be an arbitrary set and $<$ a linear order on X. For convenience in defining the linear-order topology, we prepend a point $-\infty$ to X which precedes all points of X, and we append a point $+\infty$ to X which follows all points of X. The linear-order topology on X has as basis the open intervals $(a, b) = \{x \in X \mid\ a < x < b\}$, where $a, b \in \{-\infty\} \cup X \cup \{+\infty\}$. In this subsection we recall the standard facts about compact, linearly-ordered spaces.

Exercise 4.1. Every linearly-ordered space is Hausdorff and normal. (The proof of regularity is easier than the proof of normality.)

4.1.1. *Subspaces.* A subset Y of a linearly-ordered space inherits both a subspace topology T_s and a linear order from which it derives a linear-order topology $T_<$.

Exercise 4.2. Show that the subspace topology and the linear-order topology of Y need not coincide.

Theorem 4.3. *If Y is compact, then the subspace topology T_s and the linear-order topology $T_<$ coincide.*

Proof. Since $T_< \subset T_s$, the identity map from (Y, T_s) to $(Y, T_<)$ is continuous. Therefore we have a continuous bijection from a compact Hausdorff space onto another Hausdorff space. It is therefore a homeomorphism. □

4.1.2. *Constructions.* We need to show the existence of arbitrarily large compact connected linearly-ordered spaces.

Example 4.3.1. Dedekind cut spaces. Let X be an arbitrary linearly-ordered set. A *Dedekind cut* in X is a subset C of X such that, if $x < c \in C$, then $x \in C$. We denote the set of all cuts in X by $\mathrm{Cut}(X)$. The set $\mathrm{Cut}(X)$ is naturally linearly-ordered by inclusion, with first cut $\emptyset$ and last cut X.

Theorem 4.4. *The linearly-ordered space* $\mathrm{Cut}(X)$ *is totally disconnected and compact.*

Proof. We see that the space $\mathrm{Cut}(X)$ is totally disconnected as follows. Let $c_1 < c_2$ be cuts. Let $x \in X$ be a point which lies in c_2 but not in c_1. There are two natural cuts associated with x, namely $(-\infty, x)$ and $(-\infty, x]$. The first clearly precedes the second and no point lies between them. Hence the space is separated between the two, hence also between c_1 and c_2. We conclude easily that $\mathrm{Cut}(X)$ is totally disconnected.

Compactness: Let $U = \{U_\alpha \mid \alpha \in A\}$ be an open cover of $\mathrm{Cut}(X)$ by basic open intervals $U_\alpha = (x_\alpha, y_\alpha)$. Any subset of $\mathrm{Cut}(X)$ which lies in the union of some finite subcollection of U is said to be *finitely covered*. Define $Y = \{c \in \mathrm{Cut}(X) \mid [\emptyset, c] \text{ is finitely covered}\}$. The first cut, the empty cut, is covered by a single element of the cover; hence the empty cut, the first element of $\mathrm{Cut}(X)$, lies in Y. The union c_0 of the cuts which are elements of Y is a cut. We first show that $c_0 \in Y$, then that $c_0 = X$, the last cut in X. Once we establish these two facts, we will conclude that U contains a finite subcover so that $\mathrm{Cut}(X)$ is compact.

$c_0 \in Y$: If $c_0 = \emptyset$, we already know that $c_0 \in Y$. Otherwise, there is an interval (a, b), with $a \geq \emptyset$, such that $c_0 \in (a, b)$, and (a, b) is covered by a single element of U. Since $a < c_0$, there is an element $x \in c_0 \setminus a$; and since $x \in c_0$, there exists $c \in Y$ such that $x \in c$. We now clearly have $a < (-\infty, x] \leq c$. Since $c \in Y$, the interval $(-\infty, c]$, which contains a, is finitely covered. Adding (a, b) to the finite cover, we see that $(-\infty, c_0]$ is also finitely covered, so that $c_0 \in Y$.

$c_0 = X$: Let (a,b) be an interval covered by a single element of U as in the previous paragraph. If $b = +\infty$, we obtain a finite cover of all of $(-\infty, c_0] \cup (a,b) = X$, and we are done. Otherwise b is a cut. If there is no cut between c_0 and b, then a finite cover of $(-\infty, c_0]$ and an element covering b together form a finite cover of $(-\infty, b]$, a contradiction to the fact that c_0 is the last element of Y. If there is a cut c between c_0 and b, then $(-\infty, c]$ is clearly finitely covered, which also contradicts the fact that c_0 is the last element of Y. The only noncontradictory possibility is that $c_0 = X$. □

Example 4.4.1. Well-ordered spaces form another excellent source of compact, linearly ordered spaces.

Theorem 4.5. *Suppose that X is a well-ordered space with a last point. Then X is compact.*

Proof. Again take an open cover by intervals. Again define a subspace Y consisting of those points $x \in X$ such that $(-\infty, x]$ are finitely covered. Let c_0 denote the least upper bound of the set Y.

First we show that $c_0 \in Y$: Clearly the first point of X is in Y. The point c_0 is covered by one interval (a,b), where we may assume that $-\infty < a$. Since $a < c_0$, $a \in Y$. Hence $(-\infty, a]$ is finitely covered. Adding the interval (a,b), we find that $(-\infty, c_0]$ is finitely covered. Hence $c_0 \in Y$.

Now c_0 is the last point of X – the proof is exactly like the last part of the proof of Theorem 4.4. □

Example 4.5.1. Connected spaces Both of our constructions above in Example 4.3.1 and Example 4.4.1 can be extended to create connected spaces. The following theorem succeeds for both cases.

Theorem 4.6. *Suppose that X is a compact, linearly-ordered space. Extend the space X and its linear order in the following way: if $x < y$ in X, and if there are no points of X between x and y, then add a real, open interval $(0,1)_x$ with its natural order between x and y. The resulting space X_+ is compact, linearly-ordered, and connected.*

Proof. The space X_+ is linearly ordered by hypothesis. X_+ is compact: It is easy to prove that X_+ has a first point, a last point, and satisfies the least upper bound axiom. Then the proof of compactness follows the lines of proofs in Theorem 4.4 and Theorem 4.5

X_+ is connected: Let H and K be disjoint closed sets whose union is X_+, with the first point of X_+ in H. Let Y be the set of points x such that $(-\infty, x] \subset H$. Let c_0 be the least upper bound of the set Y. There is an interval about c_0 that lies either entirely in H or entirely in K. Since every interval about c_0 intersects H, this interval lies in H. The right hand part of the interval starting at c_0 cannot contain a point of K since the entire interval is in H. It also cannot contain a point c of H since otherwise $c \in Y$. Hence this half interval must be empty. It follows that c_0 cannot be an element of one of the added open intervals. Hence $c_0 \in X$. Since no interval was added immediately following c_0, the point c_0 must either be a limit point of successors in X, an impossibility, or c_0 must be the final point of X_+. We conclude that $K = \emptyset$ and that X_+ is connected.

□

Remark 4.6.1. What are the point pairs that have no points between them? In the case of the Dedekind cut space $\mathrm{Cut}(X)$, two cuts are adjacent if and only if they differ by precisely one point. Thus the intervals added in forming $\mathrm{Cut}_+(X)$ from $\mathrm{Cut}(X)$ correspond precisely to the points of the original linearly ordered set X. In the case of the well-ordered space, every point has an immediate successor, hence is joined by an interval to that successor.

4.2. **Quotients.** It is very important in our proof that the big fundamental group is a set rather than just a class that we know how to pass from one compact linearly-ordered space to another via monotone quotient.

Definition 4.7. Monotone quotients. A *monotone decomposition* of a compact, linearly-ordered space X is an equivalence relation $\sim$ on X such that each equivalence class is either a single point or a closed interval in X. (Note that a closed interval in X is a set of the form $[x, y] = \{z \in X \mid x \leq z \leq y\}$ where $x < y \in X$; it may or may not be connected.) There is an associated identification space $X/\!\sim$ whose elements are the equivalence classes, and there is an associated projection map $\pi : X \longrightarrow X/\!\sim$ which takes each point to its equivalence class. A set U of $X/\!\sim$ is open if and only if $\pi^{-1}U$ is open in X. Note that $X/\!\sim$ also inherits a linear order from X so that $X/\!\sim$ can be endowed with a linear-order topology.

We have the following theorem analogous to Theorem 4.3.

Theorem 4.8. *Let X be a compact and linearly–ordered space. Let $\sim$ be a monotone decomposition of X. Let T_s denote the quotient topology and $T_<$ the linear-order topology on the set $X/\!\sim$. Then the topologies T_s and $T_<$ coincide.*

Proof. Suppose that we can show that the identity map from $(X/\!\sim, T_s)$ to $(X/\!\sim, T_<)$ is continuous. Then, since $T_<$ is Hausdorff, T_s is also; and, since T_s is compact, $T_<$ is also. Thus we will have a continuous bijection between compact Hausdorff spaces, necessarily a homeomorphism.

Consider therefore a subbasic open set of $(X/\sim, T_<)$ of the form $(-\infty, [b])$, where $[b]$ is the equivalence class of a point $b \in X$. Let b' be the first point of the closed interval $[b]$. Then $\pi^{-1}((-\infty, [b])) = (-\infty, b')$ which is open in X. Hence $(-\infty, [b])$ is open in the identification topology T_s. Similarly, the inverse of every other subbasic open set in $T_<$ is in T_s. Thus the identity map is continuous as required. □

Corollary 4.9. *If X is compact and linearly-ordered, and if $\sim$ is a monotone decomposition of X, then the space $X/\sim$ is compact and linearly-ordered, and the projection map $\pi : X \longrightarrow X/\sim$ is a closed map.*

Proof. The space $X/\sim$ is compact since the topology is the identification topology T_s. The space is linearly-ordered since $T_s = T_<$. The space is Hausdorff since linearly-ordered spaces are Hausdorff. The map π is closed because it is a continuous function from a compact space Hausdorff space to a Hausdorff space. □

4.2.1. *Quotients of Dedekind cut spaces.*

Theorem 4.10. *If X is a linearly ordered set and $Y \subset X$, then* $\mathrm{Cut}(Y)$ *is a monotone decomposition of* $\mathrm{Cut}(X)$.

Proof. Define $q : \mathrm{Cut}(X) \longrightarrow \mathrm{Cut}(Y)$ as follows. If C is a cut in X, then $q(C) = C \cap Y$ is a cut in Y. Clearly q is order preserving since $c_1 \leq c_2$ implies $c_1 \cap Y \leq c_2 \cap Y$.

The function q is onto: indeed, if C' is a cut in Y, define

$$c_1 = \{x \in X \mid \exists y \in C' \text{ such that } x \leq y\}$$

and

$$c_2 = \{x \in X \mid \text{ if } y \in Y \text{ and } y \leq x, \text{ then } y \in C'\}.$$

Then $q^{-1}(C') = [c_1, c_2]$, where, if $c_1 = c_2$, then $[c_1, c_2] = \{c_1\} = \{c_2\}$. We conclude that we can identify $\mathrm{Cut}(Y)$ with the equivalence classes of a monotone decomposition of $\mathrm{Cut}(X)$ in a natural way. Since $\mathrm{Cut}(Y)$ has the linear-order topology, it follows from Theorem 4.8 that the topologies coincide as well. □

It will be important to us to know the potential size of a monotone quotient space in terms of the size of a dense subset. Hence we consider the following definition.

Definition 4.11. We say that a nonempty subset D of the compact, linearly-ordered space X is *self-separating* in X if the following condition is satisfied: if $d_1 < d_2$ in D and if X intersects the open interval (d_1, d_2), then D intersects the open interval (d_1, d_2). If D is self-separating in X, we declare distinct points x and y of X to be equivalent, written $x \sim y$, if and only if at most one point of D intersects the closed interval from x to y.

Theorem 4.12. *Suppose D is self-separating in the compact linearly-ordered space X. Then the associated relation $\sim$ is a monotone decomposition of X. Furthermore, the image of D is dense in $X/\sim$.*

Proof. We first show that $\sim$ is an equivalence relation on X. Since $\sim$ is obviously reflexive and symmetric, we need only prove transitivity. Suppose therefore that distinct points x, y, z satisfy $x \sim y \sim z$. If z is between x and y, or x between y and z, then x and z are clearly equivalent. We therefore need only consider the case $x < y < z$. Suppose there are two points $d_1 < d_2$ of D in the closed interval $[x, z]$. Since $x \sim y$ and $y \sim z$, we must have $x \leq d_1 < y < d_2 \leq z$. But since D is self-separating in X and $d_1 < y < d_2$, there is a third point d of D in the interval $[d_1, d_2]$. But this third point contradicts either the equivalence $x \sim y$ or the equivalence $y \sim z$. We conclude that $[x, z]$ contains at most one point of D so that $x \sim z$. This argument proves the transitivity of $\sim$ and completes the proof that $\sim$ is an equivalence relation.

We next show that every equivalence class U which contains more than one point is in fact a closed interval. Let $a \in X$ be the greatest lower bound of U, $b \in X$ the least upper bound. It is easy to see that $(a, b) \subset U \subset [a, b]$; we simply need to show that $a, b \in U$. If $a \notin U$, then a is a limit point of U. Let $x \in U$. Since a and x are not equivalent, two points of D must lie in the interval $[a, x]$. For one of them, call it $d(x)$, we must have $a < d(x) \leq x < b$ so that $d(x) \in U$. It follows easily that D has infinite intersection with U, an obvious contradiction. Thus $a \in U$. Similarly, $b \in U$.

We conclude that $\sim$ is a monotone decomposition of X. It remains only to prove that the projection $\pi : X \longrightarrow X/\sim$ takes D to a dense subset of $X/\sim$. Suppose $\alpha < \beta < \gamma$ in $X/\sim \cup \{\pm\infty\}$. We need to show that $\pi(D)$ intersects (α, γ). This desired fact is obvious if $\alpha = -\infty$ and $\gamma = +\infty$ since $\pi(D) \neq \emptyset$. If $\alpha > -\infty$, let a be the last point of $\pi^{-1}(\alpha)$, b the first point of $\pi^{-1}(\beta)$. Since a and b are inequivalent, there are points d_1 and d_2 of D such that $a \leq d_1 < d_2 \leq b$. Thus $\pi(d_2) \in (\alpha, \gamma)$. If $\gamma < +\infty$, then we may proceed similarly. We conclude that $\pi(D)$ is dense. □

4.2.2. *The size of a linearly-ordered space.* We shall need to know that, up to homeomorphism, there is only a set's worth of spaces of a given size. If X is a set, define $P(X)$ to be the set of subsets of X. Inductively define $P^0(X) = X$ and $P^{n+1}(X) = P(P^n(X))$.

Lemma 4.13. *There are, up to homeomorphism, at most $|P^2(X)|$ topological spaces of cardinality $\leq |X|$. There are, up to homeomorphism, at most $|P^4(X)|$ regular Hausdorff spaces having a dense set of cardinality $\leq |X|$.*

Proof. If X' is a topological space of cardinality $\leq |X|$, then, up to homeomorphism, we may assume that $X' \subset X$. But this makes the topology on X' a set of subsets of X. That is, the topology is an element of $P^2(X)$. Hence there are, up to homeomorphism, at most $|P^2(X)|$ such topologies.

Let S be a regular Hausdorff space having as dense set a subset X' of X. Fix a point $p \in S$. Let $\{U_\alpha \mid \alpha \in A\}$ be the set of open sets containing p. Let $X_\alpha = U_\alpha \cap X'$. Note that X_α may equal X_β for $\alpha \neq \beta \in A$. However, X_α does determine the closure $\mathrm{cl}(U_\alpha)$ of U_α; and, since S is regular Hausdorff,

$$\bigcap_{\alpha \in A} \mathrm{cl}(U_\alpha) = \bigcap_{\alpha \in A} U_\alpha = \{p\}.$$

Thus the collection $\{X_\alpha \mid \alpha \in A\}$ determines p. But $\{X_\alpha \mid \alpha \in A\} \in P^2(X)$. We conclude that S has at most $|P^2(X)|$ points. Hence by the first assertion of the lemma, there are up to homeomorphism at most $|P^4(X)| = |P^2(P^2(X))|$ such spaces S. □

Among compact, connected, linearly-ordered spaces, the real interval $[0, 1]$ is easily characterized.

Theorem 4.14. *A compact, connected, linearly-ordered space X is homeomorphic with the real unit interval $[0, 1]$ if and only if it has a countably infinite dense set.*

Proof. The condition is clearly necessary. Suppose it is satisfied. Let $D = \{d_0, d_1, d_2, \dots\}$ be a countable dense set in X. We may assume that d_0 is the first point of X, d_1 the last point. Since X is connected, it easily follows that there is a point of D between each pair of points of D. It is easy to construct a linear-order-preserving bijection between the points of D and the rational points in $[0, 1]$, d_0 corresponding to 0 and d_1 corresponding to 1. Then one extends the correspondence to limit points by continuity. We leave the details to the reader. □

Lemma 4.15. *If S is a countable, linearly-ordered set, then* $\mathrm{Cut}(S)$ *embeds in the real line.*

Proof. Each element of S has two natural images in $\mathrm{Cut}(S)$ each of which is dense in $\mathrm{Cut}(S)$. If we add open intervals to form $\mathrm{Cut}_+(S)$, then we are adding only countably many intervals so that $\mathrm{Cut}_+(S)$ also has a countable dense set. It follows immediately from the previous lemma that $\mathrm{Cut}_+(S)$ is either a single point or a closed real metric interval. □

4.3. **Constructing the big fundamental group.**

4.3.1. *Big intervals, paths, loops, rectangles, homotopies.*

Definition 4.16. A *big interval* is a compact, connected, linearly-ordered space. A *big path* is a continuous function whose domain is a big interval. A *big loop* is a big path where the first and last points of the big interval are mapped to the same point, the *base point* of the big loop. A *big rectangle* is the product of two big loops. A *big homotopy* between two functions $f, g : X \longrightarrow Y$ is a continuous function $F : X \times [a, b] \longrightarrow Y$, where $[a, b]$ is a big interval with first point a and last point b, the map F equals the map f on the set $X \approx X \times \{a\}$, and the map F equals the map g on the set $X \approx X \times \{b\}$. A map is said to be *big contractible* if it is big homotopic to a constant map. A space is said to be *big contractible* if the identity map is big homotopic to a constant map.

Theorem 4.17. *Any two maps $f, g : X \longrightarrow [a, b]$ from a space X into a big interval $[a, b]$ are big homotopic by a big homotopy which fixes all points on which f and g agree.*

Proof. We require two simple lemmas.

Lemma 4.18. *The function $h = \max(f, g) : X \longrightarrow [a, b]$ is continuous, where, for each $x \in X$, $h(x) = \max(f(x), g(x)) \in [a, b]$.*

Proof. Suppose x is a point at which $f(x) > g(x)$. Then $h(x) = f(x)$ near x, and consequently h is continuous at x. Similarly h is continuous at points x where $g(x) > f(x)$. If $h(x) = f(x) = g(x)$ and if U is a neighborhood of $h(x)$, then there exist neighborhoods V and W about x such that $f(V)$ and $g(W)$ lie in U. Thus

$$h(V \cap W) \subset f(V) \cup g(W) \subset U,$$

and h is continuous at x. □

Lemma 4.19. *If, for each $x \in X$, $f(x) \leq g(x)$, then f and g are big homotopic. For each $x \in X$ such that $f(x) = g(x)$, the homotopy may be chosen to be constant.*

Proof. Let $[a, b]$ itself be the parameter space. Hence we want to construct a continuous function $F : X \times [a, b] \longrightarrow [a, b]$ such that $F \mid X \times \{a\} = f$ and $F \mid X \times \{b\} = g$. Define F by the formulas

$$F(x,t) = \begin{cases} f(x) & \text{for } y \leq f(x) \\ t & \text{for } f(x) \leq t \leq g(x) \\ g(x) & \text{for } g(x) \leq t. \end{cases}$$

We need to show that F is continuous. Define

$$\begin{aligned} A &= \{(x,t) \in X \times [a,b] \mid t \leq f(x)\} \\ B &= \{(x,t) \in X \times [a,b] \mid f(x) \leq t \leq g(x)\} \\ C &= \{(x,t) \in X \times [a,b] \mid g(x) \leq t\}. \end{aligned}$$

Since F is the composite

$$X \times [a,b] \longrightarrow X \xrightarrow{f} [a,b]$$

on A, the projection

$$X \times [a,b] \longrightarrow [a,b]$$

on B, and the composite

$$X \times [a,b] \longrightarrow X \xrightarrow{g} [a,b]$$

on C, F is continuous on each of the three pieces. We claim that each of these sets is closed. Indeed, if $(x, t) \in X \times [a, b] \setminus A$, then $f(x) < t$. Since $[a, b]$ is connected, there is a point t' with $f(x) < t' < t$. Since f is continuous, there is a neighborhood U of x such that $f(U) < t'$. Then $U \times (t', t)$ is a neighborhood of (x, t) in $X \times [a, b] \setminus A$. We conclude that $X \times [a, b] \setminus A$ is open and A closed. The same argument shows C closed. Finally, B is the intersection of two such closed sets. Thus F is defined piecewise, continuously, and compatibly on closed sets, hence is continuous. (Lemma 4.18 is a special case of this theorem about functions defined piecewise, continuously, and compatibly on finitely many closed sets.) □

Completion of the proof of Theorem 4.17. By Lemma 4.19, both f and g are big homotopic to the continuous function $h = \max(f, g)$ by homotopies which are constant on the points at which f and g, hence h, agree. Thus f and g are big homotopic since big homotopy is obviously an equivalence relation. □

4.3.2. *Mapping big spaces into small spaces.* Even if X is the one point space, there is more than a set's worth of big paths in X because there is more than a set's worth of big intervals, all of which can be mapped into X. We will be saved from set-theoretic difficulties only by the remarkable fact that all maps defined on big intervals into a fixed space X factor through maps defined on relatively small big intervals.

Theorem 4.20. *If X is a Hausdorff space, then there is a cardinal number $\alpha(X)$ having the following property. Suppose $[a,b]$ is a big interval, C is a compact Hausdorff space, and $f : [a,b] \times C \longrightarrow X$ is a continuous function. Then there is a monotone decomposition $\pi : [a,b] \longrightarrow [a',b'] = [a,b]/\sim$ with $[a',b']$ of cardinality $\leq \alpha(X)$ such that f factors as a composite*

$$[a,b] \times C \xrightarrow{\pi\times id} [a',b'] \times C \longrightarrow X.$$

Proof. First we carry out a completely standard covering argument for $[a,b] \times C$. Fix an open cover U of X and a point $(t,u) \in [a,b] \times C$. Since f is continuous, there is a neighborhood $(a(t,u), b(t,u)) \times C(t,u)$ of (t,u) in $[a,b] \times C$ whose image lies in a single element of U. As we leave t fixed and vary u, the sets $C(t,u)$ cover C. Since C is compact, some finite subcollection $C(t,u_i)$, for $i = 1,\ldots,k$, covers C. Define

$$(a(t), b(t)) = \bigcap_i (a(t,u_i), b(t,u_i)).$$

Since $[a,b]$ is compact, there exists a finite subcollection $(a(t_j), b(t_j))$, for $j = 1,\ldots,m$, which covers $[a,b]$. Let $S(U)$ denote the finite point set consisting of the $a(t_j)$ and $b(t_j)$ not equal to $\pm\infty$. Let $S = \cup\{S(U) \mid U \text{ covers } X\}$. Now extend S to a self-separating set D of $[a,b]$. As noted in the section on monotone decompositions, D induces a monotone decomposition $\pi : [a,b] \longrightarrow [a',b'] = [a,b]/\sim$ of $[a,b]$. It remains to prove two things: $[a',b']$ has cardinality which is bounded by a number $\alpha(X)$ depending only on X, and f factors as claimed.

The cardinality of $[a',b']$ is bounded by a cardinal depending only on X: The set S has cardinality $\leq |P^2(X)| \cdot \aleph_0$ since each U is an element of $P^2(X)$ and each $S(U)$ is finite. Making S self-separating requires adding at most $\aleph_0$ points for each point already in S. That is $|D| \leq |P^2(X)| \cdot \aleph_0 \cdot \aleph_0$. But $\pi(D)$ is dense in $[a',b']$ by Theorem 4.12. Thus $[a',b']$ has cardinality $\leq |P^2(D)|$ by the proof of Lemma 4.13. But $|P^2(D)| \leq P^2(|P^2(X)| \cdot \aleph_0^2)$.

The map f factors as claimed: Let $(t,u) \in [a',b'] \times C$. Then we must show that $\pi^{-1}(t) \times \{u\}$ has constant image under f. Let $x < y$ in $\pi^{-1}(t)$. Then at most one point s of of the pre-self-separating set

S lies in the interval $[x, y]$, say $x \le s \le y$. For every open cover U of X, f maps $[x, s] \times \{u\}$ into a single element of U, one which contains $f((x, u))$. Let $z \in X \setminus \{f((x, u))\}$ and choose an open set V in X which contains $f((x, u))$ and misses z. Take $U = \{V, X \setminus \{f((x, u))\}\}$. Then $f([x, s] \times \{u\}) \subset V$. We conclude that $f([x, s] \times \{u\}) = f((x, u))$. Similarly, $f([s, y] \times \{u\}) = f((y, u))$. We conclude that $f((x, u)) = f((y, u))$, hence that $\pi^{-1}(t) \times \{u\}$ has constant image. □

Scholium 4.21. *If X is metric, then we may take $[a', b']$ to be the real interval* $[0, 1]$.

Proof. One need not use all open covers U in the proof of the previous theorem, but only enough to separate $f((x, u))$ from each $z \in X \setminus \{f(x, u)\}$ as in the previous argument. In a metric space, one may take the sequence of $1/i$ covers. Thus S, hence D will be countable. Hence $[a', b']$ will have a countable dense set, hence will be homeomorphic with $[0, 1]$ by Theorem 4.14. □

4.3.3. *Equivalence of big paths (rel endpoints).* We want to be able to compare big paths homotopically even if they are defined on different big intervals.

Definition 4.22. We define $f_0 : [a_0, b_0] \longrightarrow Y$ and $f_1 : [a_1, b_1] \longrightarrow Y$ to be *equivalent paths (rel endpoints)* if there is a big rectangle $[c, d] \times [c', d']$ and continuous functions

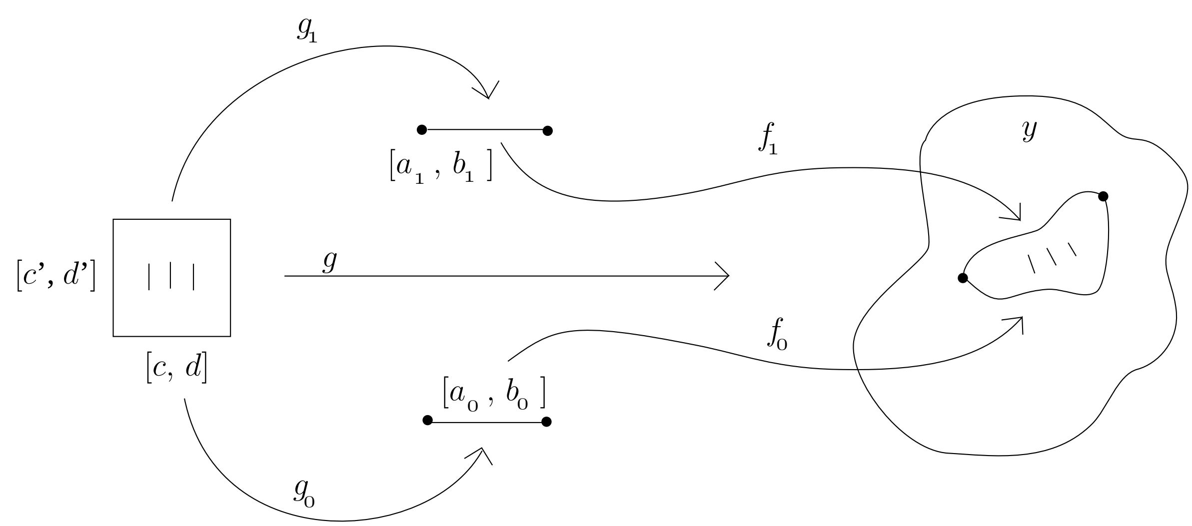

FIGURE 2. A big homotopy of big paths

$g_0 : [c, d] \longrightarrow [a_0, b_0]$, $g_1 : [c, d] \longrightarrow [a_1, b_1]$, and $g : [c, d] \times [c', d'] \longrightarrow Y$ satisfying all of the following compatibility conditions:

1. $g_0(c) = a_0,\ g_0(d) = b_0,\ g_1(c) = a_1,\ g_1(d) = b_1$;
2. $\forall t \in [c,d],\ f_0 \circ g_0(t) = g(t,c')$ and $f_1 \circ g_1(t) = g(t,d')$;
3. $g \mid \{c\} \times [c',d']$ and $g \mid \{d\} \times [c',d']$ are constant maps.

Lemma 4.23. *Equivalence (rel endpoints) is an equivalence relation on big paths.*

Proof. Suppose $f_0 \sim f_1 \sim f_2$. Then we have big rectangles $[c,d]\times[c',d']$ and maps

$$g_0 : [c,d] \longrightarrow [a_0,b_0], g_1 : [c,d] \longrightarrow [a_1,b_1], \text{and } g : [c,d] \times [c',d'] \longrightarrow Y$$

demonstrating that $f_0 \sim f_1$. Similarly we have big rectangles $[\gamma,\delta] \times [\gamma',\delta']$ and maps

$$h_0 : [\gamma,\delta] \longrightarrow [a_1,b_1], h_1 : [\gamma,\delta] \longrightarrow [a_2,b_2], \text{and } h : [\gamma,\delta]\times[\gamma',\delta'] \longrightarrow Y$$

demonstrating that $f_1 \sim f_2$.

We form a new big rectangle R as follows. As horizontal interval we take the concatenation $[c,d] \vee [\gamma,\delta]$ formed by identifying d and γ. As vertical interval we take a concatenation $[c',d'] \vee [c'',d''] \vee [\gamma',\delta']$, where $[c'',d'']$ remains to be chosen and d' is identified with c'', d'' with γ'. The picture divides into five regions:

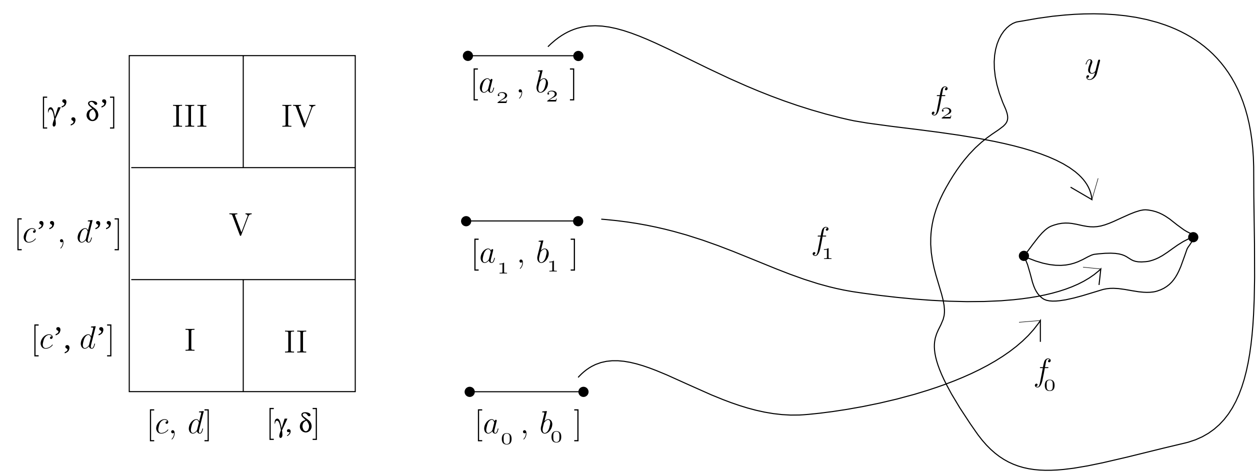

FIGURE 3. Five easy regions

Define $F_0 : [c,d] \vee [\gamma,\delta] \longrightarrow [a_0,b_0]$ by the formula

$$F_0 \mid [c,d] = f_0 \text{ and } F_0([\gamma,\delta]_{=}b_0.$$

Define $F_1 : [c,d] \vee [\gamma,\delta] \longrightarrow [a_2,b_2]$ by the formula

$$F_1([c,d]) = a_2 \text{ and } F_1 \mid [\gamma,\delta] = h_1.$$

Define $F : R \longrightarrow X$ piece by piece as follows.

$$F \mid I = g : [c, d] \times [c', d'] \longrightarrow Y.$$
$$F(II) = f_0(b_0) = g((d, c')).$$
$$F(III) = f_2(a_2) = h((\gamma, \gamma')).$$
$$F \mid IV = h : [\gamma, \delta] \times [\gamma', \delta'] \longrightarrow Y.$$

We map V into $[a_1, b_1]$ and then follow it by the map $f_1 : [a_1, b_1] \longrightarrow Y$ as follows. Map the top of I to $[a_1, b_1]$ by the map $g_1 : [c, d] \longrightarrow [a_1, b_1]$. Map the top of II to b_1. Map the bottom of III to a_1. Map the bottom of IV to $[a_1, b_1]$ by $h_0 : [\gamma, \delta] \longrightarrow [a_1, b_1]$. Then the two resulting maps of $[c, d] \vee [\gamma, \delta]$ onto $[a_1, b_1]$ are homotopic fixing endpoints by Theorem 4.17. That theorem determines $[c'', d'']$. This completes the construction of F which shows that $f_0 \sim f_2$. □

4.3.4. *Multiplying equivalence classes of big paths.* Clearly big paths which share last endpoint to first endpoint can be concatenated. That is, suppose $f : [a, b] \longrightarrow Y$ and $g : [\alpha, \beta] \longrightarrow Y$ are big paths with $f(b) = g(\alpha)$. Then there is a product $f \vee g : [a, b] \vee [\alpha, \beta] \longrightarrow Y$.

Lemma 4.24. *If $f_0 \sim f_1$ and $g_0 \sim g_1$, and if $f_0 \vee g_0$ exists, then $f_1 \vee g_1$ exists and $f_0 \vee g_0 \sim f_1 \vee g_1$. That is, concatenation of big paths is well-defined on equivalence classes of big paths.*

Proof. Clearly $f_1 \vee g_1$ exists since the terminal endpoints of f_0 and f_1 in Y are equal and coincide with the initial endpoints of g_0 and g_1 in Y. Let

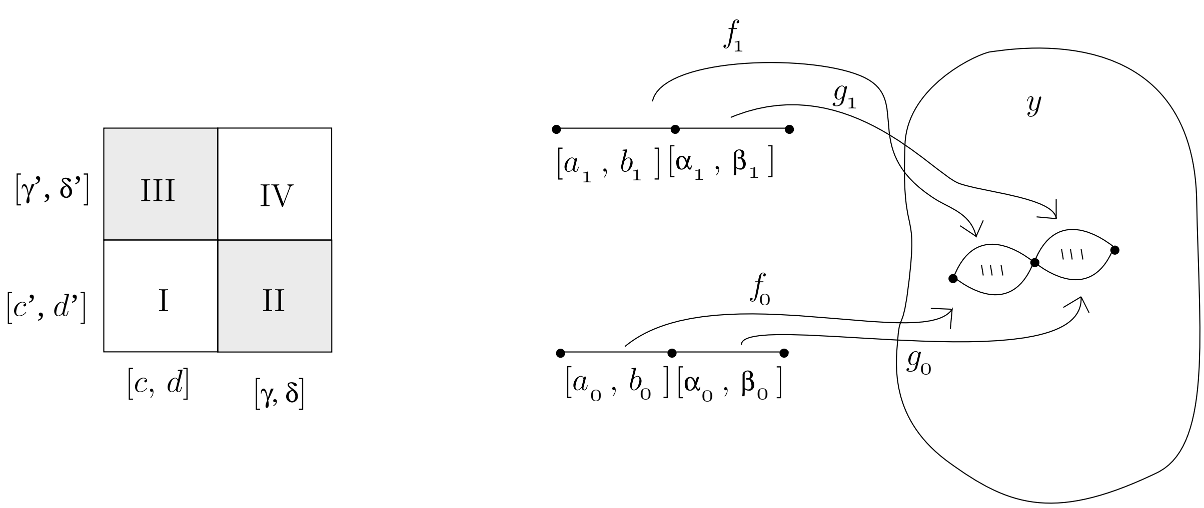

Figure 4. Concatenation is well-defined on homotopy classes

$$F_0 : [c, d] \longrightarrow [a_0, b_0], F_1 : [c, d] \longrightarrow [a_1, b_1], F : [c, d] \times [c', d'] \longrightarrow Y$$

demonstrate that $f_0 \sim f_1$. Let

$$g_0 : [\gamma, \delta] \longrightarrow [\alpha_0, \beta_0], G_1 : [\gamma, \delta] \longrightarrow [\alpha_1, \beta_1], \text{and } G : [\gamma, \delta]\times[\gamma', \delta'] \longrightarrow X$$

demonstrate that $g_0 \sim g_1$. Form Figure 4.

Map I into Y by F, map each horizontal slice in II into Y by $g_0 \circ G_0$. Map each horizontal slice in III into Y by $f_1 \circ F_1$. Map IV into Y by G. Map the bottom into $[a_0, b_0] \vee [\alpha_0, \beta_0]$ by $F_0 \vee G_0$. Map the top into $[a_1, b_1] \times [\alpha_1, \beta_1]$ by $F_1 \vee G_1$. These maps demonstrate that $f_0 \vee g_0 \sim f_1 \vee g_1$. □

4.3.5. *Definition of the big fundamental group.* In the previous two sections we saw that equivalence (rel endpoints) is an equivalence relation on big paths and that concatenation, when defined on representatives, is well-defined on equivalence classes. We define the *big fundamental group* $\Pi_1(Y, y_0)$ to be the class of equivalence classes of big loops in Y based to y_0 with multiplication given by concatenation.

Theorem 4.25. *If Y is a Hausdorff space, then the big fundamental group is a group.*

Proof. The hardest part of the proof is that the big fundamental group is a set in the sense that there is a set of big loops such that every big loop is equivalent to one of the big loops from this set. To that end, we make use of Lemma 4.13 and Theorem 4.20. Let $f : [a, b] \longrightarrow Y$ be any big loop based at y_0. By Theorem 4.20, the loop f factors through a big loop $f' : [a', b'] \longrightarrow Y$ where $[a', b']$ has cardinality $\leq \alpha(Y)$, a cardinality depending only on Y and not on $[a, b]$. It is easy to see that $f \sim f'$: take rectangle $R = [a, b] \times [c, d]$ with $[c, d]$ arbitrary; map R to Y by first projecting onto $[a, b]$ and then applying f; map $[a, b] \times \{c\}$ to $[a', b']$ by $\pi : [a, b] \longrightarrow [a', b']$; map $[a, b] \times \{d\}$ to $[a, b]$ by the identity map. Lemma 4.13, there is only a set's worth of spaces of cardinality $\leq \alpha(Y)$ up to homeomorphism. Hence one need only use a set's worth of representative spaces $[a', b']$ in order to represent all equivalence classes of loops. We conclude that $\Pi_1(Y)$ is a set.

The class of a constant loop serves as the identity for multiplication of classes. Indeed, if $[a', b']$ is the representative big interval for some element, and if $[c', d']$ is the representative big interval for some constant loop, then the wedge $[a', b'] \vee [c', d']$ can be collapsed back to $[a', b']$ exactly as in the previous paragraph without changing the equivalence class.

Multiplication is exactly associative on representatives since we need do no reparameterizations. Thus multiplication is associative.

The reverse $\bar{f} : [b, a] \longrightarrow Y$ of a big loop $f : [a, b] \longrightarrow Y$ has class which serves as inverse to the class of f. The concatenation $f \vee \bar{f} : [a, b] \vee [b, a] \longrightarrow Y$ and the constant function $g : [a, b] \vee [b, a] \longrightarrow Y$ both factor naturally through $[a, b]$. For example, we have $f \vee \bar{f}$ as the composite

$$[a, b] \vee [b, a] \longrightarrow [a, b] \xrightarrow{f} Y.$$

But any two maps into $[a, b]$ are big homotopic. It follows that $f \vee \bar{f}$ is big nullhomotopic. We conclude that $\Pi_1(Y)$ is a group. □

Theorem 4.26. *The big fundamental group reduces to the fundamental group in a metric space.*

Proof. If Y is a metric space, then the small big paths $[a', b']$ representing an equivalence class of big loops may be chosen to be the real interval $[0, 1]$ by the Scholium to Theorem 4.20 Similarly, the big rectangles used in representing homotopies collapse to real disks $[0, 1] \times [0, 1]$. The equivalence relation looks different from the standard since it is not required that the bottom and top of a rectangle map into Y exactly by the original maps one wants to show homotopic. But they must map in by functions that are themselves homotopic to the originals since they factor through the original intervals. We conclude that the big fundamental group is indeed the classical fundamental group. □

4.3.6. *Simple examples.* The big fundamental group and the classical fundamental group can each be nontrivial even if the other is trivial. For example, join the ends of the long interval. Then the classical fundamental group is trivial while the big fundamental group is the integers Z. On the other hand, form the big cone over the standard circle in the sense that radii are long intervals. Then the classical fundamental is nontrivial if one takes as basepoint a point of the outer circle, while the big fundamental group is trivial. This last example is not path connected. In order to make it path connected, one can collapse one radius to a point.

5. The big fundamental group of a big Hawaiian earring is a big free group

We fix a cardinal number c and examine the big fundamental group of the generalized Hawaiian earring $E(c)$. The big fundamental group was defined in Section 4.3.5. The generalized Hawaiian earring was defined in Section 3. The big free group was defined in Section 2.

Theorem 5.1. *The big fundamental group $\Pi_1(E(c))$ of the generalized Hawaiian earring $E(c)$ and the corresponding big free group $BF(c)$ are isomorphic.*

Proof. We map transfinite words used in defining $BF(c)$ to loops in $E(c)$ as follows:

Let A be an alphabet of cardinality c. For each $a \in A$, let $I(a)$ be a real open interval. Let the generalized Hawaiian earring $E(c)$ be realized as the one-point compactification of the disjoint union of these open intervals $I(a)$. Let $\alpha : S \to A \cup A^{-1}$ be a transfinite word in the alphabet A. Let $\mathrm{Cut}(S)$ denote the space of Dedekind cuts in the set S; this space was defined in Example 4.3.1. Let $[x(\alpha), y(\alpha)] = \mathrm{Cut}_+(S)$ denote the compact, connected, linearly-ordered space formed from $\mathrm{Cut}(S)$ as in section Example 4.5.1 by inserting a real open interval between any two points of $\mathrm{Cut}(S)$ which are not separated in $\mathrm{Cut}(S)$ by any point of $\mathrm{Cut}(S)$. These open intervals, as noted in the remark at the end of section Example 4.5.1, correspond precisely to the elements of S so that we can index them as intervals $J(s)$, for $s \in S$. Map $\mathrm{Cut}(S)$ to the basepoint ∞ in $E(c)$. If $s \in S$, then $\alpha(s) \in A \cup A^{-1}$. There is therefore an element $\alpha'(s) \in A$ such that $\alpha(s)$ and $\alpha'(s)$ are either equal or inverses of one another. Map the interval $J(s)$ linearly around the circle in $E(c)$ containing the interval $I(\alpha'(s))$ in the "counterclockwise" direction if $\alpha(s) = \alpha'(s)$ and in the "clockwise" direction if $\alpha(s) = \alpha'(s)^{-1}$. Since no letter is used more than finitely often in the word α, only finitely many open intervals are mapped into any $I(a) \subset E(c)$; it follows easily from Theorem 3.2 that we have defined a continuous big loop in $E(c)$. Note that our map on transfinite words respects concatenation.

Note also that our linearly–ordered space $[x(\alpha), y(\alpha)]$ and the accompanying map into $E(c)$ are completely determined by the word α; we denote the map by f_α.

We next show that every equivalence class of big loops is represented by some one of our maps f_α:

Let $f : [x, y] \to E(c)$ be a big loop based at $\infty \in E(c)$. Consider the open intervals $I(a)$, with $a \in A$, used in forming $E(c)$. Let $p(a)$ denote the midpoint of $I(a)$. The principal facts to be noted are

(1) we may make independent homotopy modifications of f in the intervals $I(a)$ by Theorem 3.2 and
(2) we may take a monotone decomposition of $[x, y]$

as in Theorem 4.8, Theorem 4.20 and Theorem 4.26 and their proofs.

The set $f^{-1}(p(a))$ is closed, hence compact, hence is covered by finitely many disjoint open intervals from $[x, y]$ whose closures $J_1, \ldots, J_k$ are mapped by f into the metric open interval $I(a)$. By Theorem 4.20 and its Scholium, the function f factors through a map into a smaller compact, connected, linearly-ordered space where each of the intervals

J_i has been replaced by a real metric interval. This replacement may, by Theorem 3.2, Theorem 4.8, and the proof of Theorem 4.26 be made in each of the open intervals $I(a)$ simultaneously without affecting the continuity or equivalence class of f. In each individual $I(a)$ we may then put the metric intervals in general position with respect to the point $p(a)$ so that the set $f^{-1}(p(a))$ becomes finite, with $f([x, y])$ locally crossing $p(a)$ linearly at each point of intersection. Then one may deform a neighborhood of ∞ in $I(a)$ down to ∞. At that point the map f will look exactly like the kind of map we constructed above except for the fact that the preimage of ∞ may contain intervals in $[x, y]$. We collapse each of these intervals in $[x, y]$ to a point, obtain a new $[x, y]$, a monotone decomposition of the old, and the old f factors through the new $[x, y]$. The new f is precisely of the type constructed above.

Finally, we show that loops f_α and f_β are equivalent if and only if the corresponding transfinite words α and β are equivalent.

If α and β are equivalent, then f_α and f_β are equivalent:

In order to be more explicit, we take $\alpha : S(\alpha) \to A \cup A^{-1}$. We may assume that β is the reduced form of α. Thus there is a maximal cancellation $* : T \to T$ of α so that $\beta = \alpha/*$, where $T \subset S$, so that $\beta = \alpha|[S(\beta) = (S \setminus T)]$. By Theorem 4.10, $\mathrm{Cut}(S(b))$ is a monotone decomposition of $\mathrm{Cut}(S(a))$. It follows that $[x(\beta), y(\beta)]$ is a monotone decomposition of $[x(\alpha), y(\alpha)]$, say by a map $\pi : [x(\alpha), y(\alpha)] \to [x(\beta), y(\beta)]$.

We let $P(*)$ denote the set of unordered pairs (t, t^*), where $t \in T$. We linearly order $P(*)$ in such a way that, if the points t and t^* are between the points u and u^*, then the pair (t, t^*) precedes the pair (u, u^*). Let $\mathrm{Cut}(P(*))$ denote the space of Dedekind cuts in $P(*)$, and let $\mathrm{Cut}_+(P(*))$ denote the big interval formed from $\mathrm{Cut}(P(*))$ by filling in real open intervals between adjacent points. These connecting intervals correspond precisely to the pairs appearing in $P(*)$.

We form a big rectangle $R = [x(\alpha), y(\alpha)] \times \mathrm{Cut}_+(P(*))$ and use it in showing that α and β are equivalent. We must construct functions $F : R \to E(c)$, $F_\alpha : [x(\alpha), y(\alpha)] \to [x(\alpha), y(\alpha)]$, and $F_\beta : [x(\alpha), y(\alpha)] \to [x(\beta), y(\beta)]$ which show that α and β are equivalent. For F_α we take the identity map. For F_β be take the monotone projection map π. For F we proceed as follows. For each letter $s \in S(\alpha)$ in the word α we construct a certain closed set in R which we map in a very explicit way into $E(c)$. It is because of the special order on $P(*)$ that the sets on which we define our function do not intersect except on their boundaries, all of which are mapped to the base point. There are two cases.

Case 1. Suppose in this case that the letter s does not lie in T so that it is not paired by $*$. Let $I(s)$ denote the closed interval in $[x(\alpha), y(\alpha)]$ corresponding to s. Then f_α wraps the interval $I(s)$ linearly around one circle in $E(c)$. In this case, map the entire product $I(s) \times \mathrm{Cut}_+(P(*))$ by first projecting onto $I(s)$ and then applying f_α.

Case 2. Suppose in this case that the letter s does lie in T so that it is paired with an element s^* in T. We may choose the notation so that $s < s^*$. The pair (s, s^*) lies in $P(*)$ so that there is a closed connecting interval $J(s, s^*)$ in $\mathrm{Cut}_+(P(*))$ corresponding to the pair $(s, s^*) \in P(*)$. There are of course closed connecting intervals $I(s)$ and $I(s^*)$ in $[x(\alpha), y(\alpha)]$ corresponding to s and s^*, respectively. For convenience, we name the endpoints of the intervals we are considering: $\mathrm{Cut}_+(P(*)) = [c, d]$, $J(s, s^*) = [c', d']$, I(s) = [a,b], and $I(s^*) = [b^*, a^*]$. We take the region shaped like an inverted U which is the union of the sets $[a, b] \times [c, c']$, $[a, a^*] \times [c', d']$, and $[b^*, a^*] \times [c, c']$.

We map $[a, b] \times [c, c']$ into $E(c)$ by first projecting onto $I(s) = [a, b]$ and then applying f_α. We map $I(s^*) \times [c, c']$ into $E(c)$ by first projecting onto $I(s^*) = [b^*, a^*]$ and then applying f_α. We map $[b, b^*] \times [c, d]$ into $E(c)$ by first projecting onto $[c, d]$, then rotating $[c, d]$ counterclockwise and mapping it linearly onto $[a, b]$, and then applying f_α. Two little rectangles, $[a, b] \times [c', d']$ and $[b^*, a^*] \times [c', d']$, remain to be mapped. Each is a metric rectangle. Taking the point (b, c') as center, one maps each radial segment from (b, c') in $[a, b] \times [c', d']$ linearly onto $[a, b]$ and applies f_α. Taking the point (b^*, c') as center, one maps each radial segment from (b^*, c') in $[b^*, a^*] \times [c', d']$ linearly onto $[b^*, a^*]$ and then applies f_α.

Having dealt with these two cases, we map all other points of the rectangle R to the base point $\infty \in E(c)$. By Theorem 3.2 our map F is continuous since the inverse image of each fundamental open interval forming $E(c) \setminus \{\infty\}$ is a finite union of (open) inverted U's, and the function on the closure of this finite union is obviously continuous.

It is easy to check that F, F_α, and F_β show that f_α and f_β are equivalent.

If f_α and f_β are equivalent, then α and β are equivalent transfinite words:

By Theorem 2.7, it suffices to show that for each finite collection A' of letters, the projection $\pi(A')[\alpha] = \pi(A')[\beta]$. We project the generalized earring $E(c)$ onto the finite wedge of circles corresponding to the letters of A'. Since f_α and f_β are equivalent in $\Pi_1(E(c))$, their images in the finite wedge are also equivalent in the big fundamental group of the finite wedge. But the big fundamental group of the finite wedge is the classical fundamental group of the finite wedge by Theorem 4.26.

Hence the letters cancel one another by the classical case. The proof is therefore complete. □

References

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