ON THE FUNDAMENTAL GROUPS OF ONE–DIMENSIONAL SPACES

November 26, 1998

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Abstract. We study the fundamental group of one–dimensional spaces. Among the results we prove are that the fundamental group of a second countable, connected, locally path connected, one–dimensional metric space is free if and only if it is countable if and only if the space has a universal cover and that the fundamental group of a compact, one–dimensional, connected metric space embeds in an inverse limit of finitely generated free groups.

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1. INTRODUCTION

This paper is the last in a series of three. The first paper in the series [CC1], gives a combinatorial description of the Hawaiian earring group, introduced the notion of *big free groups* as a generalization of both finite rank free groups and the Hawaiian earring group and studies their group theoretic properties.

In the second paper, [CC2], we introduce the *big fundamental group* of a topological space. This group is similar to the standard fundamental group, but rather than being based on maps of the unit interval into a given space, the big fundamental group is based on maps of *big intervals* (compact totally ordered connected spaces). We then define spaces known as the *big Hawaiian earrings*, and prove that the big fundamental group of a big Hawaiian earring is a big free group.

The current paper studies the fundamental groups of one–dimensional spaces. For example, we prove there that the fundamental group of a second countable, connected, locally path connected, one–dimensional metric space is free if and only if it is countable if and only if the space has a universal cover, and we prove that the fundamental group of a compact, one–dimensional, connected metric space embeds in an inverse limit of finitely generated free groups and thus is locally free, residually free, and residually finite.

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[1] This research was supported in part by an NSF research grant.

The notion of the fundamental group of a space first appeared when Poincaré first used it to construct the first example of a space whose homology was that of a 3-sphere and yet was not a 3-sphere. Since then the fundamental group has been used as a standard tool in many branches of topology, geometry, and analysis. Among the tools for calculating the fundamental group are covering spaces, the Seifert–van Kampen theorem and the theory of nerves of covers. However these tools were designed to study spaces whose local homotopy structure is uncomplicated. For instance, the theory of covering spaces can only be used to its full potential on spaces which are semilocally simply-connected and useful information can be obtained from the Seifert-Van Kampen theorem or nerve theory when applied to spaces which admit an open cover satisfying strong intersection conditions. While many of the spaces which topologists study, such as manifolds, simplicial complexes and CW– complexes, are locally trivial with respect to homotopy, there are many examples of interesting spaces which are much more complicated. In this article we concentrate our interest on one–dimensional spaces (but from time to time consider general metric spaces). The basic principle we wish to illustrate is that those spaces admitting universal covers have relatively small, well-understood fundamental groups whereas those spaces not admitting universal covers have large, ill-understood fundamental groups. To bring the issue into focus we mention the following elementary lemma which appears in the last section of this paper.

Lemma 7.3. *If X is a one–dimensional metric space which is connected, locally path connected and semilocally simply-connected (equivalently if X admits a universal covering space) then $\pi(X)$ is a free group.*

However, general one–dimensional metric spaces have a much more complicated homotopy structure. Consider, for instance, the Hawaiian earring, which is defined to be the union of planar circles of radius $1/n$, for $n \in \mathbb{N}$, tangent to the x-axis, and passing through the origin. It is an interesting and somewhat surprising fact that the fundamental group of this relatively simple compact one–dimensional space is not a free group, and, incidentally, is not countable (see [CC1]). Even more complicated are the fundamental groups of the Sierpinski and Menger curves. These are spaces are important topologically, but not well-understood homotopically. In this paper we study the fundamental groups of one–dimensional metric spaces and peripherally the fundamental groups of metric spaces in general.

We will focus our attention on second countable spaces since the structure of "larger" spaces is more readily understood using other tools such as the "big fundamental group" (see [CC2] for details), so we make the convention that all spaces discussed in this paper will be second countable unless otherwise noted. One obvious question to ask about one–dimensional spaces is under what conditions are their fundamental groups free. We will show that a one–dimensional metric space has a free fundamental if and only if its fundamental group is countable, if and only if it has a universal cover. We will also show that the fundamental group of a one–dimensional space embeds in an inverse limit of free groups, and thus is locally free, residually free and residually finite. The main tools which we use are Theorem 4.4, and the theory of nerves of covers. The study of the relationships between fundamental groups and nerves is particularly relevant to the study of one–dimensional spaces since one–dimensionality is a property of nerves and is, in fact, defined as such. Rudiments of such a theory appear in the last section Section 7 and will be referred to as needed. Theorem 4.4 shows that if X is a first countable space, and $x \in X$ then any countable factor group, L of $\pi(X, x)$ has the property that the images, in L, of the homotopy groups of descending chains of open neighborhoods of x is Artinian (i.e. any descending sequence is eventually constant). It also shows that the limit is trivial in the case where L is Abelian with no *infinitely divisible* elements.

To understand the homotopy groups of spaces which are not semilocally simply–connected one must realize that in such spaces the fundamental group enjoys an *infinite* product structure. As an example consider the Hawaiian earring, H, and a loop which traverses all of the circles which comprise H. This homotopy class of this loop can be considered as the infinite product of the homotopy classes of the circles in question. Definition 4.1 and Lemma 4.2 will make this notion more rigorous.

Historically there have been a number of papers along these lines. Curtis and Fort have a series of papers [CF1, CF2, CF3] which explore the homotopy and homology groups of one–dimensional spaces and Fort has a paper [F] which studies mappings of S^1 into one–dimensional spaces. In [CF1], Curtis and Fort prove the compact case of our Theorem 5.9. Our Theorem 5.11 can be proven using results from [CF3] and our Corollary 3.5. Other papers in this general area we are aware of are a preprint of A. Zastrow [Z], where it is argued that all subsets of $\mathbb{R}^2$ are "$K(\pi, 1)$", and a preprint of W. Bogley and A. Sieradski in which they study one–dimensional metric cell complexes in which one–cells can limit on zero–cells.

This article has the following structure. In Section 2 we give examples of one–dimensional spaces with interesting fundamental groups. In Section 3, we give an introduction to the homotopy theory of paths in one–dimensional spaces. This includes proving the existence of normal forms and the injectivity of homomorphisms between fundamental groups one–dimensional spaces induced embeddings. In Section 4 we rigorously define the notion of infinite multiplication in fundamental groups and prove Theorem 4.4. In Section 5 we prove a number of results including

Theorem 5.1. *If X is a second countable, locally path connected metric space then*

1. *Any free abelian factor group of $\pi(X)$ is countable, consequently any free factor group of $\pi(X)$ has countable rank.*
2. *If X is compact then any free abelian factor group of $\pi(X)$ is finitely generated, consequently any free factor group of $\pi(X)$ has finite rank.*
3. *If $\pi(X)$ is free abelian then X has a universal cover.*

Theorem 5.9. *If X is a second countable, connected, locally path connected , one–dimensional metric space then the following are equivalent: $\pi(X)$ is free, $\pi(X)$ is countable, X is locally simply connected, X has a universal cover.*

Theorem 5.11. *If X is a compact, one–dimensional, connected metric space, and $x_0 \in X$, then $\pi_1(X, x_0)$ embeds in an inverse limit of finitely generated free groups and thus is locally free, residually free, and residually finite.*

We also introduce the notion of as space being *homotopically Hausdorff*. This key property allows us to effectively study one–dimensional spaces, but is also enjoyed by many locally simple spaces such as manifolds and CW–complexes. A space X is homotopically Hausdorff if given any point, x_0 in X and any nontrivial homotopy class c in $\pi(X, x_0)$ there is a neighborhood U of x_0 which contains no representative for c. We prove the following results:

Lemma 5.6. *If X is a second countable, connected, locally path connected , metric space which is homotopically Hausdorff (e.g. X is one–dimensional) then $\pi(X)$ is countable if and only if X has a universal cover.*

Corollary 5.7. *If X is a compact, connected, locally path connected, metric space which is homotopically Hausdorff then the following are equivalent : $\pi(X)$ is countable, $\pi(X)$ is finitely generated, $\pi(X)$ is finitely presented, X has a universal cover.*

Theorem 5.10. *If X is a path connected, locally path connected, homotopically Hausdorff metric space then the normal covering spaces of X are in one–to–one correspondence with the countable factor groups of $\pi(X)$. Furthermore, if X is compact then the normal covering spaces of X are in one–to–one correspondence with the finitely generated factor groups of $\pi(X)$.*

We have the following interesting corollary:

Corollary 5.8. *Any compact, connected, locally path connected metric space which has a universal cover (e.g. a compact manifold) has a finitely presented fundamental group.*

In Section 6 we study the relationships between infinite multiplication and abelianizations. A topological classification of the elements of the commutator subgroup of a one–dimensional space is given. This classification analogous to a classification of the elements of the commutator subgroup of finite rank free groups and the Hawaiian earring group which we discuss in [CC1]. We also define the *strong abelianization* of the fundamental group of a space and discuss some if its properties. We conclude this section by showing that the fundamental group of a connected, locally path connected, separable metric space contains a subgroup C so that C has countable image in the abelianization (or equivalently homology) and that dividing by C leaves a group which becomes trivial by twice factoring out all infinitely divisible elements.

In Section 7 we prove results relating covering spaces to the theory of nerves. We have already mentioned Lemma 7.3. These results also tell us that spaces which are locally well-behaved have covering nerves which have fundamental groups which are isomorphic to that of the space. They also give us information about the size of the fundamental group of a "well-behaved" space. For instance

Lemma 7.6. *If X is a connected, locally path connected, second countable metric space which is* locally trivial with respect to $f : \pi(X) \longrightarrow K$ *then $f(\pi(X))$ is countable, and furthermore is finitely generated if X is compact.*

2. Examples of one–dimensional spaces and their fundamental groups

Example 2.0.1 (The Hawaiian earring). The *Hawaiian Earring*, H, is defined to be the subspace of $\mathbb{R}^2$ consisting of the union of planar circles c_i of radius $1/i$, tangent to the x-axis at the origin for $i \in \mathbb{N}$.

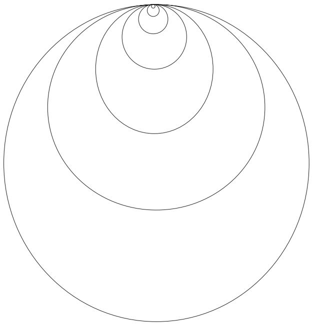

Figure 1. Hawaiian Earring

This space is well-known to have interesting properties, in particular it is compact, but its fundamental group is uncountable. Also it is one dimensional, but its fundamental group is not free. See [CC1] for details. Later in the current article (Theorem 5.9) we will show that any second countable, one–dimensional, locally path connected space has a fundamental group which is either countable and free or uncountable and not free. In [CC1] we show that the *Hawaiian Earring group*, $\pi(H)$ is not free by proving that each of its free factor groups have finite rank. In this article, Theorem 5.1 proves this result for general compact locally path connected metric spaces. In [CC1] we give a combinatorial description of the Hawaiian Earring group as a *big free group* in the sense that its elements are infinite words and its cancellations are also *bigger than those in a standard free group.*

Example 2.0.2 (The doubled cone over the Hawaiian earring). Consider $C = H \times [0,1]/H \times \{1\}$, the cone over the Hawaiian Earring. Let c_0, the basepoint of C, be the point $((0,0),0)$. Then the doubled cone over the Hawaiian earring is the space $D = C \vee_{c_0} C$, in other words two copies of the cone C glued together at the basepoint.

The following results are from [CC1]. Note that C is contractible. However D is not contractible since its fundamental group is uncountable! If we were to attach the two copies of c_0 with an arc rather than

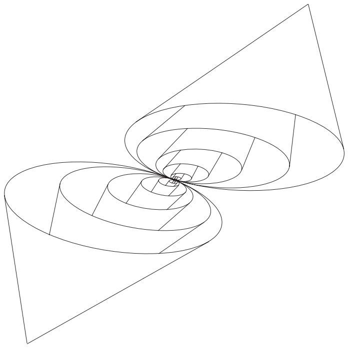

Figure 2. The doubled cone over the Hawaiian earring

identifying them, the resulting space would be contractible. D is particularly misbehaved since there is one loop (namely going around the first copy of c_1 then the second, then the first copy of c_2 then the second, etc) which is homotopic into any neighborhood of the basepoint and yet is not nullhomotopic. Thus this space is not *homotopically Hausdorff*. An easy application of Theorem 4.4 shows that the fundamental group of D has no nontrivial countable free abelian factor groups and thus this group cannot embed in an inverse limit of finite rank free groups.

Example 2.0.3 (The Sierpinski and Menger Curves). In some senses the Sierpinski curve and Menger curve are generic one dimensional continua, and so their fundamental groups are good candidates for study. Applying Theorem 5.1, and Theorem 5.9 we know quite a bit about the Sierpinski and Menger curve groups. In particular, they are uncountable, not free, and each of their free factor groups has finite rank. Theorem 5.10 classifies their countable factor groups in as being in one–to–one correspondence with their covering spaces. However unlike the Hawaiian Earring group these groups have no known combinatorial (word) structure and so, in a combinatorial sense, are not at all well–understood.

Conjecture 2.0.4. The Hawaiian Earring, Sierpinski curve and Menger curve groups are distinct groups, in other words no two are isomorphic.

3. The Homotopy Theory of Paths in One–Dimensional Spaces

3.1. Embeddings of fundamental groups of one–dimensional spaces.

Theorem 3.1. *If X is a subset of Y with X compact, connected, locally connected metric, with Y compact metric, and with $Y \setminus X$ one–dimensional, then Y retracts to X.*

Proof. Take an open cover U of $Y \setminus X$ with mesh going to 0 as one approaches X. (Use, for example, open balls $B(y, r)$ of radius r centered at $y \in Y \setminus X$ such that $r < (1/2)d(y, X)$.) Let V denote a locally finite refinement of U with nerve N which is one–dimensional. Let $\{\rho_v : Y \setminus X \to [0, 1] \mid v \in V\}$ be a partition of unity on $Y \setminus X$ subordinate to V; that is, each ρ_v is continuous, $\rho_v(y) \neq 0$ implies $y \in v$, and $y \in Y \setminus X$ implies that $\sum_v \rho_v(y) = 1$. Then the ρ_v are barycentric coordinates for a continuous function $\rho : Y \setminus X \to N$. We define a continuous function $\sigma : N \to X$ as follows. A vertex of N is an element $v \in V$. Map v to $\sigma(v) \in X$ such that $d(v, \sigma(v)) \leq 2d(v, X)$. Each 1-cell of N corresponds to a pair (v, w) of elements of V such that $v \cap w \neq \emptyset$. Map (v, w) to a path in X which connects $\sigma(v)$ and $\sigma(w)$ and has diameter no more than twice the infimum of all possible diameters of paths in X from $\sigma(v)$ to $\sigma(w)$. Define $r : Y \to X$ by $r \mid X = \mathrm{id}$ and $r \mid Y \setminus X = \sigma \circ \rho$. We Leave to the reader the proof that r is continuous. ☐

Corollary 3.2. *Suppose that $f : B^2 \to Z$ is a continuous function from the disk B^2 into a one–dimensional Hausdorff space. Then f can be retracted into the image of $S^1 = \partial B^2$.*

Proof. The map f is necessarily a closed function since Z is Hausdorff. Hence the image Y of f is compact metric and, as a subset of Z, one–dimensional. The image X of S^1 under f is compact, connected, locally connected metric. Hence the previous theorem applies, and Y retracts to X. ☐

Corollary 3.3. *If $g : X \to Y$ is an embedding of one one–dimensional Hausdorff space into another, then the induced map on fundamental group is injective.*

Proof. Let $f : S^1 \to X$ be a continuous function such that the composite $g \circ f$ is nullhomotopic. Let $h : B^2 \to Y$ be a continuous extension of $g \circ f$. Let $r : h(B^2) \to g \circ f(S^1)$ be a retraction whose existence is ensured by the previous corollary. Then $g^{-1} \circ r \circ h$ is a contraction of f in X. ☐

Corollary 3.4. *Suppose that X is a pathwise connected subset of the 2-sphere S^2 such that the complement of X is dense in S^2. Then the fundamental group of X embeds in the fundamental group of the Sierpinski curve Y.*

Proof. Pick a countable dense set C in $S^2 \setminus X$. By G.T. Why burn's topological characterization of the Sierpinski curve, [GTW], there is a continuous function $\pi : Y \to S^2$ from the Sierpinski curve Y onto S^2 so that the nondegenerate point preimages are precisely the preimages of the points of C, and such preimages are precisely the boundary circles of Y. Then π^{-1} embeds X in the inaccessible part of Y. The previous corollary then implies that the fundamental group of X is thereby embedded in the fundamental group of Y. □

Corollary 3.5. *If X is any one–dimensional, path-connected, compact metric space. Then the fundamental group of X embeds in the fundamental group of the one–dimensional Menger curve in E^3.*

Proof. It is well-known that X embeds in Y. Corollary 3.3 then implies the desired result. □

Question 3.5.1. Does the Menger-curve group embed in the Sierpinski-curve group? Does the Sierpinski-curve group embed in the Hawaiian earring group? We conjecture not.

3.2. **One–dimensional relations.**

We fix a compact, pathwise connected, one–dimensional metric space X for the remainder of this section. We answer the following questions:

- Is there a somewhat combinatorial way or recognizing a trivial (nullhomotopic) loop in X?
- Can one give a well-defined notion of *reduced loop* in X?
- Is every loop homotopic to a reduced loop? If so, to what extent is the reduced loop unique? Is there tree–like object that might be considered the topological Cayley graph of the fundamental group?

Some of our theorems will involve the notion of *dendrite.*

Definition 3.6. A *dendrite* is a connected, locally connected, compact metric space that contains no simple closed curve.

Every dendrite is contractible.

Our semi-combinatorial description of nullhomotopic loops depends on the following theorem.

Theorem 3.7. *Let $f : S^1 \to X$ be a nullhomotopic loop. Then f factors through a surjective map $f' : S^1 \to D$, where D is a planar dendrite.*

Proof. Let $f'' : B^2 \to X$ denote a contraction of f. By Corollary 3.2, we may assume that the image of f'' lies in the image of f.

We can modify f'' so that no point preimage of f'' separates $E^2 \supset B^2$. Indeed, we may order those components C_α of point preimages which separate E^2 so that $C_\alpha < C_\beta$ if C_α lies in a bounded complementary domain of C_β. It is easy to prove that every C_α either is maximal or lies in a bounded complementary domain of a maximal element. For each maximal element C_α, map C_α and all of its bounded complementary domains to the same point to which C_α was originally mapped.

We now let G denote the collection of components of point preimages of f''. We extend G to all of E^2 by adding in the individual points of $E^2 \setminus B^2$. Then G, as so extended, forms a cellular, upper-semicontinuous decomposition of E^2. By R.L. Moore's theorem [RLM], the decomposition space E^2/G is homeomorphic with E^2.

Let D' denote the image of B^2 under the projection map $\pi : E^2 \to E^2/G$. We assert that D' is one dimensional. Indeed, there is a map $f''' : D' \to X$ such that $f'' = f''' \circ \pi$ and each point preimage of f''' is totally disconnected. Hence f''' cannot lower dimension by Theorem VI.7 of Hurewicz and Wallman, [HW]. Since the image of D' is one dimensional, it follows that D' is one dimensional.

We conclude that $D = \pi(S^1)$, which lies in D', is a compact, locally connected, one–dimensional planar continuum. Since D' does not separate E^2, D can also not separate E^2. We conclude that D is a planar dendrite, as required. The map f clearly factors through the map $f' = \pi \mid S^1 : S^1 \to D$. □

3.2.1. *A combinatorial description of the dendrite D.* For each $x \in D$, let H_x denote the convex hull of ${f'}^{-1}(x)$ in B^2. If $x \neq y$, then $H_x \cap H_y = \emptyset$. Indeed, there are disjoint continua C_x and C_y in B^2 intersecting S^1 precisely in ${f'}^{-1}(x)$ and ${f'}^{-1}(y)$ because $D \subset D'$ which is realized by such a decomposition. Then $D = B^2/\{H_x \mid x \in D\}$. One can analyze the structure of D completely in terms of the geometry of the decomposition $\{H_x \mid x \in D\}$ of B^2.

3.3. **One–dimensional Relators.**

Definition 3.8. We say that a nonconstant loop $f : (S^1, 1) \to (X, x_0)$ is *reducible* if there is an open arc $I = (x, y)$ in $S^1 \setminus \{1\}$ such that $f(x) = f(y)$ and the resulting loop based at $f(x) = f(y)$ defined by $f \mid [x, y]$ is nullhomotopic. We say f is *reduced* if it is not reducible. A constant loop is, by definition *reduced.*

Theorem 3.9. *Every loop* $f : (S^1, 1) \to (X, x_0)$ *is homotopic to a reduced loop. This reduced loop is unique up to reparameterization of* S^1*.*

Proof. Existence: Suppose f is not nullhomotopic. Consider the open sets G_α of S^1 such that no component of G_α contains the base point 1 of S^1 and such that each component of G_α represents a nullhomotopic loop under f. Order the G_α by inclusion. Let $\{G_\alpha \mid \alpha \in A\}$ be a totally ordered subcollection; that is, if α_1 and α_2 are in A, then either $G_{\alpha_1} \subset G_{\alpha_2}$ or $G_{\alpha_2} \subset G_{\alpha_1}$. We assert that $G = \cup\{G_\alpha \mid \alpha \in A\}$ is also such an open set. In order to see this, we must show that, if I is a component of G, then $f \mid I$ defines a nullhomotopic loop. Each compact subset of I is covered by some single G_α, $\alpha \in A$. Hence there is a sequence $I_{\alpha_1}, I_{\alpha_2}, \ldots$ of interval components from $G_{\alpha_1} \subset G_{\alpha_2} \subset \cdots$, with $I_{\alpha_1} \subset I_{\alpha_2} \subset \cdots \subset \cup_i I_{\alpha_i} = I$. There are three cases to consider:

Case 1: $I = I_{\alpha_i}$ for some i. Then I is nullhomotopic as required.

Case 2: I shares one end, say the first point, with some I_{α_i}, say with I_{α_1}. Let α_i denote the initial point of I_{α_i}, β_i the terminal point. Then $\alpha_1 = \alpha_2 = \cdots = \alpha_i \cdots$ and $\beta_1 \le \beta_2 \le \cdots$. Then all of α_1, β_1, β_2, $\ldots$ have the same image under f, and each of the loops $f \mid [\alpha_1, \beta_1]$, $f \mid [\beta_1, \beta_2]$, $f \mid [\beta_2, \beta_3]$, $\ldots$ is nullhomotopic. Shrinking each in its own image yields a shrinking of the loop $f \mid I$.

Case 3: $I = [\alpha, \beta]$, with $\alpha < \alpha_i$ and $\beta_i < \beta$ for every i. Then the loop $f \mid [\alpha_1, \beta_1]$ is nullhomotopic, as is each of the loops $f \mid [\alpha_{i+1}, \alpha_i] \wedge [\beta_i, \beta_{i+1}]$. Shrinking each of these loops in its own image results in a nullhomotopy of I.

We conclude that each chain $\{G_\alpha \mid \alpha \in A\}$ has an upper bound. Thus by Zorn's lemma, there is a maximal element G.

Taking a maximal element G, we homotop f on each component of G to the constant map at the ends of that interval. If we perform the homotopies always within the original image of the closed interval, then all of these homotopies together form a homotopy of f. Let f' denote the new map.

We claim that, if G is maximal, then no two components of G have a common endpoint in $S^1 \setminus \{1\}$. For otherwise, those two components could be joined into a single component, thereby enlarging G. Consequently, G defines a monotone decomposition G' of S^1 whose elements are the closures of the components of G. If two of these closed components intersect at 1, they are to be joined into a single interval. The decomposition is still monotone. The decomposition space is still a circle since f is not nullhomotopic. Let $\pi : S^1 \to (S^1 = S^1/G')$ be the decomposition map.

The map $f'' = f' \circ \pi^{-1} : S^1 \to X$ is the desired reduced loop homotopic to f.

Uniqueness: If g_1 and g_2 are two such maps, then $g_1 g_2^{-1}$ is nullhomotopic. Let D be a dendrite through which $g_1 g_2^{-1}$ factors. It must be an arc reparameterizing g_1 to g_2, for otherwise one finds an unallowable cancellation in either g_1 or g_2. □

3.4. **Is there tree–like object that might be considered the topological Cayley graph of the fundamental group?** There is certainly a candidate for the point set of such a space. For each reduced loop, one takes an (ordered) interval $[0, 2\pi]$ with each point labeled by its image in X. For two such reduced loops, one identifies the largest initial segments that, after possible reparameterization, are identical.

4. Infinite Multiplication in fundamental groups

Definition 4.1 (Continuous representative, Legal product)**.** If X is a topological space, $x_0 \in X$, B is a countable ordered set and $\{p_\beta\}_{\beta\in B}$ is a set of closed paths in X based at x_0 then we say the path p is a *continuous representative* of $\prod_{\beta\in B} p_\beta$ if there are subintervals $\{I_\beta\}_{\beta\in B}$ of I with disjoint interiors whose natural order agrees with that of B and which are dense in I so that $p\,|_{I_\beta}$ is equal to p_β up to reparameterization. We say that $\prod_{\beta\in B} p_\beta$ is a *legal product* if it has a continuous representative.

We leave the proof of the following elementary lemma to the reader.

Lemma 4.2. *If X is a topological space which has a countable basis at X_0, B is a countable ordered set and $\{p_\beta\}_{\beta\in B}$ is a set of closed paths in X based at x_0 then there is a continuous representative of $\prod_{\beta\in B} p_\beta$ if and only if for each countable local basis $U_1 \supseteq U_2 \supseteq \cdots$ for X at x_0, and for all $i \in \mathbb{N}$ the images of all but finitely many of the p_β are contained in U_i.*

Definition 4.3 (Infinitely Divisible)**.** An element h of a group H is said to be *infinitely divisible* if there are infinitely many integers e and corresponding nonidentity elements h_e of H so that $(h_e)^e = h$. The subgroup of H generated by infinitely divisible elements is normal. A group with no infinitely divisible elements is torsion–free.

4.1. **An Artinian Property of homomorphisms of fundamental groups.**

Theorem 4.4. *Let X be a topological space, let $f : \pi(X, x_0) \longrightarrow L$ be a homomorphism to the group L, $U_1 \supseteq U_2 \supseteq \cdots$ be a countable local basis for X at x_0, and G_i be the image of the natural map of $\pi(U_i, x_0)$ into $\pi(X, x_0)$. Then*

1. *If L is countable then the sequence $f(G_1) \supseteq f(G_2) \supseteq \cdots$ is eventually constant.*
2. *If L is abelian with no infinitely divisible elements then $\bigcap_{i\in\mathbb{N}} f(G_i) = \{0_L\}$.*
3. *If L is countable abelian with no infinitely divisible elements then $f(G_i) = \{0_L\}$ for some $i \in \mathbb{N}$.*

Proof. Let $G = \pi(X, x_0)$. Since we may replace L by the image of f, we will assume f is surjective. To prove the first part of the result we will proceed using a diagonalization argument. By way of contradiction, assume $f(G_1) \supseteq f(G_2) \supseteq \cdots$ is not eventually constant. Then there exists a sequence of increasing natural numbers $(i_j)_{j=0}^{\infty}$ such that the index of $f(G_{i_{j+1}})$ in $f(G_{i_j})$ is greater than $j+1$. Also, since L is countable, we may enumerate its elements in a sequence $(l_i)_{i=0}^{\infty}$, and denote by the *subscript* of an element of L its position in this sequence.

We will construct a sequence of elements, $(k_j)_{j=0}^{\infty}$, in G such that each k_j lies in G_{i_j} and so that the legal products

$$w_j = \prod_{i=0}^{j} k_j$$

satisfy the condition that every element of the coset $f(w_j)f(G_{i_{j+1}})$ has subscript greater than j.

Deferring the construction for a moment, we complete the argument by pointing out that

$$w = \prod_{i=0}^{\infty} k_j$$

is a legal product in G and that $f(w)$ is an element of $f(w_j)f(G_{i_{j+1}})$ for every j and thus has subscript larger than j for every natural number j, yielding a contradiction.

Choose $k_0 = 1_L$. If k_{j-1} (and thus also w_{j-1}) has been defined, we define k_j as follows. The index of $f(G_{i_{j+1}})$ in $f(G_{i_j})$ is greater than $j+1$ implies there is a set $\{r_1, r_2, \ldots, r_t\}$ of elements of G_{i_j} so that $\{f(r_1), f(r_2), \ldots, f(r_t)\}$ is a set of at least j+2 representatives of distinct left cosets of $f(G_{i_{j+1}})$ in $f(G_{i_j})$. It is evident that

$$\{f(w_{j-1}r_1), f(w_{j-1}r_2), \ldots, f(w_{j-1}r_t)\}$$

is also a set of representatives of distinct left cosets of $f(G_{i_{j+1}})$ in $f(G_{i_j})$. Since there only $j+1$ elements in L with subscript at most $j+1$, at least one of the cosets $f(w_{j-1}r_s)f(G_{i_{j+1}})$ contains no element whose subscript is less than $j+2$. Choose $k_j = r_s$.

We now proceed to prove the second part of the result. Suppose

$$\bigcap_{i \in \mathbb{N}} f(G_i)$$

contains a nonidentity element, a. For each $i \in \mathbb{N}$ choose $g_i \in G_i \cap f^{-1}(a)$.

By Lemma 4.2, for each $n, i \in \mathbb{N}$ we may choose elements $w_{n,i}$ of G_i which satisfy the following conditions,

$$\begin{gathered} w_{n,1} = g_1 w_{n,2}^n \\ \vdots \\ w_{n,i} = g_i w_{n,i+1}^n \\ \vdots \end{gathered}$$

Now,

$$\begin{aligned} f(w_{n,1}) &= f(g_1) + n \cdot f(w_{n,2}) = a + n \cdot (a + n \cdot f(w_{n,3})) \\ &= a + na + n^2 \cdot f(w_{n,3}) \\ &\vdots \\ &= a + na + n^2 a + \cdots + n^{k-1} a + n^k \cdot f(w_{n,k+1}) \\ &= a \left(\frac{n^k - 1}{n - 1} \right) + n^k \cdot f(w_{n,k+1}), \ \forall n, k. \end{aligned}$$

So,

$$\begin{aligned} (n-1) \cdot f(w_{n,1}) + a &= an^k + (n-1)n^k \cdot f(w_{n,k+1}) \\ &= n^k (a + (n-1) \cdot f(w_{n,k+1})), \ \forall n, k. \end{aligned}$$

However, L has no infinitely divisible elements. It follows that $(n-1) \cdot f(w_{n,1}) + a = 0_L$ for all n. Therefore $a = (1-n) \cdot f(w_{n,1})$ for all n and thus $a = 0_L$.

The third part of the lemma follows immediately from the first two. □

5. How Nearly Free are the Fundamental Groups of One–Dimensional Spaces

5.1. **A One–Dimensional Metric Space admits a Universal Cover if and only if its Fundamental Group is Free.** In [CF1] Curtis and Fort prove that a one-dimensional separable locally connected continuum has a universal cover if and only if it has a free fundamental group. We give some related results and generalizations in this section.

Theorem 5.1. *Suppose X is a second countable, locally path connected metric space then*

1. *Any free abelian factor group of $\pi(X)$ is countable, consequently any free factor group of $\pi(X)$ has countable rank.*
2. *If X is compact then any free abelian factor group of $\pi(X)$ is finitely generated, in consequently any free factor group of $\pi(X)$ has finite rank.*
3. *If $\pi(X)$ is free abelian then X has a universal cover.*

Proof. We begin by proving 2. We note that if $\pi(X)$ has a free factor group of infinite rank then there is a surjective homomorphism $f : \pi(x) \longrightarrow A$, where A is an countably infinite rank free abelian group. Thus it suffices to prove that any free abelian factor group of $\pi(X)$ has finite rank. Since A is free abelian, it contains no infinitely divisible elements.

Fix $x \in X$, let $U_1 \supseteq U_2 \supseteq \cdots$ be a path connected countable local basis for X at x, and G_i be the image of the natural map of $\pi(U_i, x)$ into $\pi(X, x)$. By Theorem 4.4, $f(G_i) = \{1_L\}$ for some $i \in \mathbb{N}$. Thus for each $x \in X$ there is an open neighborhood U_x of x such that the image of the natural map from $\pi(U_x)$ to $\pi(X)$ induced by inclusion lies in the kernel of f . Applying Lemma 7.6 we get that $f(\pi(X))$ is finitely generated.

We will now use 2. to prove 1. As above, suppose fix $x \in X$, let $U_1 \supseteq U_2 \supseteq \cdots$ be a path connected local basis for X at x, and G_i be the image of the natural map of $\pi(U_i, x)$ into $\pi(X, x)$, and suppose there is a surjective homomorphism $f : \pi(x) \longrightarrow A$, where A is a free abelian group.

Suppose $f(G_i)$ has infinite rank for all i. We will now define a continuous map from the Hawaiian Earring, H, to X. Map the basepoint of the Hawaiian Earring to x. Since each $f(G_i)$ has infinite rank, we can find a sequence l_i of homotopy classes in $\pi(X)$ so that each l_i lies in G_i and that the set of elements $\{f(l_i)\}$ of A form a basis for a countable rank free abelian subgroup of A. Recall that H is comprised of the union of planar circles c_i of radius $1/i$ each tangent to the x-axis at the origin. Map each circle c_i in H to a closed curve in U_i based at x and which represents l_i. Note that map $P : H \longrightarrow X$ we have just constructed is continuous, since any open neighborhood of x contains the images of all but finitely many of the c_i. Now we have a map $f \circ P^*$ from the Hawaiian Earring group to A whose image is free abelian of countably infinite rank. Since this is impossible by 2., we see that there is an $i \in \mathbb{N}$ such that $f(G_j)$ has finite rank for all $j > i$.

We notice that the composition of f with the map from $\pi(U_i)$ to $\pi(X)$ induced by inclusion is a map $h : \pi(U_i) \longrightarrow f(G_i)$ from $\pi(U_i)$ to the countable free abelian group $f(G_i)$. Applying Theorem 4.4 to the map h, we see that $f(G_j)$ must be trivial for some $j > i$.

Thus for each $x \in X$ there is an open neighborhood U_x of x such that the image of the natural map from $\pi(U_x)$ to $\pi(X)$ induced by inclusion lies in the kernel of f . Applying Lemma 7.6 we get that $f(\pi(X))$ is countable.

Finally we prove 3. If $\pi(X)$ is free abelian, we can choose f as above to be the identity map on $\pi(X)$. By the above argument, X is *semilocally simply connected*, and thus (see, for instance, [M]) has a universal cover.

□

Definition 5.2 (Homotopically Hausdorff)**.** We say that the topological space X is *homotopically Hausdorff at the point* $x_0 \in X$ if for every $g \in \pi(X, x_0) - \{1\}$ there is an open neighborhood of x_0 which contains no path representing g. We say that X is *homotopically Hausdorff* if X is homotopically Hausdorff at x_0 for every $x_0 \in X$.

The moniker *homotopically Hausdorff* is motivated by the fact that the space of homotopy classes of based paths $\Omega(X, x_0)$ of a space X based at $x_0 \in X$ is Hausdorff if and only if X is homotopically Hausdorff.

Remark 5.2.1. It is immediate that standard "locally well-behaved" spaces such as manifolds and CW-complexes are homotopically Hausdorff as well as are all other semilocally simply–connected spaces (i.e. spaces with universal covers). In [CL] it is shown that planar sets are homotopically Hausdorff.

If the following conjecture were true it would allow us to remove the homotopically Hausdorff hypotheses from Lemma 5.6 ,Corollary 5.7, and Theorem 5.10.

Conjecture 5.2.2. The fundamental group of a connected, locally path connected space which is not homotopically Hausdorff is uncountable.

The following lemma follows immediately from Theorem 3.7 and Theorem 3.9.

Lemma 5.3. *If X is a one–dimensional topological space and p is a closed path in X based at x_0 then p is homotopic to a unique (up to reparameterization) closed path based at x_0 which is either constant (if p is null-homotopic) or has no proper null-homotopic subpaths. Furthermore, the image of the homotopy can be chosen to be contained in the image of p.*

Corollary 5.4.

1. *A connected, locally arcwise connected, Hausdorff, one–dimensional space has a universal cover if an only if it is locally simply connected.*
2. *Every one–dimensional topological space is homotopically Hausdorff.*

Proof. It is clear from the above lemma that a one dimensional space is locally simply connected if and only if it is semilocally simply connected. Since a connected, locally path connected Hausdorff space has a universal cover if and only if it is semilocally simply connected we have that 1. is true. It is immediate from the preceding lemma that 2. is true. □

Lemma 5.5. *A path connected, locally path connected, first countable, homotopically Hausdorff space, X, with countable fundamental group has a universal cover.*

Proof. Let $x \in X$, let $f : \pi(X, x) \longrightarrow \pi(X, x)$ be the identity homomorphism, $U_1 \supseteq U_2 \supseteq \cdots$ be a countable local basis for X at x_0, and G_i be the image of the natural map of $\pi(U_i, x)$ into $\pi(X, x)$. Since $\pi(X, x)$ is countable we may apply Theorem 4.4 to get that the sequence $f(G_1) \supseteq f(G_2) \supseteq \cdots$ is eventually constant. Thus the sequence $G_1 \supseteq G_2 \supseteq \cdots$ is eventually constant. Choose $j \in \mathbb{N}$ so that $\bigcap_{i\in\mathbb{N}} p_i = p_j$. Since X is homotopically Hausdorff, $p_j = \{1\}$. Allowing x to vary over all points of X we see that X is semilocally simply connected and thus has a universal cover. □

Lemma 5.6. *If X is a second countable, connected, locally path connected , metric space which is homotopically Hausdorff (e.g. X is one–dimensional) then $\pi(X)$ is countable if and only if X has a universal cover.*

Proof. We will first show that if $\pi(X)$ is countable then X has a universal cover. We will then show if X has a universal cover it follows that $\pi(X)$ is countable.

Suppose $L = \pi(X)$ is countable. Fix $x \in X$, let $U_1 \supseteq U_2 \supseteq \cdots$ be a path connected countable local basis for X at x, and G_i be the image of the natural map of $\pi(U_i, x)$ into $\pi(X, x)$. Let $f : \pi(X, x) \longrightarrow L = \pi(X, x)$ be the identity map. By Theorem 4.4 the sequence $G_1 = f(G_1) \supseteq G_2 = f(G_2) \supseteq \cdots$ is eventually constant. Since X is homotopically Hausdorff, $\bigcap_{i \in \mathbb{N}} G_i = \{1_L\}$. Thus, $G_i = \{1_L\}$ for some $i \in \mathbb{N}$. Now, every $x \in X$ has an open neighborhood U_x of x such that the natural map from $\pi(U_x)$ to $\pi(X)$ induced by inclusion is trivial, in other words X is *semilocally simply connected.* We recall [M] that a locally path connected space is semilocally simply connected if and only if it has a universal cover.

Now if X has a universal cover, X must be semilocally simply connected and so by choosing $f : \pi(X, x) \longrightarrow L = \pi(X, x)$ to be the identity map we may apply Lemma 7.6 we get that $\pi(X)$ is countable. ☐

Corollary 5.7. *If X is a compact, connected, locally path connected , metric space which is homotopically Hausdorff then the following are equivalent:*

1. *$\pi(X)$ is countable*
2. *$\pi(X)$ is finitely generated*
3. *$\pi(X)$ is finitely presented*
4. *X has a universal cover.*

Proof. The proof follows from the proof of the above lemma except that one applies Lemma 7.7 in this case to learn that $\pi(X)$ is finitely presented. ☐

The following is a restatement of Lemma 7.7, but we include it here for completeness.

Corollary 5.8. *Any compact, connected, locally path connected metric space which has a universal cover (e.g. a compact manifold) has a finitely presented fundamental group.*

Theorem 5.9. *If X is a second countable, connected, locally path connected , one–dimensional metric space then the following are equivalent.*

1. *$\pi(X)$ is free*
2. *$\pi(X)$ is countable*
3. *X is locally simply connected.*
4. *X has a universal cover.*

Proof. (1) $\Rightarrow$ (2) by Theorem 5.1. (2) $\Leftrightarrow$ (3) follows from Lemma 5.6. (3) $\Leftrightarrow$ (4) follows from Corollary 5.4. Finally (4) $\Rightarrow$ (1) follows from Lemma 7.3. ☐

Example 5.9.1. The fundamental groups of the Hawaiian earring, the Sierpinski curve, and the Menger curve are uncountable and not free since these spaces are not locally simply connected.

Theorem 5.10. *If X is a path connected, locally path connected, homotopically Hausdorff metric space then the normal covering spaces of X are in one–to–one correspondence with the countable factor groups of $\pi(X)$. Furthermore, if X is compact then the normal covering spaces of X are in one–to–one correspondence with the finitely generated factor groups of $\pi(X)$.*

Proof. Suppose K is normal subgroup of $\pi(X)$ having countable index. As in the above proofs we use Theorem 4.4 to construct a cover of X so that each point $x \in X$ is contained in an element, U_x, of the cover so that the inclusion map $i : \pi(U_x, x) \longrightarrow \pi(X, x)/K$ is trivial. Applying Lemma 7.8, we obtain that X admits a normal covering space $(\widetilde{X}, p)$ such that $p_*(\pi(\widetilde{X})) = K$.

Now suppose that X admits a normal covering space $(\widetilde{X}, p)$ such that $p_*(\pi(\widetilde{X})) = K$. Then by Lemma 7.8, X is *locally trivial* (see the definition of locally trivial preceding Lemma 7.6) with respect to $\pi(X)/K$. By Lemma 7.6, $\pi(X)/K$ is countable, and is finitely generated in the case that X is compact. □

5.2. **Fundamental groups of one–dimensional spaces are locally free and residually finite.**

The goal of this section is to prove the following theorem.

Theorem 5.11. *Let X be a compact, one-dimensional, connected metric space, and $x_0 \in X$. Then $\pi_1(X, x_0)$ embeds in an inverse limit of finitely generated free groups and thus is locally free, residually free, and residually finite.*

This result also follows from Corollary 3.5 and results in [CF3] since, in that paper, Curtis and Fort prove that the fundamental group of the Menger Curve can be embedded in an inverse limit of free groups. Recently, Eda and Kawamura have offered another proof of this result in [EK]. We include our proof because of its relative shortness and because we hope that similar constructions will lead to other results.

Proof of Theorem 5.11. By standard dimension theory, X embeds in three–dimensional Euclidean space as the intersection of handlebodies $H_1 \supset H_2 \supset \cdots \supset H_n \supset \cdots \cap H_i = X$, where the handlebodies H_n have the following properties:

(1) Each H_n has a handle decomposition into finitely many disjoint 0–handles joined by finitely many disjoint 1-handles, each 0- and

1-handle having diameter $< 1/n$. Each 1-handle h is attached to its adjacent 0-handles along disks $D_{-1}(h)$ and $D_{+1}(h)$ called its *attaching disks*, and $D_{-1}(h)$ and $D_{+1}(h)$ are separated by an intermediate spanning disk $D_0(h)$ called its *belt disk*. See figure Figure 3.

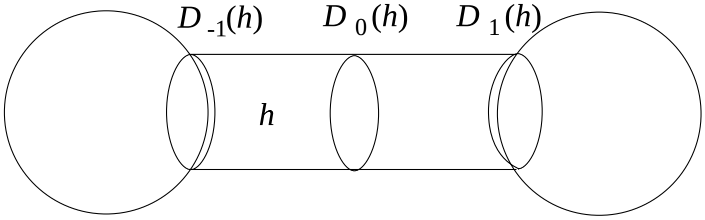

FIGURE 3. Attaching Disks

(2) Each H_{n+k} lies in the interior of H_n, and if a handle h' of H_{n+k} intersects the belt disk $D_0(h)$ of a 1-handle h of H_n, then h' is a 1–handle of H_{n+k} and $h' \cap D_0(h) = D_0(h')$. See figure Figure 4.

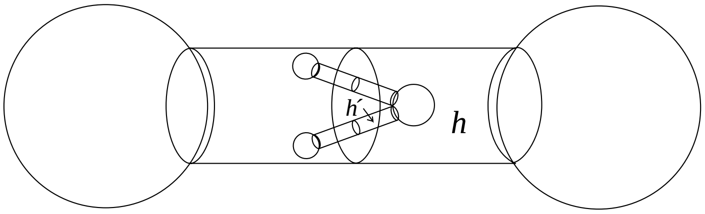

FIGURE 4. Belt disks contain only belt disks

We have an obvious commutative diagram of groups and homomorphisms induced by inclusions:

$$\begin{tikzcd}
 & \pi_1(X,x_0) \arrow[dl, "\phi_1"] \arrow[d, "\phi_2 \cdots"] \arrow[drr, "\phi_n"] & & & \\
\pi_1(H_1,x_0) & \pi_1(H_2,x_0) \arrow[l] & \arrow[l] & \pi_1(H_n,x_0) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Hence there is an induced homomorphism,

$$\phi : \pi_1(X,x_0) \to [\lim_{\leftarrow} \pi_1(H_n,x_0)].$$

Our task is to show that ϕ has trivial kernel. That is, if $f : S^1 \to X$ is a loop in X based at x_0, and if $\phi_n([f]) = 1$ for each n (that is, f is nullhomotopic in each H_n), then we must show that f is nullhomotopic in X. The difficulty is, of course, that the hypothesized shrinkings of f in the various H_n need not be closely related.

We take a first step toward making the problem combinatorial by fixing n and choosing disjoint arcs $A_1(n)$, ..., $A_{k(n)}(n)$ in S^1 having the following properties.

(3) Each arc $A_i(n)$ has image $f(A_i(n))$ which lies in the interior of a 1–handle of H_n.

(4) The inverse in S^1 of the union of the belt disks of H_n lies in the union of the interiors of the arcs $A_i(n)$.

As the second step, we take a disk $F : B^2 \to H_n$ bounded by f and move F slightly near the belt disks of H_n so that F is in general position with respect to those belt disks. The inverse image under F of the union of the belt disks of H_n will then be a family $B_1(F)$, ..., $B_{\ell(F)}(F)$ of disjoint spanning arcs of B^2 joining various points of the union $A_1(n) \cup \cdots \cup A_{k(n)}(n)$.

There are a number of important properties satisfied by the arcs $B_1(F), \ldots, B_{\ell(F)}(F)$.

The first involves the flexibility with which F can be chosen: we can require that $F(B^2)$ lie in any given fixed H_{n+k}. Hence we have

(5) smallness of image: If we require, as we may, that $F(B^2) \subset H_{n+k}$, then each individual arc $B_i(F)$ has image $F(B_i(F))$ which lies in a single belt disk of H_{n+k}, hence has image of diameter less than $1/(n+k)$.

The second important property is a separation property in B^2.

(6) separating distant points by individual arcs: Let $x, y \in S^1$ be such that $f(x)$ and $f(y)$ are far enough apart in H_n to be separated from one another by the 1-handles of H_n containing neither $f(x)$ nor $f(y)$. Then there is an arc $B_i(F)$ whose image lies in a 1–handle containing neither $f(x)$ nor $f(y)$ such that $B_i(F)$ separates x from y in B^2.

Proof of (6). Let B be the union of those arcs $B_i(F)$ whose images lie in a 1-handle missing $f(x)$ and $f(y)$. If A is any arc in B^2 from x to y, then by hypothesis, $F(A)$ must traverse some 1–handle h which misses $f(x)$ and $f(y)$ from attaching disk to attaching disk. Hence $F(A)$ hits $D_0(h)$, and therefore $A \cap B \neq \emptyset$. We conclude that B separates x from y. Hence some $B_i(F)$ in B separates x from y since the simply connected space B^2 is unicoherent. □

The final important property is a stability condition.

(7) stability of separation: Let $x, y \in S^1$, and suppose that the arc $B_i(F)$ separates x from y in B^2 and its image $F(B_i(F))$ lies in

a 1–handle h of H_n which contains neither x nor y. Then $B_i(F)$ joins two distinct arcs $A_{j_1}(n)$ and $A_{j_2}(n)$. Furthermore any line segment in B^2 joining $A_{j_1}(n)$ to $A_{j_2}(n)$ separates x and y in B^2.

Proof of (7). An arc $A_j(n)$ containing an endpoint of $B_i(F)$ must map into the 1–handle h. Hence x and y miss each such $A_j(n)$. But no segment joining $A_j(n)$ to itself can separate two points of $S^1 \setminus A_j(n)$, hence $B_i(F)$ joins distinct arcs $A_{j_1}(n)$ and $A_{j_2}(n)$. But any two segments joining $A_{j_1}(n)$ and $A_{j_2}(n)$ separate the same points of $S^1 \setminus (A_{j_1}(n) \cup A_{j_2}(n))$ in B^2. ☐

We use properties (5), (6), and (7) to establish the following additional property.

(8) minimal collections: We call a minimal subcollection $C(F)$ of $B_1(F), \dots, B_{\ell(F)}(F)$ satisfying (5) and (6) a *minimal cancellation* in H_n. Minimal cancellations clearly exist and may be characterized as follows. Consider any subcollection $C(F)$ of $B_1(F), \dots, B_{\ell(F)}(F)$ such that each element of $C(F)$ joins different $A_j(n)$'s and such that, if there is an arc $B_i(F)$ joining two different arcs $A_{j_1}(n)$ and $A_{j_2}(n)$, then there is exactly one such arc in $C(F)$. Then the subcollection $C(F)$ still satisfies (5) and (6).

Proof of (8). If $B_1(F), \dots, B_{\ell(F)}(F)$ satisfies (5), then so does any subcollection $C(F)$. By (7), separation properties of (6) only depend on saving at least one arc joining each of the pairs joined by

$$B_1(F), \dots, B_{\ell(F)}(F).$$

☐

(9) finiteness condition on minimal cancellations: Call two minimal cancellations equivalent if one can be obtained from the other by moving endpoints within the arcs $A_j(n)$. It follows immediately from (7) and (8) that there are, up to equivalence, only finitely many distinct minimal cancellations in H_n.

The proof of the embedding theorem is complete given the following two lemmas. ☐

Lemma 5.12. *There is a countable collection B_1, B_2, … of spanning arcs in B^2 possibly sharing endpoints but having disjoint interiors which satisfy the following conditions:*

$(5)'$ *Each of the sets $f(\partial B_i)$ is a single point.*

$(6)'$ *If $x, y \in S^1$ have distinct images $f(x) \neq f(y)$, then there is an arc B_i which separates x from y in S^1 such that $f(x) \neq f(\partial B_i) \neq f(y)$.*

The collection $\{B_i\}$ *is infinite if the map* $f : S^1 \to X$ *is not constant.*

Proof. Each of the shrinkings $F_n : B^2 \to H_n$ gives rise to a minimal cancellation in H_1. Since there are only finitely many equivalence classes of minimal cancellations in H_1, some equivalence class occurs infinitely often. Passing to a subsequence if necessary, we may assume that all give rise to the same equivalence class. Iterating, we find that we may assume that for each n and for each k, F_{n+k} gives rise to the same equivalence class of minimal cancellations in H_n. That is, we may choose minimal cancellations $C(F_n)$ in H_n such that each collection $C(F_{n+k})$ contains a minimal cancellation in H_n equivalent to $C(F_n)$. Passing to further subsequences if necessary, we may assume that, for each n, the equivalent minimal cancellations for H_n from the collections $C(F_{n+k})$ converge geometrically to a cancellation $C_\infty(n)$ for H_n where the arcs of $C_\infty(n)$ may share endpoints but have disjoint interiors. By (5), each arc of $C_\infty(n)$ has endpoints which map to the same point of X, whence $(5)'$. Again by (9) we may assume that the limits $C_\infty(1)$, $C_\infty(2)$, $\dots$ fit together to form $\{B_1, B_2, \dots\}$ so that arcs may share endpoints but not interior points. Condition $(6)'$ follows from (6) and (7). $\square$

Lemma 5.13. *Let* $\{B_i\}$ *be the collection whose existence is promised by Lemma 5.12. Define two points* x *and* y *of* S^1 *to be equivalent if there is no arc* B_i *which separates* x *from* y *in* B^2 *such that* $f(x) \neq f(\partial B_i) \neq f(y)$. *Let* G *be the set of equivalence classes. Then* G *respects* f, *is upper semicontinuous, is noncrossing, and is filling. Hence* f *factors through the contractible planar continuum* S^1/G *and is nullhomotopic.*

Proof. The relation is an equivalence relation. Indeed, it is clearly reflexive and symmetric. We see transitivity as follows. Suppose x is equivalent to y and y to z. Then by $(6)'$, $f(x) = f(y) = f(z)$. Suppose that there is an arc B_i with $f(\partial B_i) \neq f(x)$ which separates x from z. Then y cannot be a point of B_i. Consequently A separates one of the pairs $\{x, y\}$ and $\{y, z\}$, which contradicts either the equivalence of x and y or the equivalence of y and z. We conclude that x is equivalent to z and that the relation is transitive, hence an equivalence relation.

The decomposition G clearly respects f by $(6)'$.

In order to show that G is upper semicontinuous, it is enough to show that the projection map $\pi : G \to S^1/G$ is a closed map. Therefore let C be a closed subset of S^1. We need to show that $U = \pi^{-1}((S^1/G)\setminus\pi(C))$ is open. If U is not open, then there is a sequence $x_1, x_2, \dots$ of elements of $S^1 \setminus U$ which converges to a point x of U. Since the points x_i are not in U, their corresponding elements g_i of G intersect C, say in points y_i of C. We may assume that y_i converges to y in C since C is compact. Since $x_i \to x$ and $y_i \to y$, and since x_i and y_i both lie in g_i, we must have $f(x_i) = f(y_i)$, and consequently $f(x) = f(y)$. Since $x \in U$ and U is π-saturated, the entire element g containing x is in U. Since $y \in C$, which misses U, x and y are not equivalent. Hence there is an arc B_j which separates x and y such that $f(x) \neq f(\partial B_j) \neq f(y)$. But, for large i, x_i will be so close to x and y_i will be so near to y that B_j will separate x_i from y_i in B^2 and $f(x_i) \neq f(\partial B_j) \neq f(y_i)$ so that x_i and y_i are not equivalent, a contradiction. We conclude that U is open and that G is upper semicontinuous.

We see that G is noncrossing as follows. Suppose that, to the contrary, there are two elements $g_1, g_2 \in G$, points $x_1, y_1 \in g_1$, and points $x_2, y_2 \in g_2$ such that, in angular coordinates on S^1, $x_1 < x_2 < y_1 < y_2$ in $[0, 2\pi]$. Since g_1 and g_2 are distinct elements of G, there is an arc B_i which separates x_1 and x_2 in B^2 such that $f(x_1) \neq f(\partial B_i) \neq f(x_2)$. Since x_1 and y_1 are equivalent, B_i cannot separate them in B^2. Hence the endpoints of B_i must be between x_1 and x_2 and between x_2 and y_1. Hence B_i separates x_2 and y_2, a contradiction. We conclude that G is noncrossing.

We see that G is filling as follows. If g_1 and g_2 are distinct, and if $x_1 \in g_1$ and $x_2 \in g_2$, then there is an arc B_i separating x_1 and x_2 in B^2 such that $f(x_1) \neq f(\partial B_i) \neq f(x_2)$. Note that all points of g_1 are on the same side of B_i as x_1. Similarly, all points of g_2 are on the same side of B_i as x_2. The set ∂B_i must lie in a single element g of G. The element g clearly separates g_1 from g_2. Hence G is filling. $\square$

6. Abelianization in groups with Infinite Multiplication

6.1. Relating the Abelianizations of Free Groups and One–Dimensional Groups.

We have previously shown, [CC1], that the commutator subgroup of the Hawaiian earring groups naturally generalize in structure the commutator subgroup of the finitely generated free groups. We shall see in this section that the commutator subgroup of the fundamental group of a one–dimensional Hausdorff space can be similarly analyzed.

Theorem 6.1. *Suppose that X is a one–dimensional Hausdorff space and that $f : S^1 \to X$ is a reduced loop. Then f represents an element of the commutator subgroup of the fundamental group of X if and only if there are finitely many reduced paths $f_1, \dots , f_k$ in X having the following properties:*

(1) The loop f is the exact concatenation $f_1 \cdots f_k$ of $f_1, \dots , f_k$.

(2) The elements $f_1, \dots , f_k$ admit a pairing $f_i \leftrightarrow f_i^*$ such that f_i^* is the precise reverse of f_i (after possible reparametrization).

Remark 6.1.1. Note that the commutator theorem for finitely generated free groups involved cancelling letters, for the Hawaiian earring groups involved cancelling words, and for the general one–dimensional groups involves cancelling paths which are not necessarily loops. We conclude that the *groupoid* structure of the general one–dimensional fundamental groups is visible within the fundamental group itself.

Proof. It is a simple and standard exercise to prove that f represents an element of the commutator subgroup if the paths $f_1, \dots , f_k$ exist. We shall establish the more difficult converse. Assume therefore that f represents an element of the commutator subgroup. We find the paths $f_1, \dots , f_k$ as follows.

Let $[a_1, b_1] \cdots [a_\ell, b_\ell]$ be a finite product of commutators, with each of the loops a_i and b_i reduced, such that the element of the fundamental group represented by $f \cdot [a_1, b_1] \cdots [a_\ell, b_\ell] : S^1 \to X$ is trivial. Let G be a complete cancellation decomposition of S^1 which demonstrates that triviality. Form the associated cellular upper semicontinuous decomposition $H(G) = \{H(g) \mid g \in G\}$ of B^2 whose elements are the convex hulls $H(g)$ of g in B^2.

Lemma 6.2. *All but finitely many elements of G contain exactly two points of S^1.*

Proof. We first claim that no element of G can contain two points x and y on some one of the domain arcs of f, a_i, a_i^{-1}, b_i, or b_i^{-1}. For otherwise, G restricted to the interval $[x, y]$ defines a complete cancellation decomposition which shows that the appropriate one of the loops f, a_i, a_i^{-1}, b_i, and b_i^{-1} is reducible, a contradiction.

If $g \in G$ contains exactly one point, then we claim that g is in fact an endpoint of one of the domain arcs for f, one of the a_i or a_i^{-1}, or one of the b_i or b_i^{-1}. If not, then there is by upper semicontinuity of $H(G)$ an element $g' \in G$ which contains a point x' near g on one side of g and a point y' near g on the other side of g, and this contradicts the result of the preceding paragraph. We conclude that only finitely many elements of G are singletons.

If infinitely many elements of G had more than two points, then, by the first paragraph of this proof, some two of them would have three distinct points each in the same three of the domain intervals for f, a_i, a_i^{-1}, b_i, and b_i^{-1}. But it would follow that the two convex hulls would intersect, which contradicts the noncrossing nature of $H(G)$. Hence only finitely many elements of G have more than two points. The proof of this lemma is complete. ☐

Continuation of the proof of Theorem 6.1.

Attach strips S_i (respectively, T_i) to B^2 along the arcs corresponding to a_i and a_i^{-1} (respectively, b_i and b_i^{-1}) in such a way as to realize the parameterizations which show that a_i and a_i^{-1} are precise inverses (reverses) of one another. See figure Figure 5. One obtains thereby a punctured orientable surface M_ℓ of genus ℓ.

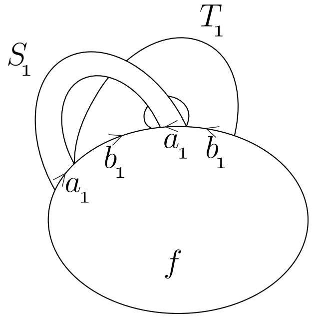

FIGURE 5. The punctured surface $M_\ell = B^2 \cup (\cup_i S_i) \cup (\cup_i T_i)$.

Realize the cancellations of a_i with a_i^{-1} geometrically by a family of parallel arcs filling S_i and joining corresponding points of the domain of a_i and the domain of a_i^{-1}. See figure Figure 6. Do the corresponding construction in the strips T_i.

Join the arcs of the previous paragraph to the decomposition elements $H(g)$ of $H(G)$ to create a new decomposition G' of the punctured

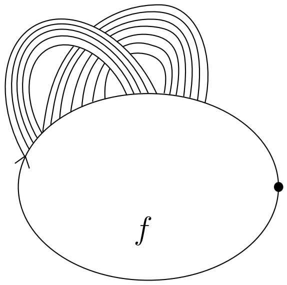

FIGURE 6. Cancellation arcs for a_i and a_i^{-1}, for b_i and b_i^{-1}.

surface M_ℓ. See figure Figure 7. Thus G' is the smallest equivalence relation on M_ℓ which declares two points to be equivalent if they lie in the same element of $H(G)$ or on one of the cancellation arcs running through one of the strips S_i or T_i.

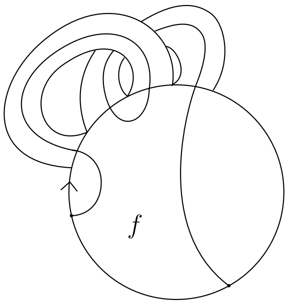

FIGURE 7. The decomposition G' for M_ℓ.

Lemma 6.3. *Let $[x, y]$ denote the domain arc for f. Let $x_0 = x < x_1 < x_2 < \cdots < x_k = y$ be the points of $[x, y]$ which lie in the finitely many elements of G' which contain an element of G with contain either:*

(1) *only one point of S^1*
or
(2) *more than two points of S^1*
or
(3) *an endpoint of a domain arc of f, a_i, a_i^{-1}, b_i, or b_i^{-1}.*

Let $f_i = f \mid [x_{i-1}, x_i]$. Then the subpaths f_i satisfy the conclusions of the theorem.

Proof. Consider one of the open intervals (x_{i-1}, x_i). The elements of G' which begin in (x_{i-1}, x_i) can be traced across B^2 by means of the elements of $H(G)$ which have an endpoint in (x_{i-1}, x_i); each of these elements is an arc, and together these arcs form a product of (x_{i-1}, x_i) with an arc. The other endpoints must lie in one of the domain arcs of a_i, a_i^{-1}, b_i, or b_i^{-1}, hence can be traced further through one of the strips S_i or T_i. This tracing process can be continued either indefinitely or until one returns to the domain arc of f. If one returns to the domain arc of f, then one obtains an exact cancellation of reverses as required by the theorem. Thus it suffices to show that the tracing cannot be continued infinitely many steps.

If the tracing could be continued through infinitely many strips, then one would cross one of the domain arcs, say a_i, infinitely often. Each time one would cross, one would cross at a subinterval A_j of the domain arc of a_i, the subintervals A_j being disjoint. Choose one point p_j from each of the intervals A_j, and let p be a limit point in the domain arc of a_i for the points p_j. Then each element of G' which began in (x_{i-1}, x_i) would cluster at p. Hence f would, by continuity, map each point of (x_{i-1}, x_i) to $f(p)$. But the reduced path f is constant on no subinterval, a contradiction. We conclude that the tracing cannot be continued infinitely many steps. □

This completes the proof of the theorem. □

6.2. **Divisibility and Strong Abelianization in the the Fundamental Group of a Topological Space.**

Remark 6.3.1. Suppose $G = \pi(X, x_0)$ is the fundamental group of the topological space X based at the point x_0. It is not difficult to prove that if g is an element of G then g is in the commutator subgroup G' of G if and only if g has a representative $p : I \longrightarrow X$ so that the unit interval $I = [0, 1]$ can be partitioned into finitely many closed intervals I_j, $j = 1 \ldots k$ so that

1. For each j, $p|I_j$ is a closed curve based at x_0.
2. There is a permutation σ so that the product $f|I_{\sigma(1)} \cdot f|I_{\sigma(2)} \cdots f|I_{\sigma(k)}$ is nullhomotopic.

Definition 6.4 (Big commutator subgroup)**.** Let $G = \pi(X, x_0)$ be the fundamental group of the topological space X based at the point x_0. We say that an element g of G is in the *big commutator subgroup, $\widehat{G}$, of G* if and only if there is representative $p : I \longrightarrow X$ for g and a representative $t : I \longrightarrow X$ for 1_G so that the unit interval $I = [0, 1]$ can be partitioned two ways into closed intervals, $I_\gamma, \gamma \in \Gamma$ and $J_\gamma, \gamma \in \Gamma$ so that

1. For each $\gamma \in \Gamma$, I_γ and J_γ have the same length (i.e., are translates of one another).

2. For each $\gamma \in \Gamma$, $p|I_\gamma$ and $t|J_\gamma$ are closed paths based at x_0 which are equal (up to translation of their domains).

Remark 6.4.1. Clearly big commutator subgroups enjoy a number of the elementary group theoretic properties of commutator subgroups. For instance if $H_1 < H_2$ then $\overline{H_1} < \overline{H_2}$

Definition 6.5 (abelianization)**.** As is standard, if H is a group we call the group H/H' the *abelianization* of H and denote it by H^{ab}.

Definition 6.6 (strong abelianization)**.** Correspondingly, if H is a subgroup of G, we define the *strong abelianization,* $\widehat{H}$*,* of H to be $H/\overline{H}$.

Notation 6.6.1. Motivated by the fact that the elements of a group H which are perfect n-th powers generate the subgroup denoted as H^n, we will denote the group generated by those elements which can be written as arbitrarily large powers of elements of H(the infinitely divisible elements of H) as H^∞. We denote the quotient H/H^∞ by $\mathcal{D}(\mathcal{H})$.

Theorem 6.7. *If* $G = \pi(X, x_0)$ *is the fundamental group of the first countable topological space* X*, then the twice iterated quotient*

$$\mathcal{D}(\mathcal{D}(G/G')) = \mathcal{D}^2(G^{\mathrm{ab}})$$

is a quotient group strong Abelianization, $\widehat{G}$*.*

Remark 6.7.1. In [CC1] it is shown that the the abelianization of the Hawaiian earring group does not embed in an inverse limit of free abelian groups, but that the strong abelianization of the Hawaiian Earring group is the countable product of copies of the integers, $\prod_{\mathbb{N}} \mathbb{Z}$ (which obviously embeds in an inverse limit of free abelian groups), and that $\mathcal{D}^2(G^{\mathrm{ab}})$ is the strong abelianization of the Hawaiian earring group G.

This leads us to the following

Conjecture 6.7.2. If X is a compact, separable, connected, locally path connected, one–dimensional metric space with fundamental group G, then $\widehat{G}$ embeds in an inverse limit of free abelian groups and $\widehat{G} = \mathcal{D}^2(G^{\mathrm{ab}})$.

Proof of the theorem. Choose a basepoint x_0 for X, let $U_1 \supseteq U_2 \supseteq \cdots$ be a countable local basis for X at x_0, and G_i be the image of the natural map of $\pi(U_i, x_0)$ into $\pi(X, x_0)$.

Choose D to be the subgroup of G containing G' so that

$$D/G' = (G/G')^\infty.$$

Thus,

$$G/D = \frac{G/G'}{D/G'} = \mathcal{D}(G/G').$$

Choose E to be the subgroup of G containing D so that

$$E/D = (G/D)^\infty.$$

Now,

$$G/E = \frac{G/D}{E/D} = \frac{\mathcal{D}(G/G')}{(G/D)^\infty} = \frac{\mathcal{D}(G/G')}{\mathcal{D}(G/G')^\infty} = \mathcal{D}(\mathcal{D}(G/G')) = \mathcal{D}^2(G^{\mathrm{ab}}).$$

Thus we need only show that $E \supseteq \overline{G}$.

Suppose $b \in \overline{G}$. Let $f : G \longrightarrow E$ be the natural map, and let $a = f(g)$. Let p, t, Γ, I_γ, and J_γ be as in the definition of b being an element $\overline{G}$. By Lemma 4.2, all but finitely many of the $p|I_\gamma$ lie in G_i, for any $i \in \mathbb{N}$. Since t is nullhomotopic, we have that $b = [f][t]^{-1}$, and consequently that $a = [f][t]^{-1}E$. Since E is abelian, for any $i \in \mathbb{N}$ we can apply finitely many commutators to $a = [f][t]^{-1}E$ to cancel those finitely many $p|I_\gamma$ which do not lie in G_i with the corresponding $t|{J_\gamma}^{-1}$. This yields an element of the form $a = gE$, where g is an element of G_i. Thus we have that a lies in $f(G_i)$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ choose $g_i \in G_i \cap f^{-1}(a)$.

By Lemma 4.2, for each $n, i \in \mathbb{N}$ we may choose elements $w_{n,i}$ of G_i which satisfy the following conditions,

$$\begin{aligned} w_{n,1} &= g_1 w_{n,2}^n \\ &\vdots \\ w_{n,i} &= g_i w_{n,i+1}^n \\ &\vdots \end{aligned}$$

Now,

$$\begin{aligned}
f(w_{n,1}) &= f(g_1) + n \cdot f(w_{n,2}) = a + n \cdot (a + n \cdot f(w_{n,3})) \\
&= a + na + [n^2 \cdot f(w_{n,3})] \\
&\vdots \\
&= a + na + n^2 a + \cdots + n^{k-1} a + n^k \cdot f(w_{n,k+1}) \\
&= a\left(\frac{n^k - 1}{n-1}\right) + [n^k \cdot f(w_{n,k+1})],\ \forall\, n, k.
\end{aligned}$$

Rearranging terms we obtain

$$\begin{aligned}
(n-1) \cdot f(w_{n,1}) + a &= an^k + (n-1)n^k \cdot f(w_{n,k+1}) \\
&= n^k(a + (n-1) \cdot f(w_{n,k+1})),\ \forall\, n, k.
\end{aligned}$$

Since k can be chosen freely, it follows for all n that

$$(n-1) \cdot f(w_{n,1}) + a$$

is an infinitely divisible element of G/G' and thus is contained in D/G'. Therefore

$$aD = [(1-n) \cdot f(w_{n,1})]D = (1-n) \cdot (f(w_{n,1})D)$$

for all n. It follows that aD is an infinitely divisible element of G/D and thus lies in E/D. Consequently $a \in E/G'$ and so $b \in E$. Whence $E \supseteq \overline{G}$ and the theorem is proven. □

Theorem 6.8. *Suppose X is a space which is connected, locally path connected, separable, and metric. Let G denote the fundamental group $\pi(X, x_0)$ of X. Then there is a countable subgroup A, of G^{ab} such that $\mathcal{D}^2(G^{\mathrm{ab}}/A)$ is the trivial group.*

Remark 6.8.1. It is an easy exercise to prove that a group which is uncountable and satisfies the conditions of the theorem cannot be free.

Proof. We need to find subgroups C and D of G such that $D > C > G'$ satisfying the following conditions:

1. C/G' is countable;
2. every element of D/C is infinitely divisible in G/C;
3. every element of G/D is infinitely divisible.

We first construct the subgroup C. Let $c_1, c_2, \cdots : S^1 \to X$ denote a countable dense set of loops in X. (That is, the set $C(S^1, X)$ of all continuous functions from S^1 into X can be metrized by using the sup metric; this space is second countable since X is second countable, hence $C(S^1, X)$ has a countable dense subset $c_1, c_2, \cdots : S^1 \to X$.) Although the loops c_i are not based at the base point x_0 of X, they represent well-defined elements of the homology group $H_1(X) = G/G'$. Let C_0 denote the countable subgroup of $H_1(X)$ generated by these elements, and let C denote the preimage of C_0 in G. Then $C/G' = C_0$ is countable, and (1) is satisfied.

We next define D as the collection of all elements g of G whose images gC in G/C are infinitely divisible in G/C. Thus (2) is satisfied by definition.

It remains only to prove that (3) every element of G/D is infinitely divisible. To that end we consider an arbitrary loop $f : S^1 \to X$ based at x_0. We must show that $[f]D$ is infinitely divisible in G/D. Our construction will involve a sequence $f_1, f_2, \cdots : S^1 \to X$ of loops from $c_1, c_2, \cdots : S^1 \to X$ converging uniformly to f and auxiliary paths or loops $f(j,k)$, $f_k(j,k)$, $A(j,k)$, and $R(j,k)$ defined below, which we shall put together in *infinite* concatenations which form *finite* singular 1–cycles.

We first make a few general remarks about one–dimensional singular homology. Every 1-cycle is represented by a finite collection of closed curves which may be divided in any finite way into singular 1-simplexes. Into any 1-cycle, one may insert any null sequence of loops and still have a (finite) 1-cycle. Other more complicated insertions and even sequences of insertions may be made as well. Continuity is the only issue. Thus 1-cycles can be created as infinite concatenations. Such infinite concatenations can be manipulated essentially algebraically provided that one respects the following rather obvious finiteness fact: two singular 1-cycles represent the same element of $H_1(X) = G/G'$ if one can be obtained from the other by a finite reordering of a finite number of subpaths.

Let $f_1, f_2, \cdots : S^1 \to X$ be loops from the sequence $c_1, c_2, \cdots : S^1 \to X$ which converge uniformly to f. Define the following auxiliary paths and loops, for all $k \geq 1$ and for all j in the range $1, \ldots, 2^k$. We use the linear parameterization $[0,1]$ for S^1.

4. $f(j,k)$ is the restriction of f to the interval $[(j-1)/2^k, j/2^k]$.
5. $f_k(j,k)$ is the restriction of f_k to the same interval.
6. $A(j,k)$ is a path in X joining $f(j/2^k)$ to $f_k(j/2^k)$, with $\operatorname{diam}(A(j,k))$ no larger than twice the infimum of all diameters of paths with the same endpoints.
7. $R(j,k)$ is the loop which is the concatenation of the paths $f(j,k)$, $A(j,k)$, $f_k(j,k)^{-1}$, and $A(j-1,k)^{-1}$.

Note the following obvious facts.

8. $f(j,k)$ is the concatenation of the paths $f(2j-1,k+1)$ and $f(2j,k+1)$.
9. The 1-cycle $f - f_k$ is homologous to the sum $R(1,k) + R(2,k) + \cdots + R(2^k,k)$ of 1-cycles, since all of the symbols $A(j,k)^{\pm 1}$ cancel in pairs.

Notation 6.8.1. We shall make considerable abuse of notation, often confounding a cycle with the homology element it represents. Furthermore, we may manipulate the representing cycles according to the finite rearrangement principle discussed above. We give here a number of examples.

In consonance with (9), we shall use the symbol $f - f_k$ to represent the sum $R(1,k) + R(2,k) + \cdots + R(2^k,k)$. Although $f - f_k$ is clearly represented by just two relatively large 1–cycles f and f_k, we usually prefer to think of it as represented by the 2^k relatively small cycles $R(j,k)$.

Let m be a positive integer. Then we may think of the symbol $m \cdot (f - f_k)$ as representing the sum of the small cycles $m \cdot R(j,k)$. The small cycle $m \cdot R(j,k)$ may be thought of as a single loop (the mth power of $R(j,k)$) or as a sum of m distinct copies of $R(j,k)$.

If the positive integer m factors into two smaller integers $m = m_1 m_2$, then we may think of $m(f - f_k)$ as the sum of m_1 copies of the loops $m_2 R(j,k)$, etc.

Our most important special use of notation involves an *infinite* sum of the form

$$m_1(f - f_1) + m_1 m_2 (f - f_2) + \cdots + (m_1 m_2 \cdots m_k)(f - f_k) + \cdots$$

where m_1, m_2, … are nonnegative integers. We shall show how to form a certain finite 1-chain h, which is somehow represented by this infinite sum, in fact is the continuous limit of finite 1–chains h_1, h_2, … representing the finite partial sums. The remainder or tail or error term of the approximation will always be a finite 1–cycle which is divisible by the leading coefficient of the remainder.

The approximations h_i are defined recursively as follows:

$$h_1 = m_1 R(1,1) + m_1 R(2,1),$$

which is clearly homologous to $m_1(f - f_1)$ by (9). To form h_{k+1} from h_k, one replaces each occurrence of the loop $R(j,k)$ in h_k by the loop

$$\begin{aligned}m_{k+1}R(2j-1,k+1)\cdot f(2j-1,k+1)&\\ \cdot\, m_{k+1}R(2j,k+1)\cdot f(2j,k+1)&\\ \cdot\, A(j,k)\cdot f_k(j,k)^{-1}\cdot A(j-1,k)^{-1},&\end{aligned}$$

which has the same endpoints as $R(j,k)$.

Observation 6.8.2. By (9), the interpolated loops really do add up to $(m_1m_2\cdots m_{k+1})\cdot(f-f_{k+1})$, so that, by (7) and (8), the partial sums are indeed represented by the elements we have constructed. If we consider each $m_{k+1}R(j',k+1)$ as a single loop, as we may, then we do not increase the number of loops in passing from h_k to h_{k+1}. Since we do everything symmetrically with respect to each copy of each $R(j,k)$, then any of the simple and admissible homology operations which manipulate the symbols representing h_k will respect the additions used in forming h_{k+1}. Since the interpolated loops get smaller and smaller as k increases, there is obviously a limit cycle h.

6.2.1. *The Construction.* Fix an arbitrary positive integer m_1. Define m_k recursively by the formula,

$$m_k = 1 + m_1 + (m_1m_2) + \cdots + (m_1m_2\cdots m_{k-1}).$$

Define a 1-cycle $g(m_1)$ by the infinite sum

$$g(m_1) = \sum_{k=1}^{\infty}(m_1m_2\cdots m_k)(f-f_k),$$

and note that $g(m_1)$ is divisible by m_1

$$g(m_1) = m_1[\sum_{k=1}^{\infty}(m_2\cdots m_k)(f-f_k)].$$

Then note that

$$\begin{aligned}f+g(m_1) &= f[1+m_1+(m_1m_2)+\cdots+(m_1m_2\cdots m_{\ell-1})]\\ &\quad -[m_1f_1+(m_1m_2)f_2+\cdots+(m_1m_2\cdots m_{\ell-1})f_{\ell-1}]\\ &\quad +\sum_{k=\ell}^{\infty}(m_1\cdot\dots\cdot m_k)(f-f_k)]\\ &= m_\ell[f+\sum_{k=\ell}^{\infty}(m_1\cdots\hat{m}_\ell\cdots m_k)(f-f_k)]\\ &\quad -[m_1f_1+(m_1m_2)f_2+\cdots+(m_1m_2\cdots m_{\ell-1})f_{\ell-1}].\end{aligned}$$

We now analyze this final formula. Notice that it has two grouped terms, the first divisible by m_ℓ, the latter being an element of C. Since m_ℓ approaches infinity as ℓ approaches infinity, we see that the element $f + g(m_1)$ is infinitely divisible in G/C. That is, $f + g(m_1)$ lies in D. Consequently, $f = -g(m_1)$ in G/D. But $g(m_1)$ is divisible by m_1, and m_1 can be chosen to be arbitrary positive. Thus f is infinitely divisible in G/D. This completes the proof of the theorem. $\square$

7. The relationships between the Fundamental Group of a Space and its Covering Nerves

Definition 7.1. We say that an open cover of a path connected space is 2-*set simple* if each element of the cover is path connected and any loop in the space that lies in the union of two elements of the cover is contractible in the space.

The following theorem appears in [Ca, page 9].

Theorem 7.2. *Suppose X is a space that is connected and locally path connected, suppose that X is 2-set simple, and suppose U is an open cover of X which is 2-set simple. Then the fundamental group of the nerve $N(U)$ of the open covering U is isomorphic with the fundamental group of X.*

Lemma 7.3. *If X is a one–dimensional metric space which is connected, locally path connected and semilocally simply-connected (or equivalently if X admits a universal covering space) then $\pi(X)$ is a free group.*

Proof. Corollary 5.4 states that any semilocally simply-connected one–dimensional space is, in fact, locally simply connected. Thus, for each $x \in X$ we may choose a simply connected open set U_x containing x Since X is metric, it is paracompact, so we may choose an open refinement C_1 of $\{U_x\}$ which is locally finite. Since C_1 is locally finite and X is metric, we can find a refinement C_2 of C_1 so that any two intersecting elements of C_2 are both contained in a single element of C_1. Finally, since X is one–dimensional, we may choose a locally finite refinement C of C_2 whose nerve, $\mathcal{N}$, is one–dimensional. Since C is locally finite, $\mathcal{N}$ is a graph. We now have a cover of X for so that the union of any two intersecting elements is simply connected. Under these hypotheses we may apply the preceding theorem of Cannon, Theorem 7.2, which implies that $\pi(X) = \pi(\mathcal{N})$. Thus $\pi(X)$ is free. $\square$

Lemma 7.4. *Suppose X is a path connected, Hausdorff, topological space, $f : \pi(X) \longrightarrow K$ is a homomorphism of groups, and C is a cover of X by path connected open sets so that any homotopy class which has a representative which lies in the union of two elements of C is in the kernel of f. Then if $\alpha, \beta : I \longrightarrow X$ are any two closed paths based at $x_0 \in X$ so that $\alpha(t)$ and $\beta(t)$ lie in a common element of C for all $t \in I$, then $f([\alpha]) = f([\beta])$. In particular if C is locally finite then $f(\pi(X))$ is finitely generated (resp. countable) if C is finite (resp. countable).*

Proof. Fix a basepoint x_0 in X. Since I is compact we can find a finite partition $\mathcal{P} = \{I_1, I_2, \ldots, I_n\}$, of I into closed intervals with disjoint interiors so that, for each $t \in \{1, 2, \ldots, n\}$, $\alpha(I_t)$ and $\beta(I_t)$ both lie entirely in one element of C, which we will denote by c_t. Now for each t, let $\alpha_t = \alpha\ |_{I_t}, \beta_t = \beta\ |_{I_t}$ and γ_t be a path in c_t which connects the terminal point of α_t to the terminal point of β_t, and where, for convenience, we choose β_0, γ_0 and γ_n all to be the constant map at x_0. Also define $\widehat{\beta_i} = \prod_{t=0}^{i} \beta_t$. It is clear that the since the path $\overline{\gamma_{i-1}}^{1}\alpha_i\gamma_i\overline{\beta_i}^{1}$ is closed and lies entirely in $c_{i-1} \cup c_i$, $[\widehat{\beta}_{i-1}\overline{\gamma_{i-1}}^{1}\alpha_i\gamma_i\overline{\beta_i}^{1}\overline{\widehat{\beta}_{i-1}}^{1}]$ is in the kernel of f for all $i \in \{1, 2, \ldots, n\}$. Now,

$$\begin{aligned}
1_K &= \prod_{i=1}^{n} f([\widehat{\beta}_{i-1}\gamma_{i-1}\alpha_i\gamma_i\overline{\beta_i}^{1}\overline{\widehat{\beta}_{i-1}}^{1}]) \\
&= \prod_{i=1}^{n} f([\widehat{\beta}_{i-1}\gamma_{i-1}\alpha_i\gamma_i\overline{\widehat{\beta}_i}^{1}]) \\
&= (\prod_{i=1}^{n} f([\gamma_{i-1}\alpha_i\gamma_i)])f([\overline{\widehat{\beta_n}}^{1}]) \\
&= (\prod_{i=1}^{n} \alpha_i))f([\gamma_n])f(\overline{[\beta]}^{1}) \\
&= f([\alpha])f(\overline{[\beta]}^{1}).
\end{aligned}$$

Now in the case where C locally is finite, for each $c_t \in C$ choose a point $y_t \in c_t$ (for simplicity choose one of these points to be x_0) and for every pair of intersecting elements c_t, c_w in C choose an embedded arc $A_{t,w}$ from y_t to y_w which lies entirely in $c_t \cup c_w$. Since C is locally finite, we may choose the $\{A_{t,w}\}$ so that if A_1 and A_2 are two distinct elements of $\{A_{t,w}\}$, then $A_1 \cap A_2$ is either empty, a closed arc containing precisely one endpoint of each of A_1 and A_2, or a point which is an endpoint of each of A_1 and A_2. The union Y, of the paths $A_{t,w}$ is an embedding of a graph, Γ, into X. Let $E : \Gamma \longrightarrow X$ denote this embedding and let K denote $\pi(\Gamma)$. It is evident that $f(E_*(K)) = f(\pi(X))$, since any closed path in X can be shadowed by a path in Y based at x_0. However we note that if C is countable (resp. finite) then Γ has only countably (resp. finitely) many edges. Since K is generated by the edges of Γ not in a maximal tree, we see that K (and thus $f(\pi(X))$) is countably (resp. finitely) generated. ☐

Definition 7.5. If X is a path connected topological space and $f : \pi(X) \longrightarrow K$ is a homomorphism of groups, so that every point $x \in X$ is contained in an open neighborhood U such that any homotopy class, based at x, which has a representative contained in U lies in the kernel of f, then we say that X is *locally trivial with respect to* f, or that X is *locally trivial with respect to* K

Lemma 7.6. *If* X *is a connected, locally path connected, second countable metric space which is locally trivial with respect to* $f : \pi(X) \longrightarrow K$ *then* $f(\pi(X))$ *is countable, and furthermore is finitely generated if* X *is compact.*

Proof. For each x choose an open set U_x containing x such that any homotopy class which has a representative contained in U_x lies in the kernel of f. Since X is metric, it is paracompact, so we may choose an open refinement C_1 of $\{U_x\}$ which is locally finite. Since C_1 is locally finite and X is metric, we can find a refinement C_2 of C_1 so that any two intersecting elements of C_2 are both contained in a single element of C_1. Finally, since X is paracompact and second countable, we may choose a countable, locally finite refinement C of C_2. In the case where X is compact we may choose C to be finite. Now we may apply the previous lemma to finish the proof.

□

Lemma 7.7. *If* X *is a compact, locally path connected, connected, metric space which is semilocally simply connected (i.e.* X *has a universal cover) then* $\pi(X)$ *is finitely presented.*

Proof. For each $x \in X$ we may choose an open neighborhood U_x of x, so that the inclusion $i : \pi(U_x, x) \longrightarrow \pi(X, x)$ is trivial. Let C_1 denote the cover of X consisting of the open sets U_x for $x \in X$. Since X is compact metric, we can find a finite refinement C_2 of C_1 so that any two intersecting elements of C_2 are both contained in a single element of C_1. We now have a cover of X for so that fundamental group of the union of any two intersecting elements has trivial image in $\pi(X)$. Under these hypotheses we may apply Theorem 7.2 , which says that $\pi(X) = \pi(\mathcal{N})$, where $\mathcal{N}$ is the *nerve* of the cover C_2. Since $\mathcal{N}$ is a finite simplicial complex, $\pi(X)$ is finitely presented. □

The following elementary homotopy theoretical lemma is stated without proof.

Lemma 7.8. *Suppose* X *is a connected, locally arcwise connected, Hausdorff topological space,* N *is a normal subgroup of* $\pi(X)$*, and* $f : \pi(X) \longrightarrow \pi(X)/N$ *is the natural map. Then* X *has a covering space* $(\widetilde{X}, p)$ *such that* $p_*(\pi(\widetilde{X})) = N$ *if and only if* X *is locally trivial with respect to* f*.*

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