TRANSLATION NUMBERS OF GROUPS ACTING ON QUASICONVEX SPACES

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Abstract. We define a group to be *translation discrete* if it carries a metric in which the translation numbers of the non-torsion elements are bounded away from zero. We define the notion of quasiconvex space which generalizes the notion of both CAT(0) and Gromov–hyperbolic spaces. We show that a cocompact group of isometries acting properly discontinuously cocompactly on a proper quasiconvex metric space is translation discrete if and only if it does not contain an essential Baumslag-Solitar quotient. It follows that if such a group is either biautomatic or residually finite then it is translation discrete.

1. Introduction

Suppose M is a compact Riemannian manifold of nonpositive sectional curvature and $\widetilde{M}$ is its universal cover, then every element of the covering isometry group fixes at least one geodesic line in $\widetilde{M}$ and acts as a translation of each line it fixes. It turns out that the displacement of these translations, which is called the *translation number*,$\tau(g)$, of the covering isometry g, is independent of which line is chosen and that $\tau(g) = \limsup_{n\to\infty} \frac{\|g^n\|}{n}$. This notion can be extended to a larger class of groups. Let G be a group which is equipped with a metric, d, which is invariant under left multiplication by elements of G (i.e., $d(xy, xz) = d(y, z)\ \forall\, x, y, z \in G$.) We then say that d is a *left-invariant group metric* (or a *metric* for brevity) on G. Let $\|\,\| : G \longrightarrow \mathbb{Z}$ be defined by $\|x\| = d(x, 1_G)$. Examples of such metrics are a finitely generated group equipped with a word metric or the pull-back metric on a group of isometries of a metric space. For x in G, let

$$\tau(x) = \lim_{n\to\infty} \frac{\|x^n\|}{n}.$$

This limit exists (see [GS]) and is called the *translation number* of x.

This paper will discuss the properties of translation numbers of group actions on metric spaces in general and on "nonpositively curved spaces" in particular. The notion of translation number is quite a useful one since it has both algebraic and geometric aspects. It allows us to assign a "length" to each element of a group which is endowed with a group invariant metric (such as a word metric or a pull-back metric). Numerous authors have used the notion of translation numbers in their work, including Busemann [Bu], the author [C1, C2], Gersten/Short [GS], Gromov [Gr] and Gromoll/Wolf [GW]. In [C1] we define a group to be *translation discrete* if it supports a group invariant metric for which the translation numbers of the nontorsion elements are bounded away from zero and prove that solvable translation discrete groups of finite cohomological dimension are finite extensions of $\mathbb{Z}^n$. Gromov [Gr] has shown that translation numbers corresponding to a word metric in a word hyperbolic group are rational with bounded denominator (and thus are discrete), The same is shown to be true for finitely generated nilpotent groups in [C2].

Of late there has been a particularly keen interest in actions on "nonpositively curved" and "negatively curved" spaces. There are several notions of nonpositive curvature which workers use today. Chief among these is the condition CAT(0). We will call a geodesic metric space *convex* if the distance between the midpoints of two sides of any geodesic triangle is no more than half the length of the third side. This notion is more general than CAT(0) and is originally due to Busemann [Bu] for G-spaces (in his notation convex would mean straight and nonpositively curved), but we will use it this more general setting.

The standard model for combinatorial negative curvature is the notion of a *Gromov–hyperbolic* space. This is a space in which there is a global constant δ such that any point on a side of a geodesic triangle is at distance at most δ from the union of the other two sides. This notion is a quasiisometry invariant and so does not "see" the fine structure of the metric.

We propose a common generalization which shares a number of the good points of both hyperbolicity and convexity. Our generalization has neither the strong negative curvature condition of hyperbolicity nor the strong local requirements of convexity. This class of spaces behaves nicely with respect to translation numbers in a way similar to that of a convex space as seen in Theorem 3.3, but allows small local changes in the metric to stay in the class. We define a geodesic metric space to be *δ-quasiconvex* if the distance between the midpoints of and two sides of any geodesic triangle is no more than half the length of the remaining side plus δ.

We prove the following results:

Theorem 3.2. A finitely generated group has a quasiconvex Cayley graph if and only if it is word hyperbolic.

Theorem 4.1. Let the group G act properly discontinuously cocompactly on a proper δ-quasiconvex metric space M, then G is translation discrete if and only if G does not contain an essential Baumslag-Solitar quotient as a subgroup.

Recently, Poleksic [Po] has announced that he has an argument which shows that cocompact groups of isometries of quasiconvex spaces contain no Baumslag-Solitar quotients. One can combine that result with the previous result to get that cocompact groups of isometries of quasiconvex spaces are translation discrete.

Corollary 4.3. Let the group G act properly discontinuously cocompactly on a proper δ-quasiconvex metric space. Then any finitely generated residually finite normal subgroup of G is translation discrete.

2. Preliminaries

2.1. **Notation.** If M is a metric space, i a nonnegative real number and m a point in M, then $B_i(m)$ will denote the closed ball of radius i about the point m. We say that a metric space is *proper* if its closed balls are compact, and that it is *geodesic* if given any two points in the space there is a rectifiable curve joining the two points whose length is the distance between the two points. We say that a group *acts discretely* on a topological space if the orbit of each point is a discrete closed set.

Suppose M is a metric space with a basepoint, m_0, and G is a group of isometries of M. Then we can *induce the metric from M to G* by defining a left-invariant (semi)metric, d_G, on G by defining the distance between each pair of elements g and h in G to be $d_G(h, g) = d_M(h(m_0), g(m_0)\,)$. It is easily verified that d_G is, indeed, a left-invariant (semi)metric. If m_0 is not a fixed point of any element of G then d is a metric. We call the (semi)norm $\|g\| = d(g, 1_G)$ the *induced norm.*

We note that a discrete group of fixed-point-free isometries of a proper metric space M acts freely, properly discontinuously.

2.2. **Displacement.**

Definition 2.1 (Displacement Function)**.** Suppose the group G acts by isometries on the metric space M. For each $g \in G$ we define the *displacement function* of g to be the function $d_g : M \longrightarrow \mathbb{R}^*$ given by $d_g(m) = d(gm, m)$ and define $d_0(g) = \inf_{m\in M} d_g(m)$. An isometry, g, of is called *semisimple* if there is an $m \in M$ such that $d_0(g) = d(m, gm)$. For all $m \in M$ define $r_0 : M \longrightarrow \mathbb{R}^*$ by $r_0(m) = \inf_{g\in G^*} d_g(m)$ and define $\lambda_0(M) = \inf_{m\in M} r_0(m)$.

It is evident that $d_g(m) = d_{hgh^{-1}}(hm)\ \forall\, h, g \in G,\ m \in M$, that d_g is a Lipschitz-(2) continuous function, and that d_0 is a class function (i.e. its value depends only on the conjugacy class of the element).

By definition, $r_0(m) = r_0(gm)\ \forall\, g \in G$. The fact that r_0 is a Lipschitz-(2) continuous function follows immediately from the fact that d_g is Lipschitz-(2) continuous.

We leave most of the proof of the following two lemmas to the interested reader.

Lemma 2.2. *If the group G acts cocompactly as isometries on the proper metric space M, then every element of G is semisimple. In particular if D is a compact subset of M such that $M = G \cdot D$, then for each $g \in G$,*

$$d_0(g) = \min_{h\in G,\ m\in D} d_{hgh^{-1}}(m).$$

Lemma 2.3. *If the group G acts discretely cocompactly as isometries on the proper metric space M, then*

1. *For each $m \in M$ there is a $g \in G$ such that $r_0(m) = d_g(m)$.*
2. *There exists $g \in G, m \in M$ such that $d_0(g) = d(m, gm) = \lambda_0(G)$.*
3. *There exists $\epsilon > 0$ such that $g \in G$ has a a fixed-point if and only if $d_0(g) < \epsilon$.*

Proof. We omit the easy proofs of the first two parts of the statement and prove the third part. Let D be a compact set so that $G \cdot D = M$. Suppose $\forall\, \epsilon > 0$ there exists a fixed-point-free $g \in G$ such that $d_0(g) < \epsilon$. Then there is a sequence of points (x_i) in M and a sequence of fixed-point-free group elements (g_i) in G^* so that $d(g_i x_i, x_i) < 1/8^i$. By the definition of D, for each x_i there is an $h_i \in G$ so that $h_i x_i \in D$. Since D is compact we may replace (x_i) by a terminal subsequence so that $(h_i x_i)$ converges to a point x of D, and $d(h_i x_i, x) < 1/8^i$. Now we have

$$\begin{aligned}
d(h_i g_i h_i^{-1} x, x) &\leq d(h_i g_i h_i^{-1} x, h_i g_i h_i^{-1}(h_i x_i)) + d(h_i g_i h_i^{-1}(h_i x_i), h_i x_i) + \\
&\quad\ d(h_i x_i, x) \\
&= 2d(x, h_i x_i) + d(h_i g_i x_i, h_i x_i) \\
&= 2d(x, h_i x_i) + d(g_i x_i, x_i) \\
&\leq 3 \cdot 1/8^i \\
&< 1/2^i.
\end{aligned}$$

Since G acts discretely, this implies that there is a constant C such that $i > C \Rightarrow h_i g_i h_i^{-1} x = x$. This shows that $i > C \Rightarrow g_i$ has a fixed-point, which contradicts the choice of (g_i). □

Lemma 2.4. *Let G act properly discontinuously cocompactly on a proper metric space M, then $\forall\ \epsilon > 0$ there are only finitely many conjugacy classes K with $d_0(K) < \epsilon$. In particular, 0 is an isolated point of $d_0(G)$.*

Proof. Let D be a compact set such that $G \cdot D = M$. Fix $m_0 \in M$. Choose R such that $B_R(m_0) \supseteq D$. Let $B = B_{R+\epsilon}(m_0)$. Suppose K is a conjugacy class of G with $d_0(K) < \epsilon$. Now choose $k \in K$ so that there is an $x \in D$ so that $d_k(x) = d_0(k) < \epsilon$. Then $k \cdot x \in k \cdot B \cap B \neq \emptyset$. Since $\{g \mid g \cdot B \cap B\} \neq \emptyset\}$ is finite there can be only finitely many such conjugacy classes K. □

2.3. Translation Numbers.

Definition 2.5 (Translation number)**.** Suppose G is a group equipped with a norm $\| \|$. We define the *translation number*, $\tau(g)$ of an element g in G by

$$\tau(g) = \lim_{n\to\infty} \frac{\|g^n\|}{n}.$$

It is shown in [GS] that this limit exists. A group is called *translation discrete* if it supports a metric such that the translation numbers of its nontorsion elements are bounded away from zero.

Observation 2.5.1. Suppose M is a metric space, G a group of isometries of M equipped with an induced metric, then

1. τ is invariant under change of basepoint.
2. $\tau(g) \leq d_g(m)$ for all $g \in G$ and $m \in M$.
3. $\tau(g) \leq d_0(g) \leq \|g\|$ for every g in G.

The following is a result of Gersten and Short ([GS]).

Theorem 2.6. *If G is a group endowed with a left-invariant metric then for every pair of elements x and y of G we have*

1. $0 \leq \tau(x) = \lim_{n\to\infty} \frac{\|x^n\|}{n} \leq \|x\|$.
2. $\tau(x) = \tau(y^{-1}xy)$.
3. $\tau(x^n) = |n| \cdot \tau(x)\, \forall n \in \mathbb{Z}$.
4. $\tau(x) = \tau(x^{-1})$.
5. *If x and y commute then $\tau(xy) \leq \tau(x) + \tau(y)$.*

2.4. Baumslag-Solitar quotients.

Definition 2.7 (Baumslag-Solitar groups)**.** We call the the class of groups

$$B_{n,m} = \langle a, b \mid a^{-1}b^m a = b^n \rangle$$

the *Baumslag-Solitar groups*. We call the group H *an essential Baumslag-Solitar quotient* if H is a quotient group of $B_{n,m},\ m > n > 0$ such that the image of b in H has infinite order.

Observation 2.7.1. It is evident that a group contains an essential Baumslag-Solitar quotient as a subgroup if and only if the group contains an element of infinite order having two different conjugate powers. One can apply Theorem 2.6 to see that such an element must have zero translation number in any metric. It follows that a translation discrete group cannot contain an essential Baumslag-Solitar quotient as a subgroup. In [GS] it is shown that a biautomatic group equipped with a word metric contains no nontorsion element with zero translation number, thus such a group cannot contain an essential Baumslag-Solitar quotient as a subgroup.

3. Quasiconvexity

Definition 3.1 (δ-Midpoint Convex)**.** We call a geodesic triangle in a geodesic metric space δ*-midpoint convex* if the distance between the midpoints of any two sides of the triangle is at most δ more than half the length of the remaining side.

Similarly we call a function, f, on a subinterval I of the real numbers a δ*-midpoint convex* function, if for any two points a and b in I,

$$f(\frac{a+b}{2}) \leq \frac{a+b}{2} + \delta.$$

We say f is δ*-quasiconvex* if whenever $a \leq t \leq b \in I$ then

$$f(t) \leq (t-a)\frac{f(b)-f(a)}{b-a} + f(a) + \delta.$$

We will call a geodesic metric space δ*-quasi convex* if every geodesic triangle in it is δ-midpoint convex. A 0-convex space is called *convex*.

Troyanov shows in [BGHHSST] that $\mathbb{R}^2$ with the ℓ_P metric with $P > 2$ is convex but not CAT(0). It is an easy exercise to show that convex spaces are contractible (retract along unique geodesics) and that quasiconvex spaces are combable (i.e., they are CW-lipschitz contractible).

Theorem 3.2. *A finitely generated group has a quasiconvex Cayley graph if and only if it is word hyperbolic.*

Proof. It follows easily from [ECHLPT, Theorem 3.3.1 and Lemma 2.5.5] that a finitely generated group with a quasiconvex Cayley graph is biautomatic with a geodesic language. Furthermore, the definition of δ-quasiconvexity implies that such a geodesic language must have the property that any two geodesics which have the same initial point and end at distance at one apart must stay within $\delta + 1$ of each other along their entire length. In [Pa], Papasoglu shows that such groups (which are called strongly geodesically automatic groups) are word hyperbolic. □

One can show that a continuous δ-midpoint convex function on a subinterval of $\mathbb{R}$ is β-quasiconvex for some β. Thus the distance function for geodesics in a quasiconvex metric space is quasiconvex. This means that if we are given any two geodesics, and we reparameterize them both to be constant speed on the unit interval then the distance between corresponding points will be a quasiconvex function of the parameter. More precisely, if we choose γ, η of lengths l_1 and l_2 respectively, then the distance $d(\gamma(t \cdot l_1), \eta(t \cdot l_2))$ will be a quasiconvex function of t on $[0, 1]$.

It is now evident that Gromov–hyperbolic metric spaces are quasiconvex since δ-thin triangles are γ-midpoint convex for some γ, and every geodesic triangle is "almost" a tripod.

The following is the main result of this section.

Theorem 3.3. *Let G be a group of isometries of a δ-quasiconvex space M, and equip G with an induced metric. Then $d_0(\gamma) \geq \tau(\gamma) \geq d_0(\gamma) - \delta\ \forall \gamma \in G$.*

Proof. Let $m \in M$. Let η be a geodesic segment joining m and γm. Let ω be a geodesic segment joining m and $\gamma^2 m$. Let s, t be the midpoints of η and $\gamma\eta$ respectively. Since γ acts as an isometry of M it follows that $t = \gamma(s)$.

Applying the fact that the triangle $(\eta, \gamma\eta, \omega)$ is δ-midpoint convex, we have $d_0(\gamma) \leq d(s,t) \leq \delta + 1/2 \cdot d(m, \gamma^2(m))$ and thus $d(m, \gamma^2(m)) \geq 2(d_0(\gamma) - \delta)$. Since m was arbitrary, we deduce that $d_0(\gamma^2) \geq 2(d_0(\gamma) - \delta)$. Since γ was also arbitrary, we may use induction to show that $d_0(\gamma^{2^n}) \geq 2^n(d_0(\gamma) - \delta)$, and hence $\|\gamma^{2^n}\| \geq 2^n(d_0(\gamma) - \delta)$. So,

$$\tau(\gamma) \quad = \quad \lim_{n\to\infty} \frac{\|\gamma^n\|}{n} = \lim_{n\to\infty} \frac{\|\gamma^{2^n}\|}{2^n} \geq \lim_{n\to\infty} \frac{2^n(d_0(\gamma) - \delta)}{2^n} = d_0(\gamma) - \delta.$$

□

The following classical result, whose short proof we include for the sake of completeness, can be deduced from Busemann, [Bu, 4.11,6.1].

Corollary 3.4. *Let G act as a cocompact group of isometries of the convex space M, then for every $g \in G$ either $\tau(g) = 0$, in which case g fixes some point of M, or $\tau(g) > 0$ and there exists an infinite geodesic which g leaves invariant and translates by an amount $\tau(g)$.*

Proof. Let $g \in G$. By Lemma 2.2, g is semisimple. Choose $m \in M$ such that $d_g(m) = d_0(g)$. By Theorem 3.3, $\tau(g) = d_0(g)$. If $d_0(g) = 0$, then $d_g(m) = d(gm, m) = 0$ and thus $gm = m$.

Now suppose $d_0(g) > 0$. Let T be the geodesic segment joining m and gm and let $S = \langle g \rangle \cdot T$. Now,

$$d_{g^n}(m) \leq |n| \cdot \text{length}(T) = |n| \cdot d_g(m) = |n| \cdot d_0(g) = |n| \cdot \tau(g) = \tau(g^n) \leq d_{g^n}(m).$$

Using a convexity argument, one can show that S is an infinite geodesic and that every point of S is translated by g an amount $\tau(g)$. □

Definition 3.5 (Infinite Quasigeodesic)**.** Suppose M is a metric space. Then an *infinite* (λ, ϵ)*-quasigeodesic* in M is a (λ, ϵ)-quasi-isometric embedding of $[0, \infty)$ into M.

The following is the result analogous to Corollary 3.4 in a quasiconvex setting

Corollary 3.6. *Let G act as a cocompact group of isometries of the quasiconvex space M, then for every $g \in G$ such that $\tau(g) > 0$ there exists an infinite quasigeodesic, S_g, which g leaves invariant and translates by an amount $d_0(g)$.*

Proof. Let $g \in G$. By Lemma 2.2, g is semisimple. Choose $m \in M$ such that $d_g(m) = d_0(g)$. By Theorem 3.3, $d_0(g) \leq \tau(g) + \delta$, where δ is the quasiconvexity constant for M. Let T_g be the geodesic segment joining m and gm and let $S_g = \langle g \rangle \cdot T$. Clearly one can parameterize S_g as a (nonembedded) line by $L_g : \mathbb{R} \to S_g$ where L_g maps the interval $[n \cdot d_0(g), (n+1) \cdot d_0(g)]$ to the geodesic segment $g^n T_g$. Now, $|n| \cdot \tau(g) = \tau(g^n) \leq d_0(g^n) \leq d_{g^n}(m) \leq |n| \cdot \text{length}(T_g) = |n| \cdot d_g(m) = |n| \cdot d_0(g) \leq |n| \cdot (\tau(g) + \delta)$, and thus L is a quasigeodesic embedding of $\mathbb{R}$ into M. Now, if x is in the image of L_g, choose n such that $x \in g^n T_g$, so that $d_0(g) \leq d(x, g(x)) \leq d(x, L_g(n+1)) + d(L_g(n+1), g(x)) = d(x, L_g(n+1)) + d(L_g(n), x) = \text{length}(g^n(T_g)) = d_0(g)$.

□

We now state two well-known results which follow immediately from Theorem 3.3 and Corollary 3.4.

Corollary 3.7. *Let G act as a discrete cocompact group of isometries of the convex space M then $g \in G$ has finite order if and only if g fixes a point of M.*

Corollary 3.8. *If G is a group of isometries acting properly discontinuously and cocompactly on a convex metric space M then G is translation discrete. Furthuremore, if G acts freely on M then G is torsion–free and $G \backslash M$ is a $K(G, 1)$*

Proof. Applying Theorem 3.3 we have that $\tau = d_0$. By Lemma 2.3, there is an $\epsilon > 0$ so that for every $g \in G, d_0(g) < \epsilon$ only if g fixes a point. By Corollary 3.7, we see that an element of G has translation number less than ϵ if and only if it has finite order, and thus G is translation discrete. If, in addition, the action of G is fixed–point–free we note that G acts as covering isometries of M and thatCorollary 3.7 implies that G is torsion-free. Then $G \backslash M$ has fundamental group G and has as its universal cover the contractible space M. Thus $G \backslash M$ is a $K(G, 1)$. □

4. Main Results

The following are the main results of this article.

Theorem 4.1. *Suppose the group G acts properly discontinuously cocompactly on a proper δ-quasiconvex metric space M, then G is translation discrete if and only if if does not contain an essential Baumslag-Solitar quotient as a subgroup.*

Proof. By Observation 2.7.1 we see that if G is translation discrete then G does not contain an essential Baumslag-Solitar quotient.

On the other hand, suppose G contains no essential Baumslag-Solitar quotients. By Lemma 2.4, if every element of G has nonzero translation number, then G is translation discrete. By Theorem 2.6, Theorem 3.3 we need only show that every non-torsion element g of G has a power g^n so that $d_0(g^n) > \delta$. Let $g \in G$ and consider the set $\{d_0(g^n) \mid n \in \mathbb{N}\}$. If each element of this set is less than δ we see that two different positive powers of g must be conjugate since there are only finitely many conjugacy classes K with $d_0(K) < \delta$. Considering Observation 2.7.1, we see that since G contains no essential Baumslag-Solitar quotients as subgroups, g must be torsion. □

The next result follows from the above and Observation 2.7.1.

Corollary 4.2. *A biautomatic group which acts properly discontinuously cocompactly on a proper δ-quasiconvex metric space is translation discrete.*

Corollary 4.3. *If the group G acts properly discontinuously cocompactly on a proper δ-quasiconvex metric space. Then any finitely generated residually finite normal subgroup of G is translation discrete.*

Proof. Let H be a finitely generated residually finite normal subgroup of G. By Theorem 2.6 and Theorem 3.3 we need only show that every non-torsion element h of H has a power h^n so that $d_0(h^n) > \delta$. Since H is finitely generated residually finite we choose a characteristic subgroup H' of H of finite index so that H' misses the non-trivial conjugacy classes K with $d_0(K) \leq \delta$. Since H' has finite index in H, every non-torsion element, h, of H has a power, say h^n, which lies in $H' \backslash \{1\}$ and thus $d_0(h^n) > \delta$. □

5. Applications

The purpose of this section is to give the reader a flavor of how the results in this paper can be applied.

Note 5.0.1. In [C1] it is shown that a finite cohomological dimension solvable subgroup of a translation discrete group is a finite extension of $\mathbb{Z}^n$ fore some n.

Theorem 5.1. *If G is a discrete linear group acting freely as a cocompact, properly discontinuous group of isometries of a quasiconvex space, then G is translation discrete and any solvable subgroup of G is a finite extension of $\mathbb{Z}^n$ for some n.*

Proof. Since G is a linear group it is residually finite and thus is translation discrete by Corollary 4.3, and has also that G is a discrete linear group implies that it has finite cohomological dimension. The result now follows from the above note. □

The following classical result follows immediately from [C1] and Corollary 3.8

Corollary 5.2 (Gromoll–Wolf Theorem)**.** *Let M be a compact Riemannian manifold of nonpositive sectional curvature. Then any solvable subgroup of $\pi_1(M)$ is abelian-by-finite. In particular, $\pi_1(M)$ is a finite extension of $\mathbb{Z}^n$ where $n \leq \dim(M)$.*

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