EVEN GALOIS REPRESENTATIONS AND THE COHOMOLOGY OF \( GL(2, \mathbb{Z}) \)

AVNER ASH AND DARRIN DOUD

ABSTRACT. Let \( \rho \) be an even two-dimensional representation of the Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) which is induced from a character \( \chi \) of odd order of the absolute Galois group of a real quadratic field \( K \). After imposing some additional conditions on \( \chi \), we attach \( \rho \) to a Hecke eigenclass in the cohomology of \( GL(2, \mathbb{Z}) \) with coefficients in a certain infinite-dimensional vector space \( V \) over an arbitrary field of characteristic not equal to 2. The space \( V \) is defined purely algebraically starting from the field \( K \).

1. Introduction

One type of noncommutative reciprocity law connects a Galois representation (i.e. a continuous homomorphism \( \rho : G_{\mathbb{Q}} \to \text{GL}(n, F) \) for some field \( F \), where \( G_{\mathbb{Q}} \) is the absolute Galois group of \( \mathbb{Q} \)) with a system of eigenvalues of a Hecke algebra of some reductive \( \mathbb{Q} \)-group acting on an \( F \)-vector space \( V \). The connection consists of an equality, for almost all rational prime numbers \( \ell \), between the characteristic polynomial of the image of a Frobenius element at \( \ell \) under \( \rho \) and a “Hecke polynomial” constructed according to a simple recipe from the eigenvalues of the Hecke operators at \( \ell \). In such a case, it is standard terminology to say that \( \rho \) is “attached” to the system of Hecke operators, or to an eigenvector in \( V \) that supports the system.

We say that \( \rho \) is odd if the characteristic of \( F \) equals 2 or if the image of complex conjugation is conjugate to a diagonal matrix with alternating 1’s and −1’s on the diagonal. When \( \rho \) is odd, there are many profound theorems and conjectures concerning these reciprocity laws. For example, if \( n = 2 \), \( \rho \) is odd, and \( F \) is a finite field, Serre’s conjecture [17] (now a theorem of Khare and Wintenberger [12, 13] and Kisin[14]) states that \( \rho \) is attached to a holomorphic modular form that is an eigenform of the Hecke operators. Other papers, such as [3, 4, 11] conjecture the existence of analogous attachments for general values of \( n \), with modular forms replaced by elements of arithmetic cohomology groups. In all of these conjectures, \( \rho \) is odd.

Conversely, work of Scholze [16] proves that any eigenclass of the Hecke operators in the cohomology of a congruence subgroup \( \Gamma \) of \( \text{GL}(n, \mathbb{Z}) \) with coefficients in a finite-dimensional admissible module \( M \) over a field \( F \) has an attached Galois representation. For a field \( F \) of characteristic 0, this theorem was already proven in [10] by Harris, Lan, Taylor and Thorne. “Admissible” means that if \( F \) has characteristic 0 then \( M \) is an algebraic representation, and if \( F \) has positive characteristic, then the matrices used to define the Hecke operators act on \( M \) via reduction modulo some fixed integer.

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Caraiani and Le Hung [7] showed that for a Hecke eigenclass \( z \in H^*(\Gamma, M) \) for a congruence subgroup \( \Gamma \) of \( \text{GL}(n, \mathbb{Z}) \) and an admissible module \( M \), the representation guaranteed to exist by Scholze’s theorem is odd if the characteristic of \( \mathbb{F} \) is positive or if \( z \) is cuspidal.

It is natural then to ask about what kind of cohomological attachment to expect if \( \rho \) is not odd. Of course, any module \( M \) occurring in such a reciprocity law could not be admissible. One reason to suppose there may be some kind of theorem along these lines is that the work in [3] gives examples of odd 3-dimensional Galois representations each of which is a sum of an even 2-dimensional Galois representation \( \sigma \) and a character, and which appear to be attached to the cohomology of a congruence subgroup of \( \text{GL}(3, \mathbb{Z}) \) with admissible coefficients (over a finite field.) Some kind of cohomological attachment for \( \sigma \) may explain these phenomena.

We have no idea what may be the case for a general Galois representation. In this paper, we set \( n = 2 \), and say that \( \rho \) is even if it is not odd. Our main theorem asserts the attachment of certain even Galois representations to Hecke eigenclasses in \( H^*(\text{GL}(2, \mathbb{Z}), M) \) for some “natural” (infinite dimensional) module \( M \). We have to be careful with the exact definition of “attachment”, which we will explain, and then we can state the theorem.

Let \( f \) be a modular form of weight \( k \geq 0 \) on the upper half plane, with level \( \Gamma_1(N) \) and nebentype \( \theta \), and suppose that \( f \) is an eigenform for the Hecke operators \( T_\ell \) and \( T_{\ell, \ell} \) for all \( \ell \nmid N \). Denote the eigenvalue of \( T_\ell \) acting on \( v \) by \( a_\ell \), and the eigenvalue of \( T_{\ell, \ell} \) acting on \( v \) by \( A_\ell \). When \( k \geq 2 \), and \( f \) is holomorphic, there is a Galois representation \( \rho \) such that for all \( \ell \nmid N \),

\[
\det(I - \rho(\text{Frob}_\ell)X) = 1 - a_\ell X + \ell A_\ell X^2,
\]

where (in this case), \( A_\ell \) is easily seen to be equal to \( \ell^{k-2}\theta(\ell) \).

The cases when \( k = 0 \) or \( 1 \) are different. If \( k = 1 \) and \( f \) is holomorphic, or if \( k = 0 \) and \( f \) is a Maass form where the eigenvalue of the Laplacian is \( 1/4 \), there is an attached Galois representation (this is only conjectural in the Maass form case) with finite image. In both cases, the motivic weight of \( f \) is 0, and the characteristic polynomial of \( \text{Frob}_\ell \) equals

\[
1 - a_\ell X + \theta(\ell)X^2.
\]

These forms of the Hecke polynomials depend on the usual normalization of the Hecke operators. The correct definition of attachment to use in our situation is analogous to the case of a Maass form.

**Definition 1.1.** Let \( V \) be a Hecke module over the field \( \mathbb{F} \), occurring in the homology of \( \text{GL}(2, \mathbb{Z}) \) with a non-admissible coefficient module. Let \( v \in V \) be an eigenvector for the Hecke operators \( T_\ell \) and \( T_{\ell, \ell} \) for almost all primes. Let \( a_\ell \) be the eigenvalue of \( T_\ell \) acting on \( v \), and \( A_\ell \) the eigenvalue of \( T_{\ell, \ell} \) acting on \( v \). Let \( \rho : G_K \to \text{GL}(2, \mathbb{F}) \) be an even Galois representation. We say that \( \rho \) is attached to \( v \) if, for almost all \( \ell \),

\[
\det(I - \rho(\text{Frob}_\ell)X) = 1 - a_\ell X + A_\ell X^2.
\]

Our main theorem (Theorem 11.1) then takes the following form.

**Theorem.** Let \( K \) be a real quadratic field of discriminant \( d \), and let \( \mathbb{F} \) be a field of characteristic 0 or a finite field of odd characteristic. In the first case, set \( p = 1 \) and in the second case let \( p \) be the characteristic of \( \mathbb{F} \). Let \( \chi : G_K \to \mathbb{F}^\times \) be a character with finite image. Let \( L \) be the fixed field of the kernel of \( \chi \) and choose \( N \in \mathbb{Z} \) so
that $L/K$ is unramified outside primes of $K$ dividing $N$. Then we may consider $\chi$ as a character on the fractional ideals of $K$ relatively prime to $N$. Define the subgroup $K(M, q) \subseteq K^\times$ as in Definition 3.7. Assume that

(1) $\chi$ is trivial on the principal fractional ideals of $K$ generated by elements of $K(M, q)$.
(2) $\chi$ is trivial on the principal fractional ideals of $K$ generated by elements of $\mathbb{Q}^\times$ that are prime to $pdN$.
(3) $[L: K]$ is odd.
(4) $L/\mathbb{Q}$ is Galois.

Then $\rho: G_\mathbb{Q} \to \text{GL}(2, \mathbb{F})$ given by $\rho = \text{Ind}_{G_K}^{G_\mathbb{Q}} \chi$ is an even Galois representation and is attached to a Hecke eigenclass in $H^1(\text{GL}(2, \mathbb{Z}), \mathfrak{M}(M, q)^\ast)$, for a certain module $\mathfrak{M}(M, q)$ defined in Definition 4.2.

The module $\mathfrak{M}(M, q)$ that we use is naturally defined in terms of the field $K$. It is a countably infinite-dimensional module related to the kind of module that we used in [2] to study reducible cases of the Serre-type conjecture for $\text{GL}(3)/\mathbb{Q}$.

The conditions imposed on $\chi$ guarantee that $L$ is a totally real field, which is why the induced representation $\rho$ can be even. Examples of characters $\chi$ satisfying the conditions of the theorem include unramified characters of $G_K$ and characters of $G_K$ cutting out subfields of ring class fields of $K$ that are Galois over $\mathbb{Q}$ and of odd degree over $K$.

The Galois representations in the theorem are known to be attached to Maass forms. The results of [6] (building on [15]) thus also serve to attach the Galois representations that we study to cohomology groups. However, there are very significant differences between the construction in [6] and our construction. First, [6] uses wholly different coefficient modules than we do. Their modules are real vector spaces and their techniques are analytic. In contrast, our coefficient modules are duals of countably infinite-dimensional vector spaces over any field of characteristic not equal to 2, and our methods are purely algebraic. Second, the results of [6] hold for arbitrary Maass forms, not only those that are expected to have arithmetic significance.

The proof of our theorem goes as follows. Viewing $K$ as a two-dimensional $\mathbb{Q}$-vector space, we construct a $\text{GL}(2, \mathbb{Q})$-module $\mathfrak{M}$ consisting of formal $\mathbb{F}$-linear combinations of elements of $\mathcal{X}$, where $\mathcal{X}$ is the set of homothety classes of column vectors in $K^2$ where the ratio of the two entries is not in $\mathbb{Q} \cup \infty$. The homotheties involved here are restricted to multiplication by the elements of a carefully chosen subgroup of $K^\times$ (see sections 2, 3, and 4). The stabilizer $\Gamma_x \subset \text{GL}(2, \mathbb{Z})$ of a homothety class $x \in \mathcal{X}$ is an infinite abelian group generated by $\{ \pm I_2 \}$ and the image $\gamma_x$ of a certain unit in the ring of integers of $K$ under a certain embedding of $K$ into $\text{GL}(2, \mathbb{Q})$ as a non-split torus.

Because of the factor $\{ \pm I_2 \}$ in the stabilizers, we must assume that the characteristic of $\mathbb{F}$ is not equal to 2.
Incidentally, the matrix $\gamma_x$ also stabilizes a closed geodesic in the quotient of the upper half plane modulo $\text{GL}(2, \mathbb{Z})$. Our initial idea was to work with the fundamental classes of these closed geodesics in some appropriate homology group. In the end we work directly with certain classes in $H_1(G, \mathcal{M})$. (The construction of these homology classes is similar to a construction used in [1], but we do not see any deep connection between the two situations.)

Because $K^\times$ is abelian, it also acts on $\mathcal{M}$, commuting with the $\text{GL}(2, \mathbb{Q})$-action. We replace $\mathcal{M}$ by a related submodule $\mathcal{M}(M, q)$ on which a certain large subgroup of $\mathbb{Q}^\times$ acts via a quadratic character $q$ related to the quadratic field $K/\mathbb{Q}$. The purpose of this is to make $q$ equal the central character of the coefficient module $\mathcal{M}(M, q)$, which then causes the coefficient of $X^2$ in the Hecke polynomial to be correct.

In section 5, we obtain an isomorphism of $H_1(\text{GL}(2, \mathbb{Z}), \mathcal{M})$ with a direct sum of $H_1(\Gamma_x, \mathcal{F})$'s, where $x$ runs over a set of representatives of the $\text{GL}(2, \mathbb{Z})$-orbits in $X$. This uses Shapiro's lemma and is an algebraic version of the fundamental classes of the corresponding closed geodesics.

To complete section 5, we work out how the Hecke operators act on this direct sum. We use the method of partial Hecke operators described in [2] to get a tractable formula for the action of a Hecke operator on $H_1(\text{GL}(2, \mathbb{Z}), \mathcal{M}(M, q))$. This rather delicate analysis is what allows us to compare the Hecke operator at $\ell$ with a Laplace operator on the $\ell$-adic Bruhat-Tits tree below.

We call attention to the connection proved in Lemma 5.8 between two constants, $d_j$ and $e_j$, which are indexes of certain subgroups of units. When we compare the Hecke operator at $\ell$ with the Laplace operator on functions of $\ell$-adic lattices, at the beginning of the proof of Theorem 8.8, this connection appears in a surprising and crucial fashion.

A class in $H_1(\text{GL}(2, \mathbb{Z}), \mathcal{M}(M, q))$ has finite support, and the Hecke operators expand that support. So there will not be any Hecke eigenvectors in the homology group. Instead, we find Hecke eigenvectors in the dual space $H^1(\text{GL}(2, \mathbb{Z}), \mathcal{M}(M, q)^*)$.

A key idea is to interpret elements of the dual space $H^1(\text{GL}(2, \mathbb{Z}), \mathcal{M}(M, q)^*)$ of the homology as functions on the space of lattices in $K$, which we explain in section 6. In order to construct suitable such functions we introduce, in section 7, the Bruhat-Tits graph $T_\ell$ for $\text{GL}(2, \mathbb{Q}_\ell)$ or a certain double cover $\tilde{T}_\ell$ of $T_\ell$, depending on whether $\ell$ is split or inert in $K$.

We then relate the Hecke operators at $\ell$ to a Laplacian on $T_\ell$ (or $\tilde{T}_\ell$). We must go to a double cover in the inert case to make this work compatibly with the central character $q$ on $\mathcal{M}(M, q)$. Then, in sections 8 and 9, we construct functions on lattices that have the desired Hecke eigenvalues.

These functions on lattices are infinite products over the rational primes of the local functions we construct on the graphs. Lattices which are fractional ideals in $K$ play a special role and we call them “idealistic” lattices. The construction of the local functions depends on the distinction between idealistic and non-idealistic lattices.

To define the desired cohomology class, the infinite product has to satisfy a certain global invariance property (proved in Section 10), which follows from the fact that $\chi$ is a global character on ideals. This global requirement means that we cannot simply define the local functions on lattices any way we want to obtain any random set of Hecke eigenvalues. Instead, the situation is rather rigid.
Finally, in section 11 we prove the main theorem (Theorem 11.1) stated above. We thank Dick Gross, David Hansen, Richard Taylor, and especially Kevin Buzzard for helpful comments and answers to questions that arose during the course of this research. We also thank Henri Darmon and the anonymous referee for their feedback, which helped us improve the exposition of the paper.

2. LATTICES AND HOMOTHEITIES IN $K$

In this section and the next two sections we construct the coefficient modules we use in the homology of $\text{GL}(2, \mathbb{Z})$ which appear in our main theorem as stated in the introduction.

Fix a real quadratic field $K$, its ring of integers $\mathcal{O}$ and an element $\omega \in \mathcal{O}$ such that $\mathcal{O} = \mathbb{Z}[\omega]$. Let $d$ be the discriminant of $K/\mathbb{Q}$. Let $\epsilon$ be a fundamental unit, i.e. a unit whose image modulo $\pm 1$ generates $\mathcal{O} = \{\pm 1\}$.

Consider $K$ as a two-dimensional vector space over $\mathbb{Q}$. By a lattice in $K$, we will mean a free $\mathbb{Z}$-module of rank 2 contained in $K$. Such a module has as a $\mathbb{Z}$-basis two $\mathbb{Q}$-linearly independent elements.

Let $Y$ be the set of all column vectors $t(a, b) \in K^2$ with $b \neq 0$ and $a/b \notin \mathbb{Q}$. If we let $\bar{\omega} = t(\omega, 1) \in Y$, then every element of $Y$ is of the form $\gamma \bar{\omega}$ for some $\gamma \in \text{GL}(2, \mathbb{Q})$. In addition, given $y, y' \in Y$, there is a unique $\gamma \in \text{GL}(2, \mathbb{Q})$ with $y = \gamma y'$. There is a natural action of $K^\times$ by scalar multiplication on $Y$, which we write as a right action.

**Definition 2.1.** Let $y = t(a, b) \in Y$. Define $\Lambda_y$ to be the $\mathbb{Z}$-lattice in $K$ generated by $a$ and $b$ (i.e. the set of all integer linear combinations of $a$ and $b$).

Note that for $\alpha \in K^\times$, we have $\Lambda_{y\alpha} = \alpha \Lambda_y$.

**Definition 2.2.** Let $H \subseteq K^\times$ be a multiplicative subgroup of $K^\times$. Two lattices $\Lambda_1$ and $\Lambda_2$ in $K$ will be said to be homothetic if there is some $\alpha \in K^\times$ such that $\Lambda_1 = \alpha \Lambda_2$. If $\alpha \in H$, we will say that the lattices are $H$-homothetic.

Homothety and $H$-homothety of lattices are equivalence relations on the set of all lattices in $K$.

**Definition 2.3.** Let $H$ be a multiplicative subgroup of $K^\times$. Define $Y/H$ to be the quotient of $Y$ with respect to the right action of scalar multiplication by $H$. The left action of $\text{GL}(2, \mathbb{Q})$ on $Y$ then gives a left action of $\text{GL}(2, \mathbb{Q})$ on $Y/H$.

**Lemma 2.4.** *There is a bijection between $\text{GL}(2, \mathbb{Z})$-orbits of elements of $Y/H$ and the set $\mathcal{H}$ of $H$-homothety classes of lattices in $K$.***

**Proof.** Define a map $f: Y/H \to \mathcal{H}$ by setting $f(x)$ equal to the $H$-homothety class of $\Lambda_y$ for any $y \in Y$ representing $x \in Y/H$. This map is constant on $\text{GL}(2, \mathbb{Z})$-orbits, and is easily seen to induce a bijection between $\text{GL}(2, \mathbb{Z})$-orbits and $H$-homothety classes of lattices in $K$. \qed

**Lemma 2.5.** *Let $\Lambda$ be a lattice in $K$. Then there is a positive integer $m$ such that $\epsilon^m \Lambda = \Lambda$.***

**Proof.** Note that if $\Lambda$ and $\Lambda'$ are $K^\times$-homothetic, the lemma will be true for $\Lambda$ if and only if it is true for $\Lambda'$, with the same value of $m$ (since $K^\times$ is commutative.)

Hence, we may, without loss of generality, assume that $\Lambda$ is contained in $\mathcal{O}$. Since $\Lambda$ is a rank two $\mathbb{Z}$-submodule of $\mathcal{O}$, it must have finite index in $\mathcal{O}$. We may thus...
choose an $N \in \mathbb{Z}$ such that $N\mathfrak{O} \subseteq \Lambda \subseteq \mathfrak{O}$. Since $\mathfrak{O}/N\mathfrak{O}$ is finite and multiplication by $\epsilon$ permutes its elements, there is some positive $m \in \mathbb{Z}$ such that $\delta = \epsilon^m$ acts trivially on $\mathfrak{O}/N\mathfrak{O}$, and hence on $\Lambda/N\mathfrak{O}$. Then $\delta$ must take $\Lambda$ to itself, so $\delta \Lambda \subseteq \Lambda$. We must also have $\delta^{-1} \Lambda \subseteq \Lambda$, so $\Lambda \subseteq \delta \Lambda \subseteq \Lambda$, and therefore $\delta \Lambda = \Lambda$. □

**Definition 2.6.** Let $H$ be a subgroup of $K^\times$ containing $-1$. Given $x \in Y/H$, we define $\Gamma_x$ to be the stabilizer of $x$ in $\text{GL}(2,\mathbb{Z})$, and $\hat{\Gamma}_x$ to be the quotient $\Gamma_x/\{\pm 1\}$.

**Remark 2.7.** Note that for $y \in Y$, $(-1)y = y(-1)$, so since $-1 \in H$, we have $-I \in \Gamma_x$ for any $x \in Y/H$.

**Theorem 2.8.** Let $H$ be a subgroup of $K^\times$ such that $H \cap \mathfrak{O}^\times$ is infinite and $-1 \in H$, and let $x \in Y/H$ be represented by $y \in Y$. Then $\Gamma_x$ is generated by $\{-I, g\}$, where $g \in \Gamma_x$ satisfies

$$gy = ye^m,$$

and $m$ is smallest possible positive integer such that $e^m \in H$ and $\Lambda_y = \Lambda_{ye^m}$. Further, $\hat{\Gamma}_x$ is cyclic, generated by the image of $g$.

**Proof.** Let $x$ be represented by $y = \ell(a, b) \in Y$. Choose the smallest positive $m$ such that $e^m \Lambda_y = \Lambda_y$ and $e^m \in H$. Then $\Lambda_y = \Lambda_{ye^m}$, so $ye^m$ is a basis of $\Lambda_y$. Hence, there is some $g \in \text{GL}(2,\mathbb{Z})$ such that $gy = ye^m$. Since $e^m \in H$, we see that $gx = x$.

We now show that every element in $\Gamma_x$ is (up to a sign) a power of $g$. Let $\eta \in \Gamma_x$. Then, since $\eta x = x$, there is some $\alpha \in H$ such that $\eta y = \eta a \alpha$. Now $\alpha$ is an eigenvalue of $\eta$, and $\eta \in \text{GL}(2,\mathbb{Z})$, so $\alpha \in \mathfrak{O}^\times$. Hence, $\alpha = \pm \epsilon^r$. By the division algorithm and the minimality of $m$, we see that $\alpha = \pm (e^m)^k$ for some $k$. Hence, $\eta = \pm g^k$. □

Certain elements $x \in Y/H$ will be quite important to us; for these elements, the value of $m$ in the previous proof is determined solely by $H$.

**Definition 2.9.** Let $H$ be a multiplicative subgroup of $K^\times$. If $x \in Y/H$ can be represented by $y \in Y$ such that $\Lambda_y$ is a fractional ideal in $K$, then we say that $x$ is **idealistic**.

Note that determining whether $x$ is idealistic does not depend on the choice of $y \in Y$ representing $x$.

**Corollary 2.10.** Let $H$ be a subgroup of $K^\times$ such that $H \cap \mathfrak{O}^\times$ is infinite and $-1 \in H$. If $x \in Y/H$ is idealistic, the value of $m$ in Theorem 2.8 is equal to the smallest positive integer $k$ such that $e^k \in H$.

For any subgroup $H$ of $K^\times$ such that $H \cap \mathfrak{O}^\times$ is infinite and $-1 \in H$, then since $\hat{\Gamma}_x$ is cyclic, there is a canonical isomorphism

$$I_x : H_1(\hat{\Gamma}_x, \mathbb{F}) \to \hat{\Gamma}_x \otimes_{\mathbb{Z}} \mathbb{F}.$$ 

Since the characteristic of $\mathbb{F}$ is not equal to 2, the Hochschild-Serre spectral sequence for the exact sequence $0 \to \{\pm 1\} \to \Gamma_x \to \hat{\Gamma}_x \to 0$ yields an isomorphism

$$\phi_x : H_1(\Gamma_x, \mathbb{F}) \to H_1(\hat{\Gamma}_x, \mathbb{F}).$$

**Definition 2.11.** Let $H$ be any subgroup of $K^\times$ such that $H \cap \mathfrak{O}^\times$ is infinite and $-1 \in H$, and let $x \in Y/H$.

(a) Define $m_x$ to be the integer $m$ described in Theorem 2.8.

(b) Define $g_x$ to be the generator of $\hat{\Gamma}_x$ described in Theorem 2.8.
Lemma 2.13. If $m \neq 0$, then $m \notin \mathbb{Q}$.

Theorem 2.8, we see that $\Lambda_y = \Lambda_{y'}$. From the description of $m$ in Theorem 2.8, we see that $m_x = m_{x'}$.

Lemma 2.13. If $\gamma \in \text{GL}(2, \mathbb{Z})$, then conjugation by $\gamma$ from $\Gamma_x$ to $\Gamma_{\gamma x}$ induces a map $\hat{\Gamma}_x \to \hat{\Gamma}_{\gamma x}$ that takes $g_x$ to $g_{\gamma x}$, and induces a map on homology which takes $z_x$ to $z_{\gamma x}$.

Proof. Let $y$ represent $x$. Then $\gamma y$ represents $\gamma x$. There is a unique lift $g \in \text{GL}(2, \mathbb{Z})$ of $g_x$, such that $g_y = y c^{m_x}$. Then

$$(\gamma g \gamma^{-1}) \gamma y = (\gamma y) c^{m_x},$$

so, since $m_x = m_{\gamma x}$, we see that $(\gamma g \gamma^{-1}) = g_{\gamma x}$. 

3. $H(M, q)$-HOMOTHETY

In this section we define a subgroup of $K^\times$, and study homothety under this subgroup. We also define a space of homothety classes that we will use to define the coefficient module for our homology groups.

Definition 3.1. Let $F$ be a field of characteristic not equal to 2. If the characteristic of $F$ is 0, we will define $p = 1$, otherwise we define $p$ to be the characteristic of $F$. Let $d = \text{disc}(K)$, let $N$ be an arbitrary natural number, and let $M$ be a positive integer dividing $pdN$.

Definition 3.2. Denote by $\mathbb{Z}_{(pdN)}$ the ring consisting of elements of $\mathbb{Q}$ with denominators prime to $pdN$.

Definition 3.3. With $p, d, M, N$ defined as above, define

$$K^\times_{(pdN)} = \{c \in K^\times : (c) \text{ is relatively prime to } (pdN)\}.$$ 

and

$$K(M) = \{c \in K^\times_{(pdN)} : c \equiv 1 \pmod{M}\}.$$ 

By $c \equiv 1 \pmod{M}$, we mean that $v_\pi(c - 1) \geq v_\pi(M)$, for each prime $\pi$ of $K$ dividing $M$, where $v_\pi$ is the valuation on $K$ associated with the prime $\pi$. Note that $K^\times_{(pdN)}$ and $K(M)$ are multiplicative subgroups of $K^\times$, and that $K(M) \subseteq K^\times_{(pdN)}$ with finite index.

Next, we define a map from $K$ to the ring $M(2, \mathbb{Q})$ of $2 \times 2$ matrices with entries in $\mathbb{Q}$.

Definition 3.4. Let $y = \frac{1}{2}(a, b) \in Y$. We define an injective ring homomorphism $r_y : K \to M(2, \mathbb{Q})$ by

$$r_y(c) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ac \\ bc \end{pmatrix}$$

for $c \in K$.

Let $\theta : \mathbb{Z} \to F$ be the quadratic Dirichlet character cutting out $K/\mathbb{Q}$. Note that since $K$ is a real quadratic field, $\theta(-1) = 1$. We now extend $\theta$ to a character of $K^\times_{(pdN)}$. 

Definition 3.5. Define \( q : K_{(pdN)}^\times \to \mathbb{F}^\times \) to be the composition of the following multiplicative maps:

1. The map taking \( a \in K_{(pdN)}^\times \) to the principal fractional ideal \((a) \subset K\),
2. The map taking a fractional ideal to its prime factorization,
3. The map taking a product of powers of prime ideals to the subproduct of powers of inert prime ideals,
4. The map taking an inert prime ideal \((\ell)\) to \(\theta(\ell)\).

We note that \( q \) is a homomorphism. By the following lemma, it extends \( \theta \).

Lemma 3.6. If \( r \in \mathbb{Z} \) is relatively prime to \( pdN \), then \( q(r) = \theta(r) \).

Proof. This follows from the fact that \( q(\ell) = \theta(\ell) \) for primes \( \ell \nmid pdN \) of \( Q \), and that \( q(-1) = 1 = \theta(-1) \).

Definition 3.7. With \( p, d, N, M \) from Definition 3.1,

(a) Define \( K(M, q) \) to be the kernel of \( q|_{K(M)} \).
(b) Define \( Q(q) \) to be the kernel of \( q \) on \( \mathbb{Z}_{(pdN)}^\times \).
(c) Define \( H(M, q) \) to be the subgroup of \( K^\times \) generated by \( Q(q) \) and \( K(M, q) \).

Lemma 3.8. \( H(M, q) \cap \mathcal{Q}^\times \) is infinite and contains \(-1\).

Proof. Every unit in the ring of integers of \( K \) is in the kernel of \( q \), and has a power that is congruent to 1 mod \( M \). Further, \(-1 \in Q(q)\).

Definition 3.9. Let \( X = Y/H(M, q) \) and let \( \mathcal{M} = \mathbb{F}X \).

Since \( FY \) is a \((\text{GL}(2, \mathbb{Q}), K^\times)\)-bimodule, we obtain an isomorphism
\[
FY \otimes_{H(M,q)} F \cong FX \otimes F = FX.
\]
Because \( K^\times \) is commutative, this is an isomorphism of \((\text{GL}(2, \mathbb{Q}), K^\times)\)-bimodules.

Lemma 3.10. Let \( x \in X \) be represented by \( y \in Y \). Then

(a) \( \Gamma_x = \{ r_y(c) : c \in H(M, q) \} \cap \text{GL}(2, \mathbb{Z}) \),
(b) If \( g \in \Gamma_x \), then \( g = r_y(c) \) for some \( c \in \mathcal{Q}^\times \).

Proof. (a) Suppose \( g \in \Gamma_x \). Then we have that \( gy = yc \) for some \( c \in H(M, q) \).
Since \( yc = r_y(c)y \), and the entries of \( y \) are a \( \mathbb{Q} \)-basis of \( K \), we see that \( g = r_y(c) \).
Hence, \( g \) is in the given intersection, and any \( g \) in the given intersection fixes \( x \).

(b) Let \( g \in \Gamma_x \). Then \( g = r_y(c) \), for some \( c \in H(M, q) \), and the characteristic polynomial of \( g \) is the same as that of multiplication by \( c \) on \( K \). Since \( g \in \text{GL}(2, \mathbb{Z}) \), we see that \( c \in \mathcal{Q}^\times \).

Denote by \( \tilde{\omega} \) the image in \( X \) of \( \bar{\omega} = \iota(\omega, 1) \) and recall the definition of \( m_x \) from Definition 2.11.

Definition 3.11. Define \( i^M = m_{\tilde{\omega}} \).

Lemma 3.12. Let \( x \in X \).

(i) For any \( x \), \( i^M \mid m_x \).
(ii) If \( x \) is idealistic, then \( m_x = i^M \).
Proof. (i) For any \( x \in \mathcal{X} \) represented by \( y \in Y \), let \( \phi_x : \Gamma_x \to H(M, q) \) be the injective homomorphism defined by \( \phi_x(g) = r_y^{-1}(g) \). Then the image of \( \phi_x \) is a subgroup of \( \mathcal{O}^\times \) containing \( \{ \pm 1 \} \), and this subgroup has index \( m_x \in \mathcal{O}^\times \).

For an element \( c \in \mathcal{O}^\times \), we see (by Lemma 3.10(a)) that \( r_y(c) \in \Gamma_x \) implies that \( c \in H(M, q) \), which in turn implies that \( r_c(c) \in \Gamma_{\omega} \). Hence, \( r_y^{-1}(\Gamma_x) \leq r_c^{-1}(\Gamma_{\omega}) \).

Hence, the index in \( \mathcal{O}^\times \) of the first \( (m_x) \) is a multiple of the index of the second \( (m_\omega = i^M) \).

(ii) Since \( x \) is idealistic, \( \Lambda_y \) is a fractional ideal of \( K \). Hence, \( r_y(c) \in \text{GL}(2, \mathbb{Z}) \) for all \( c \in \mathcal{O}^\times \).

Since these subgroups are equal, \( m_x = i^M \).

\[ m_x = \begin{cases} i^M & \text{if } x \text{ is idealistic}, \\ m_x & \text{if } x \text{ is non-idealistic}. \end{cases} \]

\( m_x = 1 \) if and only if \( x \in \mathcal{X}_0 \).

Definition 3.13. For \( x \in \mathcal{X} \), set \( m_x' = m_x/i^M \).

Lemma 3.14. Let \( r \in \mathbb{Z}_{(pdN)}^\times \), let \( \alpha \in H(M, q) \), and let \( y \in Y \). If \( r_y(r\alpha) \in \text{GL}(2, \mathbb{Z}) \), then \( q(r) = 1 \).

Proof. Since the characteristic polynomial of \( r\alpha \) is the same as the characteristic polynomial of \( r_y(r\alpha) \), we see that \( r\alpha \) is a unit in \( \mathcal{O} \). Hence, \( q(r\alpha) = q(r)q(\alpha) = 1 \).

Since \( q(\alpha) = 1 \) by the definition of \( H(M, q) \), we see that \( q(r) = 1 \).

4. Defining the Coefficient Module \( \mathfrak{M}(M, q) \)

The elements of \( \mathcal{X} \) consist of \( H(M, q) \)-homothety classes of elements of \( Y \). In order to study them more fully, we fix once and for all a certain element of \( \mathbb{Z}_{(pdN)}^\times \).

Definition 4.1. Let \( \xi \in \mathbb{Z}_{(pdN)}^\times \) be any element of \( \mathbb{Z}_{(pdN)}^\times \) with \( q(\xi) = -1 \).

Since \( \xi \) is not in \( H(M, q) \), we see that for any \( x \in \mathcal{X} \) the two elements \( x\xi \) and \( x \) are distinct. In addition, since \( \xi^2 \in H(M, q) \), right multiplication by \( \xi \) induces an involution on the elements of \( \mathcal{X} \), and in fact, this involution is independent of the choice of \( \xi \).

Recall that \( \mathfrak{M} = \mathbb{F}\mathcal{X} \) is the \( \mathbb{F} \)-vector space consisting of formal \( \mathbb{F} \)-linear combinations of elements of \( \mathcal{X} \); i.e. the set of all elements of the form \( \sum_{x \in \mathcal{X}} c_x x \) with \( c_x \in \mathbb{F} \). Also, \( \mathfrak{M} \) is a \( (\text{GL}(2, \mathbb{Q}), K^\times) \)-bimodule. Since the action of \( \xi \) on the right is an involution on \( \mathcal{X} \), it induces an involution on \( \mathfrak{M} \); the eigenvalues of this involution are all either 1 or \(-1 \).

Definition 4.2. Let \( \mathfrak{M}(M, q) \) be the eigenspace in \( \mathfrak{M} \) of \( \xi \) with eigenvalue \(-1 \).

It is clear that \( \mathbb{Z}_{(pdN)}^\times \) acts on \( \mathfrak{M}(M, q) \) via the character \( q \).

Lemma 4.3. Let \( x \in \mathcal{X} \). Then \( x \) and \( x\xi \) are in different \( \text{GL}(2, \mathbb{Z}) \)-orbits.

Proof. Suppose \( x \) is represented by \( y \in Y \), and \( x\xi = \gamma x \) for some \( \gamma \in \text{GL}(2, \mathbb{Z}) \). Then there is some \( \alpha \in H(M, q) \) such that \( y\xi\alpha = \gamma y \), which implies that \( \gamma = r_y(\xi\alpha) \in \text{GL}(2, \mathbb{Z}) \). However, this contradicts Lemma 3.14.

We now consider a collection \( \mathfrak{A} \) of representatives of the \( \text{GL}(2, \mathbb{Z}) \)-orbits in \( \mathcal{X} \).

Definition 4.4. Let \( \mathfrak{A} \) be a collection of \( \text{GL}(2, \mathbb{Z}) \)-orbit representatives in \( \mathcal{X} \), chosen so that if \( x \in \mathfrak{A} \), then \( x\xi \in \mathfrak{A} \). Choose a subset \( \mathcal{A} \subset \mathfrak{A} \) such that for every \( x \in \mathfrak{A} \), exactly one of \( \{ x, x\xi \} \) is in \( \mathcal{A} \).
The following lemma then follows easily.

**Lemma 4.5.** Let $\mathcal{A}$ be the subset of $\mathfrak{A}$ defined above. Then the set
$$\{gx : x \in \mathfrak{A}\}$$
is a basis for $\mathfrak{M}$, and the set
$$\{g(x - x\xi) : x \in \mathcal{A}, g \in \text{GL}(2, \mathbb{Z})\}$$
is a basis of $\mathfrak{M}(M, q)$.

From this fact, the following lemma follows immediately.

**Lemma 4.6.** The map from
$$\bigoplus_{x \in \mathfrak{A}} F[\text{GL}(2, \mathbb{Z})] \otimes_{\text{ST}} F \to \mathfrak{M}$$
taking $g \otimes 1$ in the summand corresponding to $x \in \mathfrak{A}$ to $gx$ is a $\text{GL}(2, \mathbb{Z})$-invariant isomorphism.

5. Homology with coefficients in $\mathfrak{M}(M, q)$ and Hecke operators

In this section, we fix an element $x_0 \in \mathfrak{X}$ and a set $\mathfrak{A}$ of $\text{GL}(2, \mathbb{Z})$-orbit representatives in $\mathfrak{X}$ as in Definition 4.4 that contains $x_0$. We have fixed an element $\xi \in \mathbb{Z}_{(pdN)}$ with $q(\xi) = -1$. Using the set $\mathfrak{A}$, we will study the action of the Hecke operators on the homology of $\mathfrak{M}$, and ultimately on the homology of $\mathfrak{M}(M, q)$.

As a consequence of Lemma 4.6, we have that
$$\mathfrak{M} \cong \bigoplus_{x \in \mathfrak{A}} \text{Ind}^{\text{GL}(2, \mathbb{Z})}_{\Gamma_x} F.$$
Hence, by Shapiro’s lemma, we have
$$H_1(\text{GL}(2, \mathbb{Z}), \mathfrak{M}) \cong \bigoplus_{x \in \mathfrak{A}} H_1(\Gamma_x, F).$$

Recall that in Definition 2.11, for each $x \in \mathfrak{X}$, we have a canonical generator $z_x$ for $H_1(\Gamma_x, F)$. Then $\{z_x : x \in \mathfrak{A}\}$ is a basis for $H_1(\text{GL}(2, \mathbb{Z}), \mathfrak{M})$. Recall that $\xi$ acts as an involution on $\mathfrak{M}$, and hence on $H_1(\text{GL}(2, \mathbb{Z}), \mathfrak{M})$. From Lemma 4.6, we see that the action of $\xi$ on $\mathfrak{M}$ swaps the summands corresponding to $x, x\xi \in \mathfrak{A}$. Hence, under Shapiro’s isomorphism, we see that the action of $\xi$ on $H_1(\text{GL}(2, \mathbb{Z}), \mathfrak{M})$ is converted to an action on $\bigoplus_{x \in \mathfrak{A}} H_1(\Gamma_x, F)$ that swaps $z_x$ and $z_{x\xi}$ (the groups $\Gamma_x$ and $\Gamma_{x\xi}$ are identical, but $z_x$ and $z_{x\xi}$ are in different summands). This proves:

**Lemma 5.1.** The set
$$\{z_x - z_{x\xi} = z_x - \xi z_x : x \in \mathcal{A}\}$$
is a basis for $H_1(\text{GL}(2, \mathbb{Z}), \mathfrak{M}(M, q))$.

We will now use Section 3 of [2] to examine an individual Hecke operator $T_s$. (Warning: in [2] the modules are right-modules, but here we are using left-modules, so the formulas need to be adjusted accordingly.) What we call $\mathfrak{X}$ is there called $X$.

To follow the notation of [2], we set $\Gamma = \text{GL}(2, \mathbb{Z})$, and let $S = \text{GL}(2, \mathbb{Z}(pd))$. Recall that $\mathfrak{A}$ is a set of representatives of $\Gamma$-orbits in $\mathfrak{X}$. Let $W$ be the $S$-sheaf (in the terminology of [2]) whose stalk at each $x \in \mathfrak{X}$ is $F$. In other words, $W$ is
$\mathfrak{M}$, and it is an $S$-module. As a $\Gamma$-module, $W$ is isomorphic to a sum of induced representations

$$W = \bigoplus_{x \in \mathfrak{A}} F[\Gamma] \otimes_{\Gamma_x} F,$$

where $\Gamma_x$ is the stabilizer in $\Gamma$ of $x$.

Let $s \in S$. We choose a set $E$ of single coset representatives $s_\alpha$ for $\Gamma s \Gamma$, so that

$$\Gamma s \Gamma = \bigsqcup_{s_\alpha \in E} \Gamma_{s_\alpha}.$$

Then $E$ is a finite set. We now adjust the elements $s_\alpha \in E$ in several stages to make the computation of Hecke operators easier.

We have chosen $\mathfrak{A}$ to contain $x_0$. The proof of Theorem 3.1 of [2] immediately shows that we can choose a finite set of elements $x_1, \ldots, x_k$ of $\mathfrak{A}$ such that after possibly left-multiplying the $s_\alpha$'s by elements of $\Gamma$, we have a partition

$$E = \bigsqcup_{i=1}^k R_i$$

where $R_i = \{s_\alpha | s_\alpha x_0 = x_i\}$. Now setting

$$G_i = \{g \in \Gamma s \Gamma | gx_0 = x_i\},$$

we see that since the $x_i$ are in distinct $GL(2, \mathbb{Z})$-orbits,

$$G_i = \bigsqcup_{s_\alpha \in R_i} \Gamma_{x_0} s_\alpha.$$

The partial Hecke operator $T_{0i} = \Gamma_{x_0} \setminus G_i / \Gamma_{x_0}$ maps $H_*(\Gamma_{x_0}, M)$ to $H_*(\Gamma_{x_i}, M)$ for any $S$-module $M$.

**Theorem 5.2.** Let $\phi$ be the isomorphism given by Shapiro’s lemma:

$$\phi : H_*(\text{GL}(2, \mathbb{Z}), \mathfrak{M}) \to \bigoplus_{x \in \mathfrak{A}} H_*(\Gamma_x, F).$$

If $z \in H_*(\Gamma_{x_0}, F)$, then

$$T_s(\phi^{-1}(z)) = \phi^{-1}\left(\sum_{i=1}^k T_{0i}z\right).$$

**Proof.** This is a restatement of Theorem 3.1 of [2], where we note that $W = \mathfrak{M}$. \qed

Now we rewrite each of the partial Hecke operators $T_{0i}$ in terms of the homology of the stabilizers $\Gamma_x$ of elements of $\mathfrak{A}$.

Fix $i$ and note that for some finite set $C_i$,

$$G_i = \bigsqcup_{t \in C_i} \Gamma_{x_0} t \Gamma_{x_0}$$

is a disjoint union of double cosets.

Let $T_i : H_1(\Gamma_{x_0}, F) \to H_1(\Gamma_{x_i}, F)$ denote the Hecke operator corresponding to the double coset $\Gamma_{x_i} \setminus \Gamma_{x_0}$. 
Theorem 5.3. With notation as above,

(a) \( T_0 = \sum_{\ell \in \mathbb{C}} T_\ell \).
(b) After possibly left-multiplying the \( s_\alpha \)'s by elements of \( \text{GL}(2, \mathbb{Z}) \),
\[
\Gamma_{x_i} t \Gamma_{x_0} = \prod_{s_\alpha \in Q_{i,t}} \Gamma_{x_i} s_\alpha,
\]
and \( E \) is partitioned by the \( Q_{i,t} \), as \( i \) and \( t \) vary.
(c) After possibly left-multiplying one \( s_\alpha \) by an element of \( \text{GL}(2, \mathbb{Z}) \), \( t \in Q_{i,t} \).

Proof. Let \( F_* \) be a resolution of \( \mathbb{F} \) by free \( \text{FGL}(2, \mathbb{Q}) \)-modules. We use \( f \) to stand for an arbitrary element of \( F_* \) and \( a \) to stand for an arbitrary function with finite support \( a : F_* \rightarrow \mathbb{F} \). For each \( t \in C_i \), write its double coset as a disjoint union of single cosets:
\[
\Gamma_{x_i} t \Gamma_{x_0} = \prod_{s_\alpha \in R_i} \Gamma_{x_i} s_\alpha.
\]

The Hecke operator \( T_t \) maps the class of a cycle \( \sum f \otimes \tau x_0 \ a(f) \) to the class of \( \sum \beta \sum f \otimes \tau x_0 \ a(f) \). (Remember that \( \tau \) acts trivially on \( \mathbb{F} \)).

There is a similar formula for the action of the Hecke operator \( T_0 \). Since the set \( G_i \) is the union of the double cosets \( \Gamma_{x_i} t \Gamma_{x_0} \), \( t \in C_i \), it is the disjoint union of the single cosets \( \Gamma_{x_i} t, t \beta \) as \( t, \beta \) vary. Part (a) is now clear.

For (b), again write \( \Gamma_{x_i} t \Gamma_{x_0} = \prod_{t, \beta} \Gamma_{x_i} t, t \beta \), for some choice of \( t, \beta \). We know from the discussion preceding Theorem 5.2 that \( E \) is the disjoint union of the \( R_i \) and that
\[
G_i = \prod_{t \in C_i} \Gamma_{x_i} t \Gamma_{x_0} = \prod_{s_\alpha \in R_i} \Gamma_{x_i} s_\alpha.
\]

Therefore each coset \( \Gamma_{x_i} t, t \beta \) must equal \( \Gamma_{x_i} s_\alpha \) for some \( s_\alpha \), and so we replace \( t, \beta \) with \( s_\alpha \). We let \( Q_{i,t} \) be the set of \( s_\alpha \)'s corresponding to \( t \). Then \( R_i \) is the disjoint union of the \( Q_{i,t} \) as \( t \) varies, and (b) follows.

Finally, since \( t \in \Gamma_{x_i} t \Gamma_{x_0} \), we see that \( t \in \Gamma_{x_i} s_\alpha \) for some \( s_\alpha \in Q_{i,t} \). We may replace that \( s_\alpha \) by \( t \), proving (c).

\[
\square
\]

Definition 5.4. Set
\[
J = \sum_i |C_i|.
\]

As \( i, t \) vary, enumerate the \( t \)'s as \( t_1, \ldots, t_J \). For each \( j = 1, \ldots, J \), set
\[
B_j = Q_{i,t_j}, \quad U_j = T_{t_j}, \quad \text{and} \quad x_j = x_i,
\]
where \( (i, t) \) is the pair corresponding to \( j \).

Set \( \Gamma_0 = \Gamma_{x_0} \) and \( \Gamma_j = \Gamma_{x_j} \).

Finally, set
\[
d_j = [\Gamma_0 : t_j^{-1} \Gamma_j t_j \cap \Gamma_0], \quad \text{and} \quad e_j = [\Gamma_j : t_j \Gamma_0 t_j^{-1} \cap \Gamma_j].
\]

Remark 5.5. Since all groups involved in the definitions of \( d_j \) and \( e_j \) contain \(-I\), we note that we also have
\[
d_j = [\hat{\Gamma}_0 : t_j^{-1} \hat{\Gamma}_j t_j \cap \hat{\Gamma}_0], \quad \text{and} \quad e_j = [\hat{\Gamma}_j : t_j \hat{\Gamma}_0 t_j^{-1} \cap \hat{\Gamma}_j].
\]

Theorem 5.6. (a) \( E = \{ s_\alpha \} = \bigsqcup B_j \) and \( T_s = \sum_{j=1}^J U_j \) where we suppress mention of the isomorphism \( \phi \) from Theorem 5.2.

(b) The number of elements in \( B_j \) is \( d_j \).
(c) $U_j(z_{x_0}) = e_jz_{x_j}$, where $z_x$ is the generator of $H_1(\Gamma_x, \mathbb{F})$ chosen in Definition 2.11.

(d) $U_j(z_{x_0} - \xi z_{x_0}) = e_j(z_{x_j} - \xi z_{x_j})$ in $H_1(\text{GL}(2, \mathbb{Z}), \mathfrak{M}(M, q))$.

Proof. Part (a) follows from Theorem 5.2 and Theorem 5.3 (a) (b) and (d) combined with the new notation.

Part (b): The number of elements in $B_j$ is the number of left $\Gamma_{x_j}$-cosets contained in $\Gamma_{x_j}t_j\Gamma_{x_0}$ (in our new notation $\Gamma_{x_j}t_j\Gamma_{x_0} = \Gamma_jt_j\Gamma_0$). A standard computation with cosets shows that the number of single cosets in this double coset equals the index $[\Gamma_0 : t_j^{-1}\Gamma_jt_j \cap \Gamma_0]$, which is $d_j$.

For part (c) we use the following lemma, which is standard, and follows easily from [5, Sec. III.9]:

**Lemma 5.7.** Let $A$ be an infinite cyclic group with generator $a$, and $B \subset A$ a subgroup of index $c$, and suppose that $A$ acts trivially on $F$. For an abelian group $G$ acting trivially on $\mathbb{F}$, we identify $H_1(G, \mathbb{F})$ canonically with $G \otimes \mathbb{F}$.

(i) The transfer map $\text{tr} : H_1(A, F) \to H_1(B, F)$ takes the generator $a \otimes 1$ to the generator $a^c \otimes 1$.

(ii) The corestriction map $i : H_1(B, F) \to H_1(A, F)$ (i.e. the map induced by the inclusion $B \subset A$) takes the generator $a^c \otimes 1$ to $c$ times the generator $a \otimes 1$.

Proof of part (c) continued: $U_j$ is a Hecke operator. A standard fact about Hecke operators is that they can be written in terms of transfer, conjugation, and corestriction. (This is also easily checked on the chain level.) In our case, the following is true:

The map $U_j : H_1(\Gamma_0, F) \to H_1(\Gamma_j, F)$ is given as the composition of the three maps

$$H_1(\Gamma_0, F) \to H_1(\Gamma_0 \cap t_j^{-1}\Gamma_jt_j, F) \xrightarrow{\phi_j} H_1(t_j\Gamma_0t_j^{-1} \cap \Gamma_j, F) \to H_1(\Gamma_j, F),$$

where the first map is the transfer, the second map is the map induced on homology via conjugation by $t_j$ on the group, and the third map is corestriction. We note that all of the groups involved in this diagram contain $\{\pm I\}$, and the maps described above each commute with the isomorphism on homology induced by the quotient map by $\{\pm I\}$. Hence, the composition above translates to the following,

$$H_1(\hat{\Gamma}_0, F) \to H_1(\hat{\Gamma}_0 \cap t_j^{-1}\hat{\Gamma}_jt_j, F) \xrightarrow{\phi_j} H_1(t_j\hat{\Gamma}_0t_j^{-1} \cap \hat{\Gamma}_j, F) \to H_1(\hat{\Gamma}_j, F),$$

where all of the involved groups are cyclic, allowing us to use Lemma 5.7.

Let $g_0$ and $g_j$ be the generators of $\hat{\Gamma}_0$ and $\hat{\Gamma}_j$ corresponding to the chosen generators $z_{x_0}$ and $z_{x_j}$ of the homology groups in Definition 2.11. Identifying elements of an abelian group with elements of the homology, we can say that by Lemma 5.7(i), the transfer takes $g_0$ to the generator $g_0^{d_j}$ of $\hat{\Gamma}_0 \cap t_j^{-1}\hat{\Gamma}_jt_j$.

Since $t_jx_0 = x_j$, conjugation by $t_j$ takes $g_0^{d_j}$ to the generator $g_0^{d_j}$ of $t_j\hat{\Gamma}_0t_j^{-1} \cap \hat{\Gamma}_j$. Then by Lemma 5.7 (ii), corestriction takes this to $g_j^{d_j}$.

Part (d) follows, since $\xi$ commutes with the action of $\text{GL}(2, \mathbb{Q})$ and so commutes with $U_j$. \square

We now compute the values of $e_j$ and $d_j$ in terms of $m_0$ and $m_j$. Recall that $t_jx_0 = x_j$. For $j = 0, \ldots, J$, choose $y_j \in Y$ such that $x_j$ is represented by $y_j$, and recall that $\Gamma_j$ is the stabilizer of $x_j$ in $\text{GL}(2, \mathbb{Z})$ and $e$ is the fundamental unit of $\mathcal{O}$.
which we chose at the beginning of Section 2. Let \( g_j \in \hat{\Gamma}_j \) and \( m_j \in \mathbb{Z} \) be defined as in Definition 2.11 (with \( H = H(M,q) \)). Then \( g_j \) is a generator of \( \hat{\Gamma}_j \) and \( m_j > 0 \).

**Lemma 5.8.** With notation as above,

\[
e_j = \text{LCM}(m_0, m_j)/m_j, \quad d_j = \text{LCM}(m_0, m_j)/m_0,
\]

and \( e_j m_j = d_j m_0 \).

**Proof.** There are two lifts of \( g_j \in \hat{\Gamma}_j \) to \( \Gamma_j \); let \( h_j \in \Gamma_j \) be the unique lift of \( g_j \) satisfying \( h_j = r_{g_j}((e_j m_j)) \) (the other lift will be \( -h_j = r_{g_j}(-e_j m_j) \)). We note that \( \Gamma_j \) is generated by \( \{-I, h_j\} \), and any subgroup \( G \subseteq \Gamma_j \) containing \(-I\) is generated by \( \{-I, h_j^k\} \), where \( k \) is the smallest positive integer such that \( h_j^k \in G \). For such a subgroup \( G \subseteq \Gamma_j \), we see easily that \( [\Gamma_j : G] = k \). In particular, \( e_j \) is the smallest positive integer such that \( h_j^{e_j} \in t_j \Gamma_0 t_j^{-1} \cap \Gamma_j \).

Now, \( h_0 y_0 = y_0 e_m^m \). Since \( t_j x_0 = x_j \), we have \( t_j y_0 = y_j \alpha_j \) for some \( \alpha_j \in H(M,q) \). Hence, \( t_j^{-1} y_j = y_0 \alpha_j^{-1} \). It follows that \( t_j h_0 t_j^{-1} y_j = y_j \alpha_0 \). In addition, \( h_j y_j = y_j e_j \).

Let \( k \) be the smallest positive integer such that \( h = (t_j h_0 t_j^{-1})^k \in \Gamma_j \). Then \( \{-I, h\} \) will generate \( \Gamma_j \cap t_j \Gamma_0 t_j^{-1} \). We note that \( k \) must be the smallest positive integer such that \( (e_j m_j)^k \) is a power of \( e_j \), or in other words, the smallest positive integer such that \( k m_0 \) is a multiple of \( m_j \). Hence, \( k m_0 = \text{LCM}(m_0, m_j) \), so \( h = (t_j h_0 t_j^{-1})^{\text{LCM}(m_0, m_j)/m_0} \), and we see that

\[
h y_j = y_j \left( \text{LCM}(m_0, m_j) \right) = y_j \left( (e_j m_j)^{\text{LCM}(m_0, m_j)/m_j} \right) = h_j^{\text{LCM}(m_0, m_j)/m_j} y_j.
\]

It follows that

\[
h = h_j^{\text{LCM}(m_0, m_j)/m_j},
\]

and therefore,

\[
e_j = \text{LCM}(m_0, m_j)/m_j.
\]

Reversing the roles of \( \Gamma_0 \) and \( \Gamma_j \) and switching \( t_j \) and \( t_j^{-1} \), we obtain

\[
d_j = \text{LCM}(m_0, m_j)/m_0. \quad \Box
\]

6. **Elements of \( H^1(\text{GL}(2, \mathbb{Z}), \mathfrak{M}(M,q)^r) \) Interpreted as Functions on Lattices**

We now interpret the cohomology of the dual of \( \mathfrak{M}(M,q) \) as a collection of functions on a space of lattices.

**Definition 6.1.** Let \( \Phi \) be a function from lattices in \( K \) to \( \mathbb{F} \). We will say that \( \Phi \) is \( q \)-homogeneous if \( \Phi(\alpha \Lambda) = q(\alpha)\Phi(\Lambda) \) for all \( \alpha \in \mathbb{Z}_{\text{pdN}}^\times \) and all lattices \( \Lambda \). We further define \( \xi \Phi \) by the formula \( (\xi \Phi)(\Lambda) = \Phi(\xi \Lambda) \).

If \( H \) is a subgroup of \( K^\times \), we will say that \( \Phi \) is \( H \)-invariant if \( \Phi(\alpha \Lambda) = \Phi(\Lambda) \) for all \( \alpha \in H \) and all lattices \( \Lambda \).

**Remark 6.2.** Since \( q \) is trivial on \( H(M,q) \), a function \( \Phi \) can be both \( q \)-homogeneous and \( H(M,q) \)-invariant. Note that \( K(M,q) \)-invariance together with \( q \)-homogeneity implies \( H(M,q) \)-invariance, since \( H(M,q) = Q(q)K(M,q) \). In addition, since \( K \) is a real quadratic field, \( q(-1) = 1 \). If this were not the case, the fact that \(-\Lambda = \Lambda \) for any lattice \( \Lambda \) in \( K \) would force all \( q \)-homogeneous functions to be identically 0.
Lemma 6.3. Choose a set $\mathfrak{A}$ of $\text{GL}(2, \mathbb{Z})$-orbit representatives of $\mathfrak{X}$ as in Definition 4.4.

(a) There is a natural $\xi$-equivariant isomorphism between $H^1(\text{GL}(2, \mathbb{Z}), \mathfrak{M})$ and the vector space of $\mathbb{F}$-valued functions on lattices in $K$ that are $H(M, q)$-invariant.

(b) There is a natural isomorphism between $H^1(\text{GL}(2, \mathbb{Z}), \mathfrak{M}(M, q)^*)$ and the vector space of $\mathbb{F}$-valued functions on lattices in $K$ that are $q$-homogeneous and $H(M, q)$-invariant.

Proof. (a) The choice of $\mathfrak{A}$ yields an isomorphism of $\text{GL}(2, \mathbb{Z})$-modules

$$f : \mathfrak{M} \to \bigoplus_{x \in \mathfrak{A}} \mathbb{F}\text{GL}(2, \mathbb{Z}) \otimes_{\mathbb{F}^{\text{G}_x}} \mathbb{F}.$$ 

This induces an isomorphism (via Shapiro’s Lemma)

$$H_1(\text{GL}(2, \mathbb{Z}), \mathfrak{M}) \cong \bigoplus_{x \in \mathfrak{A}} H_1(\text{GL}(2, \mathbb{Z}), \mathbb{F}\text{GL}(2, \mathbb{Z}) \otimes_{\mathbb{F}^{\text{G}_x}} \mathbb{F})$$

$$\cong \bigoplus_{x \in \mathfrak{A}} H_1(\Gamma_x, \mathbb{F})$$

$$\cong \bigoplus_{x \in \mathfrak{F}} \mathbb{F}.$$ 

Using the natural duality between $H_1(\text{GL}(2, \mathbb{Z}), \mathfrak{M})$ and $H^1(\text{GL}(2, \mathbb{Z}), \mathfrak{M}^*)$, we see that determining an element of $H^1(\text{GL}(2, \mathbb{Z}), \mathfrak{M}^*)$ is the same as giving a function from $\mathfrak{M}$ to $\mathbb{F}$.

The space of lattices $\mathcal{L}$ in $K$ is in bijection with $\text{GL}(2, \mathbb{Z})\backslash Y$, where a lattice $\Lambda$ corresponds to the $\text{GL}(2, \mathbb{Z})$-orbit of $y = ^t(a, b) \in Y$ where $a, b$ is a $\mathbb{Z}$-basis of $\Lambda$. The set $\mathfrak{A}$ consists of a set of representatives for the $\text{GL}(2, \mathbb{Z})$-orbits in $\mathfrak{X}$ and therefore is in bijection with $\text{GL}(2, \mathbb{Z})\backslash Y/H(M, q)$. This is the same as $\mathcal{L}/H(M, q)$, i.e. the set of $H(M, q)$-homothety classes of lattices.

Therefore there is a natural isomorphism between the vector space of functions from $\mathfrak{X}$ to $\mathbb{F}$ and the vector space of $H(M, q)$-invariant functions on lattices in $K$.

For future reference, we can write down this isomorphism explicitly:

If $\Phi$ is any $H(M, q)$-invariant function on lattices in $K$, we get a function $g$ on $\mathfrak{A}$ as follows: Given $x \in \mathfrak{A}$, lift $x$ to $y \in Y$ and set $g(x) = \Phi(\Lambda_y)$, where $\Lambda_y$ is the lattice spanned by the entries of $y$.

Conversely, given a function $g$ on $\mathfrak{A}$ and a lattice $\Lambda$ in $K$, $\Lambda$ corresponds (by choosing a basis $\{a, b\}$, setting $y = ^t(a, b) \in Y$ and projecting modulo $H(M, q)$) to an element $x' \in \mathfrak{X}$, which lies in the $\text{GL}(2, \mathbb{Z})$-orbit of a unique $x \in \mathfrak{A}$. Define $\Phi(\Lambda) = g(x)$.

It is clear that this isomorphism is $\xi$-equivariant, where we define $(\xi g)(x) = g(x\xi)$.

(b) The module $\mathfrak{M}(M, q)$ is the $q$-isotypic component of $\xi$ acting on $\mathfrak{M}$. Therefore $\mathfrak{M}(M, q)^*$ is the $q$-isotypic component of $\xi$ acting on $\mathfrak{M}^*$. It follows that $H^1(\text{GL}(2, \mathbb{Z}), \mathfrak{M}(M, q)^*)$ is the $q$-isotypic component of $\xi$ acting on $H^1(\text{GL}(2, \mathbb{Z}), \mathfrak{M}^*)$.

Therefore the $\xi$-equivariant isomorphism from part (a) induces an isomorphism between $H^1(\text{GL}(2, \mathbb{Z}), \mathfrak{M}(M, q)^*)$ and the vector space of $\mathbb{F}$-valued functions on lattices in $K$ that are $q$-homogeneous and $H(M, q)$-invariant, since $Z^\xi_{(p \mid N)}$ is generated by $\xi$ and $Q(q)$, and $H(M, q) = Q(q)K(M, q)$ (see definition 3.7).
7. The Branched Bruhat-Tits Graph and the Laplacian

In order to construct functions on lattices that are eigenfunctions of the Hecke operators, we will use a modification of the Bruhat-Tits building [8, 18], in which we lift the Bruhat-Tits building to a finite branched cover.

For each prime \( \ell \) unramified in \( K \), let \( K_\ell \) denote \( K \otimes \mathbb{Q}_\ell \). Then \( K_\ell \) is a two-dimensional vector space over \( \mathbb{Q}_\ell \).

**Definition 7.1.** If \( \ell \) is inert, then \( K_\ell \) is a quadratic field extension of \( \mathbb{Q}_\ell \). We fix the integral basis \( \{1, \omega\} \) of \( K \), and we identify \( K_\ell \) with \( \mathbb{Q}_\ell^2 \) by identifying 1 and \( \omega \) with the standard basis elements \( e_1, e_2 \in \mathbb{Q}_\ell^2 \).

If \( \ell \) splits in \( K \), then \( (\ell) = \lambda \lambda' \) for prime ideals \( \lambda, \lambda' \) in \( \mathcal{O} \) lying over \( \ell \). Each of the completions \( K_\lambda \) and \( K'_\lambda \) is then isomorphic to \( \mathbb{Q}_\ell \). Restricting these isomorphisms to \( K \), we obtain two distinct Galois conjugate embeddings \( i_\lambda, i_{\lambda'} : K \to \mathbb{Q}_\ell \). We then identify \( K_\ell = K \otimes \mathbb{Q} \mathbb{Q}_\ell \) with \( \mathbb{Q}_\ell^2 \) via the map taking

\[
t \otimes 1 \mapsto (i_\lambda(t), i_{\lambda'}(t)).
\]

We abbreviate the notation by writing \( t \mapsto (t, t') \).

**Definition 7.2.** By a lattice in \( K_\ell \), we will mean a rank two \( \mathbb{Z}_\ell \)-submodule of \( K_\ell \).

If \( \Lambda \) is a lattice in \( K \), then \( \Lambda_\ell = \Lambda \otimes \mathbb{Z} \mathbb{Z}_\ell \) is a lattice in \( K_\ell \).

**Definition 7.3.** Let \( \ell \) be a prime, and \( n \) a positive integer. Denote the elements of \( \mathbb{Q}_\ell^\times \) with \( \ell \)-adic valuation divisible by \( n \) by \( V_n \). We note that \( V_n \) is a subgroup of index \( n \) of \( \mathbb{Q}_\ell^\times \).

**Definition 7.4.** Let \( \Lambda_1 \) and \( \Lambda_2 \) be lattices in \( K_\ell \). We say that \( \Lambda_1 \) and \( \Lambda_2 \) are \( n \)-homothetic if \( \Lambda_1 = \alpha \Lambda_2 \) for some \( \alpha \in V_n \). Then \( n \)-homoty is an equivalence relation, and we call an equivalence class an \( n \)-homothety class of lattices in \( K_\ell \).

**Definition 7.5.** Let \( n \) a positive integer, \( K \) a real quadratic field, and \( \ell \) a prime unramified in \( K \). The branched Bruhat-Tits graph \( T_\ell^n \) is the graph whose vertices are \( n \)-homothety classes of lattices in \( K_\ell \). Two vertices are joined by an edge if there are representative lattices \( \Lambda_1 \) and \( \Lambda_2 \) of the vertices such that \( \Lambda_2 \subseteq \Lambda_1 \) or \( \Lambda_1 \subseteq \Lambda_2 \) with index \( \ell \).

**Remark 7.6.** The Bruhat-Tits tree is a special case of the branched Bruhat-Tits graph in which \( n = 1 \). When \( n = 1 \), we may denote \( T_\ell^n \) by \( T_\ell \). When \( n > 1 \), we will typically write vertices of \( T_\ell^n \) with a superscript \( n \), i.e. \( t^n \in T_\ell^n \).

**Definition 7.7.** Let \( \Lambda \) be a lattice in \( K_\ell \). Denote the vertex of \( T_\ell^n \) represented by \( \Lambda \) by \( \varpi(\Lambda) \). Denote the vertex of \( T_\ell \) represented by \( \Lambda \) by \( \pi(\Lambda) \). Given a vertex \( t^n \in T_\ell^n \), there is a unique vertex \( s \in T_\ell \) containing \( t^n \); we write \( s = \pi(t^n) \).

Note that for any lattice \( \Lambda \) with \( \varpi(\Lambda) = t^n \), \( \pi(t^n) = \pi(\Lambda) \). To keep our notation less cluttered, if \( \Lambda \) is a lattice in \( K_\ell \), we will often denote \( \varpi(\Lambda) \) by \( \Lambda \), as long as the context makes this usage clear.

**Remark 7.8.** We note that for any vertex \( t \in T_\ell \), there are exactly \( n \) vertices \( t^n \in T_\ell^n \) with \( \pi(t^n) = t \). If \( \Lambda \) is a lattice in \( K_\ell \) representing \( t \), these \( n \) vertices of \( T_\ell^n \) are represented by

\[
\Lambda, \ell \Lambda, \ldots, \ell^{n-1} \Lambda.
\]

**Definition 7.9.** If \( t \) is a vertex of \( T_\ell \), we will call the set \( \{t^n \in T_\ell^n : \pi(t^n) = t\} \) the fiber of \( t \) and also the fiber of \( t^n \) for any \( t^n \) in that set.
Definition 7.10. A vertex $t^n$ is idealistic if $t^n$ is the $n$-homothety class of the completion $I_\ell$ of some fractional ideal $I$ of $K$.

We now review some facts about completions $\Lambda_\ell$ of lattices in $K$. Let $\ell$ be a prime of $Q$.

By [19, V.2, Corollary to Theorem 2], the operations of sum and intersection of lattices in $K$ commute with completion at $\ell$. In addition, by [19, V.3, Theorem 2], a lattice $\Lambda$ in $K$ is determined by its set of completions $\Lambda_w$ for all finite places $w$ of $Q$. In fact

$$\Lambda = \bigcap_w K \cap \Lambda_w.$$  

Finally, completion at a finite place $w$ of finitely generated $\mathbb{Z}$-modules is an exact functor [9, Theorem 7.2].

Applying these facts to fractional ideals of $K$, we note that if $I$ is an ideal of $\mathcal{O}$ of norm prime to $\ell$, then $I_\ell$ is an ideal of $\mathcal{O}_\ell$ of index prime to $\ell$, so $I_\ell = \mathcal{O}_\ell$. In addition, multiplication of relatively prime ideals (i.e. intersection) commutes with completion at $\ell$. Hence, for an ideal $I$, the completion $I_\ell$ depends only on the factors of $I$ of $\ell$-power norm.

Now suppose that $t^n \in T_\ell^n$ is idealistic. Then we may assume that $t^n$ is represented by an ideal $I_\ell$, where $I$ is an ideal in $\mathcal{O}$ whose norm is a power of $\ell$. If $\ell$ is inert in $K$, such an $I$ must be principal, so $I_\ell$ is $\mathbb{Q}_\ell^\times$-homothetic to $\mathcal{O}_\ell$. Hence, $t^n$ is idealistic if and only if $\pi(t^n)$ is represented by $\mathcal{O}_\ell$.

On the other hand, if $\ell$ splits in $K$, then $\ell \mathcal{O} = \lambda \lambda'$, where $\lambda, \lambda'$ are prime ideals of $\mathcal{O}$ lying over $\ell$. We then see that $t^n \in T_\ell^n$ is idealistic if and only if $\pi(t^n)$ is represented by an ideal of the form $\lambda^k$ or $(\lambda')^h$ for some $k \in \mathbb{Z}$. In particular, if $\pi(t^n_1) = \pi(t^n_2)$, then $t^n_1$ and $t^n_2$ are either both idealistic, or both nonidealistic.

Lemma 7.11. Suppose $n > 1$. Let $\Lambda_1 \supset \Lambda_2$ be lattices in $K_\ell$ with $[\Lambda_1 : \Lambda_2] = \ell$. Let $t^n_1 = \pi(\Lambda_1) \in T_\ell^n$. Then there are precisely two vertices $t^n_2, t^n_3 \in T_\ell^n$ with $\pi(t^n_2) = \pi(t^n_3) = \pi(\Lambda_2)$ which are connected by an edge to $t^n_1$. If we let $t^n_2$ be represented by $\Lambda_2$, then $t^n_3$ is represented by $\ell^{-1}\Lambda_2$.

Proof. Clearly, if we take $t^n_2 = \pi(\Lambda_2)$ and $t^n_3 = \pi(\ell^{-1}\Lambda_2)$, we see that $t^n_2$ and $t^n_3$ are distinct and have the desired properties. It remains to show that there is no third vertex $t^n_4$, distinct from $t^n_2$ and $t^n_3$, with $\pi(t^n_4) = \pi(\Lambda_2)$, and such that there is an edge between $t^n_1$ and $t^n_4$.

Suppose that $\pi(t^n_4) = \pi(\Lambda_2)$ and there is an edge between $t^n_1$ and $t^n_4$. Then either there is a lattice $\Lambda_4$ representing $t^n_4$ such that $\Lambda_1 \supset \Lambda_4$ and $[\Lambda_1 : \Lambda_4] = \ell$ or there is a lattice $\Lambda_4$ representing $t^n_4$ such that $\Lambda_1 \subset \Lambda_4$ and $[\Lambda_4 : \Lambda_1] = \ell$.

Now suppose $\Lambda_4$ is homothetic to $\Lambda_2$, say with $\Lambda_4 = \alpha\Lambda_2$, where $\alpha \in \mathbb{Q}_\ell^\times$.

If $\Lambda_1 \supset \Lambda_4$ has index $\ell$, then by hypothesis $\ell^{-1}\Lambda_2 \supset \Lambda_1$ has index $\ell$ and $\Lambda_1 \supset \alpha\Lambda_2$ has index $\ell$. Hence, multiplying by $\ell$, we see that $\Lambda_2 \supset \ell\alpha\Lambda_2$ with index $\ell^2$. This implies that $\nu_\ell(\alpha) = 0$, so that $\alpha\Lambda_2 = \Lambda_2$, so $t^n_4 = t^n_2$.

On the other hand, if $\Lambda_1 \subset \Lambda_4$ with index $\ell$, then $\Lambda_2 \subset \Lambda_1$ has index $\ell$ by hypothesis and $\Lambda_1 \supset \alpha\Lambda_2$ with index $\ell$, so $\Lambda_2 \supset \alpha\Lambda_2$ has index $\ell^2$. Hence $\nu_\ell(\alpha) = -1$, and we see that $\alpha\Lambda_2 = \ell^{-1}\Lambda_2$, so $t^n_4 = t^n_3$.

Corollary 7.12. Let $n > 1$, let $t^n$ be a vertex in $T_\ell^n$, and let $t = \pi(t^n) \in T_n$. Let $s \in T_n$ be a neighbor of $t$. Then there are exactly two neighbors $s^n_1$ and $s^n_2$ of $t^n$ in $T_\ell^n$ with $\pi(s^n_1) = \pi(s^n_2) = s$. If $\Lambda$ represents $t^n$, then exactly one of $s^n_1$ and $s^n_2$
is represented by a sublattice $\Lambda'$ of $\Lambda$ of index $\ell$; the other is represented by $t^{-1}\Lambda'$, which contains $\Lambda$ with index $\ell$.

**Definition 7.13.** Let $n \geq 1$, let $t^n \in T^n_\ell$ be a vertex represented by a lattice $\Lambda$ in $K_\ell$, and let $s^n$ be a neighbor of $t^n$. If $s^n$ is represented by a sublattice of index $\ell$ in $\Lambda$, we call $s^n$ a **downhill** neighbor of $t^n$; if it is represented by a lattice containing $\Lambda$ with index $\ell$, we call $s^n$ an **uphill** neighbor of $t^n$.

We note that if $n = 1$, any neighbor of $t^n$ is both an uphill and a downhill neighbor of $t^n$.

**Definition 7.14.** Let $t^n \in T^n_\ell$. We define the **tier** of $t^n$ to be the distance between $\pi(\ell^n)$ and $\pi(\ell_t)$ in $T_t$. A neighbor of $t^n$ of higher tier than $t^n$ will be called an **outer neighbor** of $t^n$: a neighbor of lower tier will be called an **inner neighbor**.

**Remark 7.15.** Each $t^n \in T^n_\ell$ has precisely $\ell + 1$ downhill neighbors and $\ell + 1$ uphill neighbors. The use of uphill and downhill matches our intuition; if $s^n$ is a downhill neighbor of $t^n$, then $s^n$ is an uphill neighbor of $s^n$. (However, as in an Escher staircase, it is possible to go uphill several times and return to your starting point without going downhill.)

Each vertex of positive tier has precisely $\ell$ downhill outer neighbors, and $1$ downhill inner neighbor. It also has precisely $\ell$ uphill outer neighbors, and $1$ uphill inner neighbor.

A vertex of tier $0$ has only outer neighbors; $\ell + 1$ of them are uphill, and $\ell + 1$ are downhill. The following definition names a particular vertex of tier $0$.

**Definition 7.16.** Let $t^n_0 \in T^n_\ell$ be the vertex represented by the lattice $\Omega_\ell$.

There is a natural action of the group $\GL(2, \mathbb{Q}_\ell)$ on $\mathbb{Q}_\ell^2$, namely matrix multiplication with elements of $\mathbb{Q}_\ell^2$ considered as column vectors. We transfer this action to $K_\ell$ via the identification that we have made between $K_\ell$ and $\mathbb{Q}_\ell^2$. The action of $g \in \GL(2, \mathbb{Q}_\ell)$ is invertible, and preserves $\mathbb{Q}_\ell$-linear combinations, so it maps bases of $\mathbb{Q}_\ell^2$ to bases, maps lattices to lattices, and preserves $n$-homothety of lattices. Hence, multiplication by $g$ defines a bijection from $T^n_\ell$ to $T^n_\ell$. We now record some properties of this action.

**Lemma 7.17.** The action of an element $\gamma \in \GL(2, \mathbb{Z}_\ell)$ on $T^n_\ell$ permutes the vertices of $T^n_\ell$, fixes vertices of tier $0$, and preserves edges (including whether the edge is uphill or downhill) and the tier of each vertex.

**Proof.** Since the action of $\gamma \in \GL(2, \mathbb{Z}_\ell)$ is invertible, it is clear that the map it induces on vertices is a bijection. In addition, if $\Lambda_1 \subset \Lambda_2$ are lattices in $K_\ell$ with $[\Lambda_2 : \Lambda_1] = \ell$, then $\gamma\Lambda_1 \subset \gamma\Lambda_2$ with index $\ell$, so edges are preserved (including whether the edge is uphill or downhill).

Since the action of $\gamma$ stabilizes $\mathbb{Z}_\ell^2$, which is identified with $\Omega_\ell$, it fixes vertices of tier $0$. Since it preserves neighbors, a simple inductive argument shows that it maps each vertex to a vertex of the same tier. \hfill $\square$

**Lemma 7.18.** Multiplication by the fundamental unit $\epsilon \in K \subset K_\ell$ induces a permutation on the vertices of $T^n_\ell$ given (on the level of $\mathbb{Z}_\ell$-lattices) by multiplication by a matrix in $\GL(2, \mathbb{Z}_\ell)$.

**Proof.** Suppose that $\ell$ is inert in $K$. In this case (see Definition 7.1), we have identified $K_\ell = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell \omega$ with $\mathbb{Q}_\ell^2$. Since multiplication by $\epsilon$ is $\mathbb{Q}$-linear on $K$
it induces a \( \mathbb{Q}_\ell \)-linear map on \( K_\ell \). Hence, multiplication by \( \epsilon \) is represented by a matrix in \( \text{GL}(2, \mathbb{Q}_\ell) \). Since multiplication by \( \epsilon \) is an automorphism of \( \mathcal{O}_\ell \) and \( \mathcal{O}_\ell \) is identified with \( \mathbb{Z}_\ell^2 \subset \mathbb{Q}_\ell^2 \), this matrix has entries in \( \mathbb{Z}_\ell \), and since \( \epsilon \) has norm \( \pm 1 \), the matrix must have determinant \( \pm 1 \), so we see that the matrix is in \( \text{GL}(2, \mathbb{Z}_\ell) \).

Now suppose that \( \ell \) is split. Referring to Definition 7.1 again, we have identified \( K_\ell \) with \( \mathbb{Q}_\ell^2 \), where \( c \in K \) is identified with \((c, c') \in \mathbb{Q}_\ell^2 \). Hence, multiplication by \( \epsilon \) is represented by the matrix \( \text{diag}(\epsilon, \epsilon') \), which is in \( \text{GL}(2, \mathbb{Z}_\ell) \).

We will define functions on lattices in \( K \) as products of local functions on lattices in \( K_\ell \). To that end, we make the following definitions.

**Definition 7.19.** Let \( F(T^n_\ell) \) be the set of \( \mathbb{F} \)-valued functions on the vertices of \( T^n_\ell \).

**Definition 7.20.** The Laplace operator \( \Delta^\ell_n \) on \( F(T^n_\ell) \) is defined by

\[
\Delta^\ell_n(f)(t^n) = \sum_{y^n} f(u^n),
\]

where the sum runs over the \( \ell + 1 \) downhill neighbors \( u^n \in T^n_\ell \) of \( t^n \in T^n_\ell \).

Now we concentrate on the Hecke operator at \( \ell \) and describe how its coset representatives interact with lattices. We assume from now on that \( \ell \) is unramified in \( K \). We choose \( s = \text{diag}(\ell, 1) \) and refer to Definition 5.4 for the sets \( B_j \) and the integers \( d_j \). List \( B_j = \{s_\beta,j \mid \beta = 1, \ldots, d_j \} \). These definitions as well as the following definition depend on various choices, such as an element \( x_0 \in X \), a lift of \( x_0 \) to \( Y \), and a choice of \( \text{GL}(2, \mathbb{Z}) \)-orbit representatives \( \mathfrak{A} \) containing \( x_0 \). We will suppress these dependencies so as not to overburden the notation.

**Definition 7.21.** Let \( x_0 \) be represented by \( y = t(a, b) \in Y \) with \( a, b \in K \), and let \( \Lambda_y \) be the \( \mathbb{Z} \)-lattice in \( K \) generated by \( a \) and \( b \). For any \( j \) and any \( s_\alpha \in B_j \), define

\[
s_\alpha \Lambda_y = \Lambda_{s_\alpha y}.
\]

Note that \( s_\alpha \Lambda_y \) depends not just on the lattice \( \Lambda_y \), but also on the choice of basis \( y = t(a_0, b_0) \in Y \) for \( \Lambda_y \).

**Lemma 7.22.** Let \( s = \text{diag}(\ell, 1) \) and let

\[
\text{GL}(2, \mathbb{Z}) s \text{GL}(2, \mathbb{Z}) = \prod_{\alpha} \text{GL}(2, \mathbb{Z}) s_\alpha
\]

with the \( s_\alpha \) chosen and partitioned into the sets \( B_j \) as described in Definition 5.4. Let \( \Lambda_y \) be a lattice in \( K \) with a chosen \( \mathbb{Z} \)-basis \( y \in Y \). Then

(i) \( \mathcal{L} = \{s_\alpha \Lambda_y \} \) consists of the \( \ell + 1 \) lattices of index \( \ell \) contained in \( \Lambda_y \).

(ii) \( \mathcal{L} \) is partitioned into the subsets

\[
\mathcal{L}_j = \{s_\beta,j \Lambda_y \mid s_\beta,j \in B_j \},
\]

and \( |\mathcal{L}_j| = d_j \).

(iii) The same is true of the completions at \( \ell \): \( \mathcal{L}_\ell = \{s_\alpha \Lambda_y \ell \} \) consists of the \( \ell + 1 \) lattices of index \( \ell \) contained in \( (\Lambda_y)_\ell \), and these are partitioned into the subsets

\[
\mathcal{L}_{\ell, j} = \{(s_\beta,j \Lambda_y)_\ell | s_\beta,j \in B_j \}
\]

and \( |\mathcal{L}_{\ell, j}| = d_j \).
Proof. (i) Since $s_o$ is an integral matrix of determinant $\ell$, it is clear that $s_o\Lambda_y$ has index $\ell$ in $\Lambda_y$, and all sublattices of $\Lambda_y$ of index $\ell$ arise this way.

(ii) Since $\{s_o\}$ is partitioned by the sets $B_j$, it is clear that the lattices are partitioned as indicated.

(iii) If $\Lambda$ has index $\ell$ in $\Lambda_y$, then the completion $\Lambda_\ell$ has index $\ell$ in $(\Lambda_y)_\ell$, since taking completions of finitely generated modules is an exact functor. Given two lattices $\Lambda \neq \Lambda'$, each having index $\ell$ in $\Lambda_y$, we note that for all places $w \neq \ell$, $\Lambda_w = \Lambda'_w = (\Lambda_y)_w$. Since a lattice is determined by its completions at all finite places, we must have $\Lambda_\ell \neq \Lambda'_\ell$. 

$\square$

**Definition 7.23.** Let $\phi^\alpha_\ell \in F(T^n_\ell)$, let $x_0 \in \mathfrak{X}$, and let $\mathfrak{A}$ be any set of $\text{GL}(2, \mathbb{Z})$-orbit representatives of $\mathfrak{X}$ containing $x_0$, as in Definition 4.4. Define the sets $B_j$ in terms of $x_0$ and $\mathfrak{A}$ as in Definition 5.4. If, for all choices of $\mathfrak{A}$ and for all $y \in Y$ representing $x_0$, and for all $j = 1, \ldots, J$, we have that $\phi^\alpha_\ell$ is constant on the set

$$\{(s_{\beta,j}\Lambda_y)_\ell | \beta = 1, \ldots, d_j\}$$

of vertices of $T^n_\ell$, then we will say that $\phi^\alpha_\ell$ is locally constant relative to $T_\ell$ and $x_0$.

If $\phi^\alpha_\ell$ is locally constant relative to $T_\ell$ and all $x_0 \in \mathfrak{X}$, then we say that $\phi^\alpha_\ell$ is locally constant relative to $T_\ell$, or just that $\phi^\alpha_\ell$ is locally constant.

**Lemma 7.24.** Let $\phi \in F(T^n_\ell)$ be a function on the vertices of $T^n_\ell$. Assume that for every vertex $t^n \in T^n_\ell$, $\phi$ is constant on the set of non-idealistic outer downhill neighbors $u^n$ of $t^n$. Then $\phi$ is locally constant relative to $T_\ell$.

Proof. Assume that $\phi$ satisfies the conditions of the lemma. Let $x_0 \in \mathfrak{X}$, choose any collection $\mathfrak{A}$ of orbit representatives containing $x_0$ as in Definition 4.4, and choose any $y \in Y$ representing $x_0$. Partition the set $\{s_o\}$ of coset representatives for the Hecke operator $T_\ell$ as in Definition 5.4.

Let $t^n$ be the vertex of $T^n_\ell$ represented by $\Lambda_y$. For each set $B_j$, we wish to show that $\phi$ is constant on the set $\{(s_{\beta,j}\Lambda_y)_\ell | s_{\beta,j} \in B_j\}$. Let $1 \leq j \leq J$, choose any $s_{\beta,j} \in B_j$, and let $u^n$ be the downhill neighbor of $t^n$ represented by $(\Lambda_j)_\ell$, where $\Lambda_j = s_{\beta,j}\Lambda_y$. Then $\Lambda_j$ is $H(M,q)$-homothetic to a lattice with a basis representing $x_j$ (where $x_j \in \mathfrak{A}$ is given in Definition 5.4). We now divide the proof into 3 cases.

Case 1: Suppose $u^n$ is idealistic. Then $\Lambda_j$ is a fractional ideal of $K$, and it is $H(M,q)$-homothetic to a fractional ideal with basis representing $x_j$. Hence, $m_j = t^M$, so $d_j = 1$ by Lemmas 3.12 and 5.8. Hence, there is only one vertex on which $\phi$ must be constant.

Case 2: Suppose that $u^n$ is the unique downhill inner neighbor of $t^n$. Recall from Theorem 2.8 that the stabilizer $\Gamma_{x_0}$ of $x_0$ in $\text{GL}(2, \mathbb{Z})$ is generated by $-I$ and an element $\gamma_0$ that acts on $\Lambda_y$ as multiplication by $\delta_0 = \epsilon^m$ where $m = m_{x_0}$. From Theorem 2.8, we see that

$$\gamma_0\Lambda_y = \delta_0\Lambda_y = \Lambda_y.$$ 

Since multiplication by $\delta_0$ fixes $\Lambda_y$, it also fixes $(\Lambda_y)_\ell = t^n$. By Lemma 7.18, multiplication by $\delta_0$ also fixes each element of the fiber of $t^n_0 = \varpi(Q_\ell)$. Therefore it fixes the unique downhill path from $t^n$ to the fiber of $t^n_0$. Hence, multiplication by $\delta_0$ must fix $u^n$.

Now, both $\Lambda_j$ and $\delta_0\Lambda_j$ are sublattices of $\Lambda_y$ of index $\ell$. Since both must represent $u^n$, we see that they are equal. Since $\delta_0\Lambda_j = \Lambda_j$, we see that $m_j | m_0$, so that $d_j = 1$. Hence, again, there is only one vertex on which $\phi$ must be constant.
Let $\ell$ be a prime of $\mathcal{O}$ that does not divide $pdN$. We say that a function $f \in F(T_\ell^n)$ is $q$-homogeneous (or just homogeneous, if $q$ is understood) if, for all lattices $\Lambda$ in $K$,
\[ f(\ell \Lambda) = q(\ell) f(\Lambda). \]

**Definition 8.1.** Let $q : K_{(pdN)}^\times \to \mathbb{F}^\times$ be the character defined in Definition 3.5, and let $\ell$ be a prime of $\mathbb{Z}$ that does not divide $pdN$. We say that a function $f \in F(T_\ell^n)$ is $q$-homogeneous (or just homogeneous, if $q$ is understood) if, for all lattices $\Lambda$ in $K$,
\[ f(\ell \Lambda) = q(\ell) f(\Lambda). \]

**Definition 8.2.** For all finite places $w$ of $\mathbb{Q}$ unramified in $K$, let $n_w = 2$ if $w$ is inert in $K$, let $n_w = 1$ if $w$ splits in $K$. Fix a prime $\ell$ of $\mathbb{Q}$ not dividing $pdN$, and let $W$ be the set of all finite places of $\mathbb{Q}$ not dividing $\ell pdN$. For $w \in W$, let $\phi_w \in F(T_w^n)$ denote a homogeneous function such that $\phi_w(\Omega_w) = 1$. We view the functions $\phi_w$ as fixed by the context, and do not include them in the following notation for $\Phi$. For any homogeneous $\phi_\ell \in F(T_\ell^n)$, define the function $\Phi(\phi_\ell)$ on lattices $\Lambda$ in $K$ by the formula
\[ \Phi(\phi_\ell)(\Lambda) = \phi_\ell(\Lambda) \prod_{w \in W} \phi_w(\Lambda_w). \]

**Lemma 8.3.** The infinite product in the definition makes sense and $\Phi(\phi_\ell)$ is $q$-homogeneous. The map $\phi_\ell \mapsto \Phi(\phi_\ell)$ is $\mathbb{F}$-linear.

**Proof.** For any given $\Lambda$, we have that $\Lambda_w = \Omega_w$ for almost all $w$, so the product is actually finite. The linearity of the map $\phi_\ell \mapsto \Phi(\phi_\ell)$ is clear. Now suppose $\alpha \in \mathbb{Z}_{(pdN)}$ and $\Lambda$ is a lattice. Then $\alpha$ is prime to $pdN$ and factors as
\[ \alpha = \ell \lambda \prod_{w \in W} w^{f_w}. \]

Then
\[ \Phi(\phi_\ell)(\alpha \Lambda) = \phi_\ell(\alpha \Lambda) \prod_{w \in W} \phi_w(\alpha \Lambda_w) = \phi_\ell(\ell \lambda \Lambda) \prod_{w \in W} \phi_w(w^{f_w} \Lambda_w). \]

Since $\phi_\ell$ and all the $\phi_w$ are homogeneous, this equals
\[ q(\ell \lambda) \phi_\ell(\Lambda) \left( \prod_{w \in W} q(w^{f_w}) \right) \left( \prod_{w \in W} \phi_w(\Lambda_w) \right) = q(\alpha) \Phi(\phi_\ell)(\Lambda). \]

We now proceed to the main theorem of this section: the comparison between the Hecke operator and the Laplace operator.

By Lemma 2.5 and the fact that $H(M, q) \cap \Omega^\times$ is infinite, we see that for any lattice $\Lambda \subseteq K$, there is a minimal positive integer $m_\Lambda$ such that both $\epsilon^{m_\Lambda} \Lambda = \Lambda$.
and \(e^{mA} \in H(M,q)\). If \(A_1\) and \(A_2\) are \(K^x\)-homothetic lattices in \(K\), it is clear that \(m_{A_1} = m_{A_2}\). Set \(m_{\Lambda}' = m_{\Lambda}/i^M\). By Theorem 2.8, if \(\Lambda = \Lambda_y\) for \(y \in Y\), and \(x\) is the image in \(X\) of \(y\) then \(m_{\Lambda} = m_x\). Therefore, by Lemma 3.12, \(i^M|m_{\Lambda}\) and \(m_{\Lambda}'\) is a positive integer.

**Definition 8.4.** Let \(\psi_\ell \in F(T^\ell_t)\). We define the transform of \(\psi_\ell\) to be the function \(\hat{\psi}_\ell \in F(T^\ell_t)\) given by the formula

\[
\hat{\psi}_\ell(t^n) = m_{\Lambda}'\psi_\ell(t^n),
\]

where \(\Lambda\) is any lattice in \(\mathcal{O}\) of \(\ell\)-power index, such that \(\Lambda_{\ell}\) represents \(t^n\).

**Lemma 8.5.** Given \(\psi_\ell \in F(T^\ell_t)\), the transform \(\hat{\psi}_\ell\) is well defined.

**Proof.** We need to show that for \(t^n \in T^\ell_t\), the value of \(m_{\Lambda}\) does not depend on the lattice \(\Lambda\) chosen to represent \(t^n\). Note that up to homothety by powers of \(\ell^n\), there is a unique lattice \(\Lambda' \subseteq \mathcal{O}_\ell\) representing \(t^n\). By [19, V.2, Theorem 2] there is a unique lattice \(\Lambda' \subseteq \mathcal{O}\) of \(\ell\)-power index such that \(\Lambda_{\ell} = \Lambda'\). Since \(\Lambda'\) is uniquely defined up to homothety by powers of \(\ell^n\), so too is \(\Lambda\). Finally, since homothety does not change the value of \(m_{\Lambda}\), we see that \(m_{\Lambda}\) does not depend on the choice of \(\Lambda\), so \(m_{\Lambda}'\) does not.

If \(\psi_\ell(\mathcal{O}_\ell) = 1\), then \(\hat{\psi}_\ell(\mathcal{O}_\ell) = 1\), since \(m_\mathcal{O}' = 1\).

**Lemma 8.6.** Let \(\ell \nmid pdN\) be prime. If \(\psi_\ell \in F(T^\ell_t)\) is homogeneous, then \(\hat{\psi}_\ell\) is also homogeneous.

**Proof.** If \(t^n \in T^\ell_t\) is represented by \(\Lambda_\ell\), with \(\Lambda\) a lattice of \(\ell\)-power index in \(\mathcal{O}\), then \(\ell t^n\) is represented by \(\ell \Lambda_\ell\). Since \(m_{\Lambda}' = m_{\Lambda_{\ell}}\), we have

\[
\hat{\psi}_\ell(\ell t^n) = m_{\Lambda}'\psi_\ell(\ell t^n) = m_{\Lambda}'q(\ell)\psi_\ell(t^n) = q(\ell)\hat{\psi}_\ell(t^n).
\]

We now fix a set \(\mathfrak{A}\) of representatives of the \(GL(2,\mathbb{Z})\)-orbits in \(X\), as in Definition 4.4. Recall from Lemma 6.3 that this choice fixes an isomorphism between the cohomology group

\[
H^1(GL(2,\mathbb{Z}), \mathfrak{M}(M,q)^*)
\]

and \(q\)-homogeneous, \(H(M,q)\)-invariant functions on lattices.

As in Lemma 6.3 and its proof, if \(\Phi(\hat{\psi}_\ell)\) is \(H(M,q)\)-invariant and \(q\)-homogeneous, view \(\Phi(\hat{\psi}_\ell)\) as an \(\mathbb{F}\)-valued functional on \(H_1(GL(2,\mathbb{Z}), \mathfrak{M}(M,q))\), via the pairing

\[
\langle \Phi(\hat{\psi}_\ell), \bullet \rangle : H_1(GL(2,\mathbb{Z}), \mathfrak{M}(M,q)) \to \mathbb{F}.
\]

The relation between \(\Phi(\hat{\psi}_\ell)\) as a function on lattices and as a functional on homology is as follows: for a basis element \(z_x\) of \(H_1(GL(2,\mathbb{Z}), \mathfrak{M})\) with \(x \in \mathfrak{A}\), we have that

\[
\langle \Phi(\hat{\psi}_\ell), z_x \rangle = \Phi(\hat{\psi}_\ell)(\Lambda_y),
\]

where \(y \in Y\) is any representative of \(x\). Then, from the definition of \(\hat{\psi}_\ell\), we have that

\[
\langle \Phi(\hat{\psi}_\ell), z_x \rangle = \Phi(\hat{\psi}_\ell)(\Lambda_y) = m_{z_x}\Phi(\psi_\ell)(\Lambda_y),
\]

a fact that we will use in the proofs of Theorem 8.8 and Corollary 8.9.
Definition 8.7. Let $z_{x_j}$ be the chosen generator of $H_1(\Gamma_{x_j}, \mathbb{F})$ from Definition 2.11, and set $z_j = z_{x_j}$ and $m_j = m_{x_j}$ for $j = 0, \ldots, J$.

Theorem 8.8. Let $\ell \nmid pdN$ be prime. For each finite place $w \in W$, fix a $q$-homogeneous function $\phi_w \in F(T_{\ell}^{w})$, as in Definition 8.2. Let $n = n_{\ell}$, and let $\psi_{\ell} \in F(T_{\ell}^{\mathbb{A}})$ be $q$-homogeneous. Assume that $\Phi(\hat{\psi}_{\ell})$ is $H(M, q)$-homothety invariant. It will be $q$-homogeneous by Lemma 8.3.

If $\psi_{\ell}$ is locally constant relative to $T_{\ell}$ and $x_0$, then

$$\langle \Phi(\hat{\psi}_{\ell})T_{\ell}, z_0 \rangle = m'_0 \Phi(\Delta_{\ell}^q \psi_{\ell})(\Lambda_{y}),$$

where $y \in Y$ represents $x_0 \in \mathbb{A}$.

Proof. Choose a $y \in Y$ representing $x_0$. Then for $j = 1, \ldots, J$, set $y_j = t_j y$, so that $y_j$ represents $x_j$. We will write $\Lambda_j$ in place of $\Lambda_{y_j}$.

By Theorem 5.6, $T_{\ell} = \sum_{j=1}^J U_j$, and for $1 \leq j \leq J$, we have

$$U_j(z_0) = e_j z_j.$$

Then

$$\langle \Phi(\hat{\psi}_{\ell})T_{\ell}, z_0 \rangle = \langle \Phi(\hat{\psi}_{\ell}), T_{\ell} z_0 \rangle$$

$$= \sum_{j=1}^J e_j \langle \Phi(\hat{\psi}_{\ell}), z_j \rangle$$

$$= \sum_{j=1}^J e_j \Phi(\hat{\psi}_{\ell})(\Lambda_j)$$

$$= \sum_{j=1}^J e_j m'_j \Phi(\psi_{\ell})(\Lambda_j).$$

We have $e_j m'_j = m'_0 d_j$ by Lemma 5.8 and Definition 3.13, so

$$\langle \Phi(\hat{\psi}_{\ell})T_{\ell}, z_0 \rangle = \sum_{j=1}^J m'_0 d_j \Phi(\psi_{\ell})(\Lambda_j).$$

Now, for a fixed $j$, we will analyze the term $\Phi(\psi_{\ell})(\Lambda_j)$. Note that by definition,

$$\Phi(\psi_{\ell})(\Lambda_j) = \psi_{\ell}(\mu(A_j)) \prod_{w \in W} \phi_w((A_j)_w).$$

Since $t_j$ is an integral matrix with determinant $\ell$, we know that $t_j \in \text{GL}(2, \mathbb{D}_w)$ for all $w \in W$. Then $(A_j)_w$ is the same as the lattice $(\Lambda_{y_j})_w$. Set

$$c = \prod_{w \in W} \phi_w((A_{y_j})_w),$$

so that

$$\Phi(\psi_{\ell})(\Lambda_j) = \psi_{\ell}(\mu(A_j)) \cdot c.$$

Hence,

$$\langle \Phi(\hat{\psi}_{\ell})T_{\ell}, z_0 \rangle = cm'_0 \sum_{j=1}^J d_j \psi_{\ell}(\mu(A_j)) \cdot c.$$
On the other hand, since \( \psi_\ell \) is assumed to be locally constant with respect to \( T_\ell \) and \( x_0 \), any \( s_{\beta,j} \) takes any vertex to a downhill neighbor, and one of the \( s_{\beta,j} \) is equal to \( t_j \) (see Theorem 5.3(c)), we have that for each \( s_{\beta,j} \),

\[
\psi_\ell((s_{\beta,j}A_y)\ell) = \psi_\ell((t_jA_y)\ell) = \psi_\ell((A_y)\ell).
\]

Hence, using the fact that \( d_j = |B_j| \), we have

\[
\Phi(\Delta_\ell^p \psi_\ell)(A_y) = (\Delta_\ell^p \psi_\ell)((A_y)\ell) \prod_{w \in W} \phi_w((A_y)w)
\]

\[
= c(\Delta_\ell^p \psi_\ell)((A_y)\ell)
\]

\[
= c \sum_{j=1}^J \sum_{s_{\beta,j} \in B_j} \psi_\ell((s_{\beta,j}A_y)\ell)
\]

\[
= c \sum_{j=1}^J d_j \psi_\ell((A_y)\ell),
\]

where we have used Lemma 7.22. Multiplying both sides of the last equality by \( m'_0 \) yields the assertion of the theorem. \( \square \)

**Corollary 8.9.** Assume that \( \psi_\ell \) is locally constant relative to \( T_\ell \), that \( \psi_\ell(\Omega_\ell) = 1 \), that \( \Phi(\psi_\ell) \) is \( q \)-homogeneous and \( H(M,q) \)-invariant, and that \( \Delta_\ell^p \psi_\ell = \mu \psi_\ell \).

Then \( \Phi(\hat{\psi}_\ell) \), viewed as an element of \( H^1(\GL(2,\mathbb{Z}),\mathfrak{M}(M,q)^\ast) \), is an eigenclass for \( T_\ell \) with eigenvalue \( \mu \) and it is an eigenclass for \( T_{\ell,\ell} \) with eigenvalue \( \theta(\ell) \).

**Proof.** First, we show that \( \Phi(\hat{\psi}_\ell) \neq 0 \). By definition, for \( \Lambda \) a lattice in \( K \),

\[
\Phi(\hat{\psi}_\ell)(\Lambda) = \hat{\psi}_\ell(\Lambda) \prod_{w \in W} \phi_w(\Lambda_w).
\]

By construction, \( \phi_w(\Omega_w) = 1 \) for every \( w \in W \) and \( \psi_\ell(\Omega_\ell) = 1 \). Since \( m'_\Omega = 1 \), also \( \hat{\psi}_\ell(\Omega_\ell) = 1 \). Therefore, \( \Phi(\hat{\psi}_\ell)(\Omega) = 1 \).

Recall from Definition 4.4 that we have chosen a subset \( \mathcal{A} \subset \mathfrak{K} \), such that for every \( x \in \mathfrak{K} \), exactly one of \( x \) and \( x\xi \) is in \( \mathcal{A} \). For any \( x \in \mathcal{A} \), write \( \bar{z}_x = z_x - z_{x\xi} \). Then by Lemma 5.1, \( \{ \bar{z}_x : x \in \mathcal{A} \} \) is a basis of \( H_1(\GL(2,\mathbb{Z}),\mathfrak{M}(M,q)) \).

By Theorem 8.8, linearity, and the centrality of \( \xi \), for each \( x \in \mathcal{A} \) we have

\[
\langle \Phi(\hat{\psi}_\ell)T_\ell, \bar{z}_x \rangle = \langle \Phi(\hat{\psi}_\ell)T_\ell, z_x \rangle - \langle \Phi(\hat{\psi}_\ell)T_\ell, z_{x\xi} \rangle
\]

\[
= m'_x(\Phi(\Delta_\ell^n \psi_\ell)(A_y) - \Phi(\Delta_\ell^n \psi_\ell)(A_y\xi))
\]

\[
= m'_x(\Phi(\mu \psi_\ell)(A_y) - \Phi(\mu \psi_\ell)(A_y\xi))
\]

\[
= \mu(\Phi(m'_x \psi_\ell)(A_y) - \Phi(m'_x \psi_\ell)(A_y\xi))
\]

\[
= \mu(\Phi(\hat{\psi}_\ell)(A_y) - \hat{\psi}_\ell(A_y\xi))
\]

\[
= \mu(\langle \Phi(\hat{\psi}_\ell), z_x \rangle - \langle \Phi(\hat{\psi}_\ell), z_{x\xi} \rangle)
\]

\[
= \langle \mu(\Phi(\hat{\psi}_\ell), \bar{z}_x) \rangle.
\]

Since \( \Phi(\hat{\psi}_\ell) \) is in the dual space to \( H_1(\GL(2,\mathbb{Z}),\mathfrak{M}(M,q)) \), and \( \{ \bar{z}_x : x \in \mathcal{A} \} \) spans \( H_1(\GL(2,\mathbb{Z}),\mathfrak{M}(M,q)) \), we are finished with \( T_\ell \).
As for $T_{\ell,t}$, its action is given by the double coset of the central element $\ell I$. This is just a single coset, and its action on homology is given by $q(\ell)$, since it acts on $\mathcal{M}(M,q)$ as multiplication by $q(\ell)$.

Since $q(\ell) = \theta(\ell)$,

$$\langle \Phi(\hat{\psi}_t)T_{\ell,t}, x \rangle = \langle \Phi(\hat{\psi}_t), T_{\ell,t}x \rangle = \langle \Phi(\hat{\psi}_t), \theta(\ell)x \rangle = \langle \theta(\ell)\Phi(\hat{\psi}_t), x \rangle.$$ 

Hence, $\Phi(\hat{\psi}_t)T_{\ell,t} = \theta(\ell)\Phi(\hat{\psi}_t)$. \qed

9. Constructing locally constant eigenfunctions

Fix an $F$-valued character $\chi$ on the group of fractional ideals of $K$ relatively prime to $N$. In this section, we will construct locally constant $q$-homogeneous functions $\psi_0^{\ell}$ on $T_{\ell}^{\mathfrak{m}}$ that are eigenfunctions of the Laplace operator with eigenvalues related to $\chi$. We do this first for inert primes $\ell$.

**Theorem 9.1.** Let $\ell$ be a prime of $\mathbb{Q}$ that is inert in $K/\mathbb{Q}$ and does not equal the characteristic of $\mathbb{F}$. Then there is a locally constant $q$-homogeneous function $\psi_0^\ell \in F(T_{\ell}^2)$ that is an eigenvector of the Laplace operator with eigenvalue 0 and satisfies $\psi_0^\ell(\Theta_\ell) = 1$.

**Proof.** We define $\psi_0^\ell$ inductively.

For vertices of tier 0, we define $\psi_0^\ell(\Theta_\ell) = 1$ and $\psi_0^\ell(\ell\Theta_\ell) = \theta(\ell) = -1$. Then $\psi_0^\ell$ is homogeneous on the vertices of tier 0.

On vertices $t^2 \in T_{\ell}^2$ of tier 1, we define $\psi_0^\ell(t^2) = 0$. Clearly $\psi_0^\ell$ is $q$-homogeneous on vertices of tier 1. In addition, since all downhill neighbors of a vertex of tier 0 have tier 1, we can now compute $\Delta_\ell^2(\psi_0^\ell)$ on vertices of tier 0; we find that its value is 0, as desired. Finally, $\psi_0^\ell$ is constant on all downhill neighbors of vertices of tier 0.

On each vertex $t^2 \in T_{\ell}^2$ of tier 2, let $u^2 \in T_{\ell}^2$ be the unique uphill neighbor of $t^2$ of tier 1, and we let $u^2$ be the unique downhill neighbor of $u^2$ of tier 0. We define $\psi_0^\ell(t^2) = -\psi_0^\ell(u^2)/\ell$. Because the unique uphill neighbor of $\ell t^2$ of tier 1 is $\ell u^2$, which has a unique downhill neighbor of tier 0 equal to $\ell v^2$, we see that with this definition, $\psi_0^\ell$ is homogeneous on vertices of tier 1. In addition, for any vertex $u^2$ of tier 1, $\psi_0^\ell$ is constant on the downhill neighbors of $u^2$ of higher tier, since its value on such vertices depends only on its value on the unique downhill inner neighbor of $u^2$. Finally, we have constructed $\psi_0^\ell$ so that

$$\Delta_\ell^2(\psi_0^\ell)(u^2) = 0$$

for each vertex $u^2$ of tier 1.

We continue; for vertices $t^2 \in T_{\ell}^2$ of odd tier, we define $\psi_0^\ell(t^2) = 0$. This guarantees that for vertices $u^2$ of even tier, $\Delta_\ell^2(\psi_0^\ell)(u^2) = 0$, and that $\psi_0^\ell$ is constant on all downhill neighbors of $u^2$ of higher tier. Further, with this definition, $\psi_0^\ell(\ell t^2) = 0 = \theta(\ell)\psi_0^\ell(t^2)$ so that $\psi_0^\ell$ is homogeneous on vertices of odd tier.

For a vertex $t^2 \in T_{\ell}^2$ of positive even tier, let $u^2$ be the unique uphill inner neighbor of $t^2$, and let $v^2$ be the unique downhill inner neighbor of $u^2$. We define $\psi_0^\ell(t^2) = -\psi_0^\ell(v^2)/\ell$. Clearly $\psi_0^\ell$ is constant on all downhill outer neighbors of $u^2$ (since its value on such neighbors depends only on its value on $v^2$). As in the case of tier 2, we see that $\psi_0^\ell(\ell t^2) = \theta(\ell)\psi_0^\ell(t^2)$, and $\Delta_\ell^2(\psi_0^\ell)(u^2) = \psi_0^\ell(v^2)+\ell(-\psi_0^\ell(v^2)/\ell) = 0$.

With this construction, we see that $\psi_0^\ell$ is homogeneous, locally constant, and is an eigenfunction of $\Delta_\ell^2$ with eigenvalue 0. \qed
Lemma 9.2. For an inert prime $\ell$, the function $\psi^0_\ell$ defined above is $GL(2, \mathbb{Z}_\ell)$-invariant.

Proof. The action of $GL(2, \mathbb{Z}_\ell)$ fixes vertices of tier 0, and preserves uphill and downhill neighbors, and the tier of each vertex (Lemma 7.17). Since these relationships determine the values of $\psi^0_\ell$, the function is $GL(2, \mathbb{Z}_\ell)$-invariant. \qed

For a prime $\ell$ that splits in $K/\mathbb{Q}$ and does not divide $N$, we now prepare to construct a locally constant homogeneous function $\psi^0_\ell \in F(\mathcal{T}_\ell)$ that is an eigenfunction of $\Delta_\ell = \Delta^1_{\ell}$. For most of the remainder of this section, we will assume that $\ell$ splits in $K$, that $(\ell) = \lambda \lambda'$, and since $\ell \nmid N$, $\chi(\lambda)$ and $\chi(\lambda')$ are defined. In this case, the function that we construct will depend on the character $\chi$. Since we work in $\mathcal{T}_\ell = \mathcal{T}_0$, the concepts of uphill and downhill neighbor coincide.

We begin by defining some terminology and notation for subsets of $\mathcal{T}_\ell$.

Definition 9.3. We take $\Omega_\ell$ as the basepoint of $\mathcal{T}_\ell$ and denote it by $t_0$. A descendant of a vertex $t \in \mathcal{T}_\ell$ is a vertex $t_1 \neq t$ such that the path from $t_0$ to $t_1$ passes through $t$. Denote by $C(t)$ the set of all descendants $t'$ of $t$ such that every vertex of the path from $t$ to $t'$ except possibly $t$ is non-idealistic, and let $\overline{C}(t) = C(t) \cup \{t\}$. We call $C(t)$ the open cohort of $t$, and $\overline{C}(t)$ the closed cohort of $t$.

Definition 9.4. A simple chain starting at a vertex $t \in \mathcal{T}_\ell$ is a collection $C$ consisting of $t$ and descendants of $t$ such that for any pair $t', t'' \in C$, one of $t', t''$ is a descendant of the other. An apartment in $\mathcal{T}_\ell$ is a union of two infinite simple chains starting at a vertex $t$ and having no other vertices in common.

For future use, we state the next lemma for all unramified primes $\ell$.

Lemma 9.5. Let $t$ be an idealistic point in $\mathcal{T}_\ell$.

1. If $\ell$ is inert, then $t = t_0$.
2. If $(\ell) = \lambda \lambda'$ splits and $t$ is a distance $k > 0$ from $t_0$, then $t = \lambda^k t_0$ or $t = \lambda'^k t_0$, and both of these points are distance $k$ from $t_0$.
3. If $(\ell)$ splits and $k > 0$, then $\lambda^k t_0$ and $\lambda'^k t_0$ define distinct points in $\mathcal{T}_\ell$.
4. No descendant of an non-idealistic point in $\mathcal{T}_\ell$ is idealistic.
5. The vertices of $\mathcal{T}_\ell$ are partitioned into the closed cohorts $\overline{C}(t_1)$ as $t_1 = I_\ell$ runs over the idealistic points of $\mathcal{T}_\ell$ (where $I_\ell$ is an ideal of $\Omega_\ell$ of $\ell$-power norm.)
6. In the split case, the set of idealistic points of $\mathcal{T}_\ell$ form an apartment, namely

$$\{\lambda^k | k > 0\} \cup \{t_0\} \cup \{\lambda'^k | k > 0\}.$$ 

Proof. In the discussion following Definition 7.10, we proved that the set of idealistic nodes of $\mathcal{T}_\ell$ is $\{t_0\}$ if $\ell$ is inert and $\{\lambda^k | k > 0\} \cup \{t_0\} \cup \{\lambda'^k | k > 0\}$ if $\ell$ is split. Since $\lambda^k$ has index $\ell$ in $\lambda^{k-1}$, and similarly for the powers of $\lambda'$, (1) and (2) are now clear. As for (3), if $\lambda^k t_0 = \lambda'^k t_0$ in $\mathcal{T}_\ell$, then $\lambda^k = \ell^m \lambda'^k$ as ideals for some integer $m$, which is absurd.

If $\ell$ is inert, (4) and (5) are obvious.

Assume that $\ell$ splits. Then the idealistic point $\lambda^k t_0$ is at the end of a path containing the nodes $t_0, \lambda t_0, \ldots, \lambda^k t_0$. A similar statement holds for $\lambda'^k t_0$. Since every non-idealistic node is a descendant of $t_0$ and $\mathcal{T}_\ell$ is a tree, no idealistic point can be a descendant of a non-idealistic point. Hence (4) holds.

For any node $u \in \mathcal{T}_\ell$ consider the path from $t_0$ to $u$ (possibly of length 0.) Let $t_I$ be the last idealistic point in this path. Then $u \in \overline{C}(t_I)$ is in the closed cohort of
functions on $\theta$ and $J$. This clearly defines a unique sequence. For \(a\) define a sequence $t, \mu$.

Proof. Suppose $I$ and $J$ are both ideals of $O$ of $\ell$-power index, and that $I_{\ell}$ and $J_{\ell}$ are homothetic in $K_{\ell}$ by a power of $\ell$.

If $(\ell) = \lambda \lambda'$, then $I = \ell^a \mu^b$ and $J = \ell^c \nu^d$ for nonnegative integers $a, b, q, r$, and $\mu, \nu \in \{\lambda, \lambda'\}$. The fact that $I_{\ell}$ and $J_{\ell}$ are homothetic implies that $\mu^a = \nu^b$, so $I$ and $J$ differ by a factor of $\ell^{a-b}$. Since $c$ is trivial on $\ell O$, $c(I) = c(J)$. \qed

Let $t \in \mathcal{T}_\ell$. For any $s$ in the open cohort $C(t)$ of $t$, all of the neighbors of $s$ are in the closed cohort $\overline{C}(t)$. Hence, the Laplace operator $\Delta_\ell$ defines a linear map from functions on $\overline{C}(t)$ to functions on $C(t)$.

Lemma 9.8. Assume that $\ell$ is not equal to the characteristic of $F$. Let $\mu \in F$, and let $t$ be an idealistic point of $\mathcal{T}_\ell$ with closed cohort $\overline{C}(t)$. Then there is a unique $F$-valued function $\theta_{t, \mu}$ on $\overline{C}(t)$ with the following properties:

(i) $\theta_{t, \mu}(t) = 1$,

(ii) $\theta_{t, \mu}(s) = 0$ for every $s \in C(t)$ that is distance 1 from $t$,

(iii) $\theta_{t, \mu}(s)$ depends only on $\ell$, $\mu$, and the distance from $s$ to $t$,

(iv) $\Delta_\ell(\theta_{t, \mu})(s) = \mu \theta_{t, \mu}(s)$ for every $s \in C(t)$.

Proof. Define a sequence $a_k \in F$ for $k \geq 0$ by the recurrence relation $a_0 = 1$, $a_1 = 0$, and for $k \geq 2$,

\[ a_k = \frac{\mu a_{k-1} - a_{k-2}}{\ell}. \]

This clearly defines a unique sequence. For $s$ a distance $k$ from $t$ in $\overline{C}(t)$, set $\theta_{t, \mu}(s) = a_k$. With this definition, $\theta_{t, \mu}$ satisfies conditions (i), (ii), and (iii).

Given a point $s \in C(t)$ a distance $k$ from $t$, $s$ has one neighbor a distance $k - 1$ from $t$, and $\ell$ neighbors a distance $k + 1$ from $t$. Hence

\[ \Delta_\ell(\theta_{t, \mu})(s) = a_{k-1} + \ell a_{k+1} \]

\[ = a_{k-1} + \ell \left( \frac{\mu a_k - a_{k-1}}{\ell} \right) \]

\[ = \mu a_k \]

\[ = \mu \theta_{t, \mu}(s), \]

so $\theta_{t, \mu}$ satisfies condition (iv).

Conversely, if $\theta_{t, \mu}$ is a function on $\overline{C}(t)$ satisfying condition (iii), then for any $s$ a distance $k$ from $t$, we may define $a_k = \theta_{t, \mu}(s)$. If in addition $\theta_{t, \mu}$ satisfies conditions
(i), (ii), (iv), the $a_k$ satisfy the recurrence relation given above. The uniqueness of $\theta_t,\mu$ follows from the uniqueness of the sequence \{a_k\}.

**Definition 9.9.** Let $\mu \in \mathbb{F}$, and assume $\ell$ does not divide $N$ and does not equal the characteristic of $\mathbb{F}$, and that $\chi(\ell \mathcal{O}) = 1$. We define $\psi^0_\ell \in F(\mathcal{T}_\ell)$ by

$$\psi^0_\ell(s) = \hat{\chi}(t)\theta_{t,\mu}(s),$$

where $t \in \mathcal{T}_\ell$ is the unique idealistic vertex with $s \in \overline{C}(t)$.

**Lemma 9.10.** Let $\mu \in \mathbb{F}$ and assume that $\ell$ does not divide $N$ and does not equal the characteristic of $\mathbb{F}$.

1. $\psi^0_\ell(\mathcal{O}_\ell) = 1$.  
2. $\psi^0_\ell$ is locally constant with respect to $\mathcal{T}_\ell$.  
3. If $\mu = \chi(\lambda) + \chi(\lambda')$, then

$$\Delta_\ell \psi^0_\ell(s) = \mu \psi^0_\ell(s).$$

**Proof.** The first assertion is immediate from the definitions.

Let $s$ be any vertex in $\mathcal{T}_\ell$. We wish to show that $\psi^0_\ell$ is constant on all non-idealistic outer downhill neighbors $u$ of $s$. Then, by Lemma 7.24, we will obtain (2). Let $s \in \overline{C}(t)$ with $t$ idealistic. Then any such $u$ will be in $C(t)$. Since $\hat{\chi}(t)$ is constant for all points in $C(t)$, we need only show that $\theta_{t,\mu}(u)$ is constant for all such $u$. Letting the distance from $t$ to $s$ be $k - 1$, the distance from $t$ to $u$ will be $k$. Hence, the desired constancy follows from Lemma 9.8(iii).

For (3), again assume $s \in \overline{C}(t)$ with $t$ idealistic. Suppose that $s = t = I_\ell$ is idealistic, where $I$ is an ideal of $\ell$-power index in $\mathcal{D}$. Then $s$ has exactly two idealistic neighbors, namely $(\lambda I)_\ell$ and $(\lambda'I)_\ell$. The nonidealistic neighbors $u$ of $s$ are all in $C(t)$ and have distance 1 from $t$; hence $\theta_{t,\mu}$ vanishes on them all. Hence

$$(\Delta_\ell \psi^0_\ell(s)) = \chi(\lambda) + \chi(\lambda'I) = (\chi(\lambda) + \chi(\lambda'))\chi(I) = \mu \psi^0_\ell(s).$$

Finally, suppose that $s$ is non-idealistic. Then it belongs to the open cohort $C(t)$. Then

$$(\Delta_\ell \psi^0_\ell(s)) = \sum_u \psi^0_\ell(u)$$

$$= \sum_u \hat{\chi}(t)\theta_{t,\mu}(u)$$

$$= \hat{\chi}(t) \sum_u \theta_{t,\mu}(u)$$

$$= \hat{\chi}(t)(\Delta_\ell \theta_{t,\mu})(s)$$

$$= \mu \hat{\chi}(t)\theta_{t,\mu}(s)$$

$$= \mu \psi^0_\ell(s),$$

by Lemma 9.8(iv), where the sums run over all neighbors $u$ of $s$. □

**10. $H(M, q)$-invariance**

In this section, we construct a function on lattices as a product of the local eigenfunctions for the Laplacian constructed in the previous section, and prove that the resulting function is $H(M, q)$-invariant and $q$-homogeneous, which implies by Theorem 8.8 and Corollary 8.9 that it corresponds to a Hecke eigenclass in $H^1(GL(2, \mathbb{Z}), \mathfrak{M}(M, q)^*)$. 

Lemma 10.1. Fix a prime $\ell$ that is unramified in $K$, and let $n = 1$ if $\ell$ is inert. Let $\Lambda$ be a $\mathbb{Z}$-lattice in $K$, and let $\alpha \in K^\times$. Let $s^n$ be the vertex in $T^n$ corresponding to $\Lambda$, and let $u^n$ be the vertex corresponding to $(\alpha \Lambda)$. Factor the fractional ideal $\alpha \mathcal{O} = I_1 I_2$, where $I_1$ has norm a power of $\ell$ and $I_2$ is prime to $\ell$.

1. There exists a matrix $g \in \text{GL}(2, \mathbb{Q}_\ell)$ depending only on $\alpha$ (independent of $\Lambda$), such that $u^n = g s^n$. If $\ell$ is inert, then $g = \ell^k g'$ with $k \in \mathbb{Z}$ and $g' \in \text{GL}(2, \mathbb{Z}_\ell)$.
2. The vertex $s^n$ is idealistic if and only if $u^n$ is idealistic. If $s^n$ corresponds to $\Lambda_\ell$ with $\Lambda$ an ideal, then $u^n$ corresponds $(I_1 \Lambda)_\ell$.
3. Suppose $\ell$ is split. Assume that $s^n$ is not idealistic, but lies in the open cohort $C(t)$ of the idealistic point $t^n = M_\ell$, where $M$ is an ideal of $\ell$-power norm. Then $u^n$ lies in the open cohort $C(t^n_1)$, where $t^n_1 = (I_1 M)_\ell$ and the distance between $s^n$ and $t^n$ is the same as the distance between $u^n$ and $t^n_1$.

Proof. (1) First, suppose that $\ell$ is inert. Via our identification of $K_\ell$ with $\mathbb{Q}_\ell^2$, multiplication by $\alpha$ is a $\mathbb{Q}_\ell$-linear isomorphism from $\mathbb{Q}_\ell^2$ to $\mathbb{Q}_\ell^2$; hence, it is given by a matrix $g \in \text{GL}(2, \mathbb{Q}_\ell)$. We can write $\alpha \in K_\ell$ as $\alpha = \ell^k \eta$ for some $k \in \mathbb{Z}$, and some unit $\eta \in \mathcal{O}_\ell^\times$; multiplication by $\eta$ is given by a matrix in $\text{GL}(2, \mathbb{Z}_\ell)$.

Now assume that $\ell$ is split. In this case, we identify $K_\ell$ with $\mathbb{Q}_\ell^2$ by mapping $\alpha$ to $(\alpha, \alpha')$. Then multiplication by $\alpha$ is defined by the matrix
\[
\begin{pmatrix}
\alpha & 0 \\
0 & \alpha'
\end{pmatrix},
\]
which is in $\text{GL}(2, \mathbb{Q}_\ell)$.

(2) $\Lambda$ is a fractional ideal if and only if $\alpha \Lambda$ is a fractional ideal. If $\Lambda = MP$ with $M$ a fractional ideal of $\ell$-power norm, and $P$ a fractional ideal prime to $\ell$, then
\[
(\alpha \Lambda)_\ell = (I_1 M)_\ell = (I_1 \Lambda)_\ell.
\]

(3) Let $g \in \text{GL}(2, \mathbb{Q}_\ell)$ be the matrix from part (1) corresponding to multiplication by $\alpha$. Multiplication by $g$ is then an automorphism of $T_\ell$ that takes idealistic vertices to idealistic vertices, and non-idealistic vertices to non-idealistic vertices. Let $R$ be a simple path from $t^n$ to $s^n$ whose only idealistic vertex is $t^n$. Then $gR$ is a simple path from $gt^n$ to $u^n$ of the same length as $R$, whose only idealistic vertex is $gt^n$. Moreover, $u^n$ lies in the open cohort $C(gt^n)$ where $gt^n = (I_1 M)_\ell$. \qed

Theorem 10.2. Let $\mathbb{F}$ be a field of characteristic 0 or of finite characteristic not equal to two. If $\mathbb{F}$ has characteristic 0, set $p = 1$, and otherwise let $p$ be the characteristic of $\mathbb{F}$. Assume that $\chi$ is trivial on principal ideals generated by elements of $\mathbb{Z}_{(pN)}^\times$. Also assume that $\chi$ is trivial on principal ideals generated by elements of $K(M, q)$. Let $\Phi$ be the function from lattices in $K$ to $\mathbb{F}$ defined by
\[
\Phi(\Lambda) = \prod_{w \mid p \mid N} \psi_w^0(\Lambda_w),
\]
where $\psi_w^0$ is given by Theorem 9.1 if $w$ is inert in $K/\mathbb{Q}$, and by Definition 9.9 if $w$ splits in $K/\mathbb{Q}$.

Then $\Phi(\alpha \Lambda) = \Phi(\Lambda)$ for all $\alpha \in H(M, q)$ and all lattices $\Lambda$ in $K$.

Moreover, $\Phi(\alpha \Lambda) = q(\alpha) \Phi(\Lambda)$ for all $\alpha \in \mathbb{Z}_{(pN)}^\times$. 

Proof. Let $\Lambda$ be a lattice in $K$ and let $\alpha \in K(M,q)$. Note that $m'_{\Lambda} = m'_{\alpha \Lambda}$, since $K^\times$ is commutative. Hence, there is a single integer $m'$ depending on $\Lambda$, such that for each prime $w \nmid pdN$, we have

$$\hat{\psi}_w^0(\Lambda_w) = m'\psi_w^0(\Lambda_w)$$

and

$$\hat{\psi}_w^0((\alpha \Lambda)_w) = m'\psi_w^0((\alpha \Lambda)_w).$$

Assume first that $w$ is inert in $K$. Then we may factor $\alpha \mathcal{O}$ as

$$\alpha \mathcal{O} = w^j I_2,$$

with $I_2$ a fractional ideal that is relatively prime to $w$. By Lemma 10.1(1), we have

$$(\alpha \Lambda)_w = w^j g \Lambda_w$$

for some $g \in \text{GL}(2, \mathbb{Z}_w)$. Since $\psi_w^0$ is homogeneous and is $\text{GL}(2, \mathbb{Z}_w)$-invariant on $\mathcal{T}_w$ (by Lemma 9.2), we have

$$\hat{\psi}_w^0((\alpha \Lambda)_w) = m'\psi_w^0(w^j g \Lambda_w) = m'\psi_w^0(\Lambda_w) = q(w^j)\hat{\psi}_w^0(\Lambda_w).$$

Now assume that $w$ splits in $K$. Let $s \in \mathcal{T}_\ell$ be the vertex corresponding to $\Lambda_w$, and let $u$ correspond to $(\alpha \Lambda)_w$.

If $s$ is idealistic, so is $u$, and we see that

$$\psi_w^0(s) = \hat{\chi}(s)\theta_{s,\mu}(s) = \hat{\chi}(s) = \chi(\Lambda) = \chi(\alpha \Lambda) = \chi(u) = \hat{\chi}(u)\theta_{u,\mu}(u) = \psi_w^0(u).$$

If $s$ is nonidealistic, then so is $u$, and $u = gs$ for some $g \in \text{GL}(2, \mathbb{Q}_w)$. Suppose $s$ lies in the open cohort $C(t)$ of the idealistic vertex $t$ corresponding to $I_w$, where $I$ is an ideal of $w$-power index in $\mathcal{O}$. By Lemma 10.1(3), $u$ is in the open cohort $C(t_1)$ of the idealistic point $t_1$ corresponding to $(I_1 I)_w$, where $\alpha \mathcal{O} = I_1 I_2$, with $I_1$ having norm a power of $w$, and $I_2$ having norm relatively prime to $w$. In addition, the distance from $s$ to $t$ is the same as the distance from $u$ to $t_1$. Hence,

$$\hat{\chi}(t_1) = \chi(I_1 I) = \chi(I_1)\chi(I) = \chi(I_1)\hat{\chi}(t)$$

and

$$\theta_{t_1,\mu}(s) = \theta_{t_1,\mu}(u).$$

Therefore,

$$\hat{\psi}_w^0((\alpha \Lambda)_w) = m'\psi_w^0(u)$$

$$= m'\hat{\chi}(t_1)\theta_{t_1,\mu}(u)$$

$$= m'\chi(I_1)\hat{\chi}(t)\theta_{t_1,\mu}(u)$$

$$= m'\chi(I_1)\hat{\chi}(t)\theta_{t_1,\mu}(s)$$

$$= \chi(I_1)\hat{\psi}_w^0((\alpha \Lambda)_w).$$

In all of this, the fractional ideal $I_1$ depends on $w$; we will call it $I_\alpha(w)$. Then $I_\alpha(w)$ is a product of powers of primes lying over $w$; if $w$ is inert, it is clear that $I_\alpha(w)$ is principal with a generator $\beta_\alpha(w)$ in $\mathbb{Z}_w^\times$, so that $\chi(I_\alpha(w)) = 1$.

Since $\alpha \in K(M,q)$, $\alpha$ is relatively prime to $pdN$, so that

$$\alpha \mathcal{O} = \prod_{w \nmid pdN} I_\alpha(w) = \left( \prod_{w \text{ inert}} I_\alpha(w) \right) \left( \prod_{w \text{ split}} I_\alpha(w) \right).$$
Setting $\beta = \prod_{w \text{ inert}} \beta_\alpha(w)$, we have

$$\beta \mathcal{O} = \left( \prod_{w \text{ inert}} I_\alpha(w) \right).$$

Since $\alpha \in K(M, q)$, $q(\alpha) = 1$. Because $q$ depends only on inert prime factors, and the powers of inert primes dividing $\alpha$ and $\beta$ are equal, we see that

$$1 = q(\alpha) = q(\beta).$$

In addition, we have that $\chi(\beta \mathcal{O}) = 1$, since $\beta$ is a product of powers of elements of $\mathbb{Z}_{(pdN)}^\times$, and we have assumed that $\chi$ is trivial on ideals generated by elements of $\mathbb{Z}_{(pdN)}^\times$. Hence, we see that

$$\prod_{w \text{ split}} I_\alpha(w)$$

is principal, with generator $\alpha/\beta$, so

$$\prod_{w \text{ split}} \chi(I_\alpha(w)) = \frac{\chi(\alpha \mathcal{O})}{\chi(\beta \mathcal{O})} = \chi(\alpha \mathcal{O}) = 1,$$

since we have assumed that $\chi$ is trivial on principal ideals generated by elements of $K(M, q)$.

Hence, we obtain

$$\Phi(\alpha \Lambda) = \prod_{w \mid pdN} \hat{\psi}_w^0((\alpha \Lambda)_w)$$

$$= \left( \prod_{w \text{ inert}} \hat{\psi}_w^0((\alpha \Lambda)_w) \right) \left( \prod_{w \text{ split}} \hat{\psi}_w^0((\alpha \Lambda)_w) \right)$$

$$= \left( \prod_{w \text{ inert}} q(\beta_\alpha(w)) \hat{\psi}_w^0(\Lambda_w) \right) \left( \prod_{w \text{ split}} \chi(I_\alpha(w)) \hat{\psi}_w^0(\Lambda_w) \right)$$

$$= q(\beta) \left( \prod_{w \text{ split}} \chi(I_\alpha(w)) \right) \prod_{w \mid pdN} \hat{\psi}_w^0(\Lambda_w)$$

$$= \Phi(\Lambda),$$

so $\Phi$ is $K(M, q)$-invariant.

Next, if $\alpha \in \mathbb{Z}_{(pdN)}^\times$, it is a product of powers of primes not dividing $pdN$. We may thus assume that $\alpha$ is such a prime. The $q$-homogeneity of $\Phi$ then follows by Lemma 8.3 from the homogeneity of the individual $\psi_w^0$ functions (see Theorem 9.1 for inert primes, and note that homogeneity is trivial for split primes).

Finally, $K(M, q)$-invariance and $q$-homogeneity imply that $\Phi$ is $H(M, q)$-invariant.\[\square\]

11. Galois representations

We now define the Galois representations to which our main theorem below applies.

As before, we let $K$ be a real quadratic field of discriminant $d$, cut out by the Dirichlet character $\theta$. Let $\mathbb{F}$ be a field of characteristic 0 (in which case we set
Let \( p = 1 \) or a field of odd characteristic \( p \), let \( G_K \) be the absolute Galois group of \( K \) (i.e. \( \text{Gal}(\overline{\mathbb{Q}}/K) \)), and let \( \chi : G_K \to \mathbb{F}_p^\times \) be a character of \( G_K \) with finite image. By class field theory, we can think of \( \chi \) as a character on the group of the nonzero fractional ideals of \( K \) relatively prime to \( N \) for some positive \( N \in \mathbb{Z} \). Let \( L \) be the fixed field of the kernel of \( \chi \). Then \( L/K \) is Galois. We fix a positive integer \( M \) that divides \( p \mathbf{d} N \) and define \( K^{(\mathbf{d} N)}_M \) and \( K(M, q) \) as in Definition 3.3, and \( K(M, q) \) as in Definition 3.7.

We place the following conditions on the character \( \chi \).

1. \( \chi \) is trivial on the principal fractional ideals of \( K \) generated by elements of \( K(M, q) \).
2. \( \chi \) is trivial on the principal fractional ideals of \( K \) generated by elements of \( \mathbb{Q} \) that are prime to \( p \mathbf{d} N \).
3. \( [L : K] \) is odd.
4. \( L/\mathbb{Q} \) is Galois.

As mentioned in the introduction, any ring class character of an order of \( K \) that cuts out a Galois extension of \( \mathbb{Q} \) of odd degree over \( K \) will satisfy these conditions, for appropriate choices of \( M \) and \( N \). As a special case of this, any unramified character of odd order will satisfy these conditions.

Let \( \rho : G_\mathbb{Q} \to \text{GL}(2, \mathbb{F}) \) be the induced representation
\[
\rho = \text{Ind}_{G_K}^{G_\mathbb{Q}} \chi.
\]
Note that this representation will factor through \( \text{Gal}(L/\mathbb{Q}) \). We have an exact sequence
\[
1 \to \text{Gal}(L/K) \to \text{Gal}(L/\mathbb{Q}) \to \text{Gal}(K/\mathbb{Q}) \to 1;
\]
since \( [L : K] \) is odd, this sequence splits, so there is an element \( \tau \) of order 2 in \( \text{Gal}(L/\mathbb{Q}) \) mapping to the nonidentity element of \( \text{Gal}(K/\mathbb{Q}) \); we can lift it to an element \( \tau \in G_\mathbb{Q} \), and we have that \( \tau^2 \) is the identity modulo \( G_L \).

With respect to a suitable basis, it is easy to see that for \( g \in G_\mathbb{Q} \), we have the following:

(a) If \( g \in G_K \), then
\[
\rho(g) = \begin{pmatrix} \chi(g) & 0 \\ 0 & \chi(g') \end{pmatrix},
\]
where \( g' = \tau^{-1} g \tau \).

(b) If \( g \notin G_K \), then \( g = h \tau \) for some \( h \in G_K \), and
\[
\rho(g) = \begin{pmatrix} 0 & \chi(h') \\ \chi(h \tau^2) & 0 \end{pmatrix},
\]
where \( h' = \tau^{-1} h \tau \).

If we now let \( g \) be a Frobenius element in \( G_\mathbb{Q} \) for some prime \( \ell \) of \( \mathbb{Q} \) not dividing \( p \mathbf{d} N \) (so that \( \ell \) is unramified in \( L/\mathbb{Q} \)), then we have the following two cases.

If \( \ell \) splits in \( K \) and \( \ell \nmid N \), then \( g \in G_K \). If we write \( \ell \mathfrak{D} = \lambda \lambda' \) with \( \lambda, \lambda' \) primes in \( K \), then we may take \( g \) to be a Frobenius element in \( G_K \) of \( \lambda \); a Frobenius element of \( \lambda' \) will be \( g' \). Hence, we have
\[
\text{Tr}(\rho(g)) = \chi(g) + \chi(g') = \chi(\lambda) + \chi(\lambda'),
\]
and
\[
\det(\rho(g)) = \chi(g)\chi(g') = \chi(\lambda)\chi(\lambda') = \chi(\lambda \lambda') = \chi(\ell \mathfrak{D}) = 1
\]
by condition (2) on the character \( \chi \).
On the other hand, if $\ell$ is inert in $K$ and $\ell \nmid N$, write $g = h\tau$ as above. Then
\[
\text{Tr}(\rho(g)) = 0
\]
and $\det(\rho(g)) = -\chi(h\tau^2)\chi(h')$ with $h' = \tau^{-1}h\tau$. We note that $g^2$ is a Frobenius element of $\ell\mathcal{O}$ in $G_K$. Hence, we have
\[
\det(\rho(g)) = -\chi(h\tau^2)\chi(h') = -\chi(h\tau h') = -\chi((h\tau)^2) = -\chi(g^2) = -\chi(\ell\mathcal{O}) = -1,
\]
where we have again used condition (2) on $\chi$.

Note that in each case, when $g$ is a Frobenius element in $G_Q$ of $\ell$, we have $\det(\rho(g)) = \theta(\ell)$.

Now we check that $\rho$ is even. Let $c \in G_Q$ be a complex conjugation. Since $c$ has order 2 and $\chi$ has odd order, $\chi(c) = \chi(\tau^{-1}ct) = 1$. From the explicit description of the matrices $\rho(g)$ above, since $c \in G_K$, $\rho(c)$ is the identity matrix.

**Theorem 11.1.** Let $K$ be a real quadratic field of discriminant $d$, let $\mathbb{F}$ be a field of characteristic 0 or a finite field of odd characteristic. In the first case set $p = 1$ and in the second case let $p$ be the characteristic of $\mathbb{F}$. Let $\chi : G_K \to \mathbb{F}^\times$ be a character with finite image. Let $L$ be the fixed field of the kernel of $\chi$ and choose $N \in \mathbb{Z}$ so that $L/K$ is unramified outside primes of $K$ dividing $N$. Let $M$ be a positive divisor of $pdN$, $\theta$ the Dirichlet character cutting out $K$, $q$ the extension of $\theta$ defined in Definition 3.5, and $\mathfrak{M}(M,q)$ the module defined in Definition 4.2. Assume

1. $\chi$ is trivial on the principal fractional ideals of $K$ generated by elements of $K(M,q)$.
2. $\chi$ is trivial on the principal fractional ideals of $K$ generated by elements of $\mathbb{Q}^\times$ that are prime to $pdN$.
3. $[L : K]$ is odd.
4. $L/\mathbb{Q}$ is Galois.

Then $\rho : G_Q \to \text{GL}(2, \mathbb{F})$ given by $\rho = \text{Ind}_{G_K}^{G_Q} \chi$ is an even Galois representation, and is attached to a Hecke eigenclass in $H^1(\text{GL}(2, \mathbb{Z}), \mathfrak{M}(M,q)^\times)$.

**Proof.** Recall that for $w$ inert in $K$ and prime to $pN$, we have constructed a function $\psi_w^0$ in the proof of Theorem 9.1, and for $w$ split in $K$ and prime to $pN$, we defined a function $\tilde{\psi}_w^0$ in Definition 9.9. Given $\chi$ satisfying the conditions of the theorem, we define an $\mathbb{F}$-valued function $\Phi$ on lattices,
\[
\Phi(\Lambda) = \prod_{w \mid pdN} \tilde{\psi}_w^0(\Lambda_w)
\]
where $\tilde{\psi}_w^0$ is the transform (see Definition 8.4) of the function $\psi_w^0$.

By Theorem 10.2, $\Phi$ is $H(M,q)$-invariant and $q$-homogeneous. Hence, by Lemma 6.3 we may consider it as an element of $H^1(\text{GL}(2, \mathbb{Q}), \mathfrak{M}(M,q)^\times)$. By Corollary 8.9, combined with Lemma 9.10 and Theorem 9.1 we see that for all $\ell$ prime to $pdN$, $\Phi$ is an eigenvector for $T_\ell$ and $T_{\ell,\ell}$, and that the eigenvalues of $T_\ell$ match the trace of $\rho(\text{Frob}_\ell)$. The $q$-homogeneity of $\Phi$ shows that the eigenvalues of $T_{\ell,\ell}$ match the determinant of $\rho(\text{Frob}_\ell)$ for all $\ell$ prime to $pdN$. Hence, $\Phi$ is attached to $\rho$. 

**References**