REDUCIBLE GALOIS REPRESENTATIONS AND THE HOMOLOGY OF $GL(3,\mathbb{Z})$

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ABSTRACT. Let $\bar{\mathbb{F}}_p$ be an algebraic closure of a finite field of characteristic p. Let ρ be a continuous homomorphism from the absolute Galois group of $\mathbb{Q}$ to $\mathrm{GL}(3,\bar{\mathbb{F}}_p)$ which is isomorphic to a direct sum of a character and a two-dimensional odd irreducible representation. Under the condition that the Serre conductor of ρ is squarefree, we prove that ρ is attached to a Hecke eigenclass in the homology of an arithmetic subgroup Γ of $\mathrm{GL}(3,\mathbb{Z})$. In addition, we prove that the coefficient module needed is, in fact, predicted by the main conjecture of [3].

1. Introduction

Fix a prime p and let $\bar{\mathbb{F}}_p$ be an algebraic closure of a finite field of characteristic p. Generalizations of Serre's conjecture [13] connect the homology of arithmetic groups Γ with Galois representations ρ. When Γ is a congruence subgroup of $\mathrm{GL}(n,\mathbb{Z})$ and the target of ρ is $\mathrm{GL}(n,\bar{\mathbb{F}}_p)$, such a conjecture was first published in [5], extended in [3] and further improved in [8]. See Section 2 for a version of the conjecture in the cases that we treat here.

A common way to study the homology of a group G is to use a resolution $C_\bullet$ of $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules. The homology of G with coefficients in a G-module M is then the homology of $C_\bullet \otimes_G M$. In this paper, where G is a congruence subgroup of $\mathrm{GL}(n,\mathbb{Z})$, we use instead a resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules that have large but well-understood stabilizers, namely the chains on the simplex spanned by the points of projective n-space (or more exactly a certain subcomplex of this, spliced with the sharbly complex of [4]). The stabilizers are parabolic subgroups whose homology is known. The homology of $C_\bullet \otimes_G M$ is replaced by a spectral sequence whose E^1 page is computable, via Shapiro's Lemma, from the homology of the stabilizers.

This spectral sequence is Hecke equivariant. We give a general formula for computing Hecke operators acting on the kind of modules that appear in the E^1 page. A packet of Hecke eigenvalues appearing in the $E^1_{p,q}$-term will appear in the homology of G, if it survives to $E^\infty_{p,q}$. We can show this if we can determine that the given packet does not occur in any of the terms of the spectral sequence that can hit the $E^1_{p,q}$-term under the differentials. (For this to work we need finite generation of the relevant E^1-terms over the ground field.) This is what we do for the reducible representations we study in this paper.

Our main theorem (Theorem 2.5) states that for any $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(3,\bar{\mathbb{F}}_p)$ with $p > 3$ prime, such that ρ is a direct sum of a two-dimensional odd irreducible representation and a character, and the Serre conductor of ρ is squarefree, there is a Hecke eigenclass in the homology $H_3(\Gamma_0(3,N), V \otimes \epsilon)$ that is attached to ρ,

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where N, V, and ϵ are predicted by Conjecture 2.4. In addition, for any such ρ, there may be several predicted values for V; we will show that all of them yield eigenclasses with ρ attached. (The definition of odd is recalled in the paragraph before Conjecture 2.4.)

Let X_n be the Borel-Serre bordification of the symmetric space for $\mathrm{GL}(n, \mathbb{R})$. The Galois representations that we consider should be attached to eigenclasses in the homology of the boundary of X_3/Γ. However, when $n > 2$, the mod p topology of the boundary of X_n/Γ is quite complicated, and becomes increasingly intricate as n increases. Our method is designed to avoid the need to compute the homology of this boundary.

A similar approach may eventually attach Galois representations to the boundary homology of congruence subgroups of $\mathrm{GL}(n, \mathbb{Z})$ for any n, but some new ideas will be needed. So far, we can only treat $n = 2, 3$ (except for the case of sums of characters in [1]). There are two reasons we cannot immediately extrapolate the methods below even to $GL(4)$. First, the spliced complex we use does not work for $n > 3$, for a reason we point out after the proof of Lemma 7.2. A more serious problem is the following: the reason that the Hecke eigenpacket of interest does not occur in any of the terms of the spectral sequence that can hit the $E^1_{p,q}$-term in which it does occur, is that these other terms all boil down to H_0's and H_1's, and it is known from earlier work that these low dimensional homology groups can only support Hecke eigenpackets whose attached Galois representations are the sums of 3 characters. Such general facts are not known for H_k, for $k > 1$ and $n > 3$.

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2. Conjectural connections between Galois representations and arithmetic homology

A Hecke pair (Γ, S) in a group G is a subgroup $\Gamma \subset G$ and a subsemigroup $S \subset G$ such that $\Gamma \subset S$ and for any $s \in S$ both $\Gamma \cap s^{-1}\Gamma s$ and $s\Gamma s^{-1} \cap \Gamma$ have finite index in Γ.

If we let R be a ring and M a right $R[S]$-module, then for $s \in S$ there is a natural action of a double coset $\Gamma s \Gamma$ on the homology $H_i(\Gamma, M)$. We denote by $\mathcal{H}(\Gamma, S)$ the R-algebra under convolution generated by all the double cosets $\Gamma s \Gamma$ with $s \in S$. We call $\mathcal{H}(\Gamma, S)$ a Hecke algebra, and the double cosets Hecke operators. The action of the double cosets on homology makes $H_i(\Gamma, M)$ an $\mathcal{H}(\Gamma, S)$-module.

We will use the following groups and semigroups in GL_n.

Definition 2.1. *Let N be a positive integer and p a prime.*

1. *$S_0(n, N)^{\pm}$ is the semigroup of matrices $s \in M_n(\mathbb{Z})$ such that $\det(s)$ is relatively prime to pN and the first row of s is congruent to $(*, 0, \dots, 0)$ modulo N.*
2. *$S_0(n, N)$ is the subsemigroup of $s \in S_0(n, N)^{\pm}$ such that $\det(s) > 0$.*
3. *$\Gamma_0(n, N)^{\pm} = S_0(n, N)^{\pm} \cap \mathrm{GL}(n, \mathbb{Z})$.*
4. *$\Gamma_0(n, N) = S_0(n, N) \cap \mathrm{GL}(n, \mathbb{Z})$.*

In the case in which we are interested, the ring R will be the algebraic closure $\bar{\mathbb{F}}_p$ of a finite field of order p, S will be $S_0(n, N)$, and Γ will be $\Gamma_0(n, N)$. We will denote the Hecke algebra $\mathcal{H}(\Gamma_0(n, N), S_0(n, N))$ by $\mathcal{H}_{n,N}$. We note that $\mathcal{H}_{n,N}$ is commutative, and contains the Hecke operators $T(\ell,k) = \Gamma D_{\ell,k}\Gamma$ where $\ell \nmid pN$ and

$$D_{\ell,k} = \mathrm{diag}(\underbrace{1,\cdots,1}_{n-k},\underbrace{\ell,\cdots,\ell}_{k}).$$

Definition 2.2. *Let V be an $\mathcal{H}_{n,N}$-module, and let $v \in V$ be a simultaneous eigenvector of all the $T(\ell,k)$ with $\ell \nmid pN$ and $0 \le k \le n$. Denote by $a(\ell,k)$ the eigenvalue of $T(\ell,k)$ acting on v.*

We say that the Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n,\bar{\mathbb{F}}_p)$ is attached to v if

$$\det(I - \rho(Frob_\ell)X) = \sum_{k=0}^{n}(-1)^k \ell^{k(k-1)/2} a(\ell,k)X^k$$

for all primes $\ell \nmid pN$ for which ρ is unramified at ℓ.

(Note that we use the arithmetic Frobenius, so that if ω is the cyclotomic character, $\omega(Frob_\ell) = \ell$.)

The $\mathcal{H}_{n,N}$ modules that we use to find Hecke eigenvectors attached to Galois representations will be the homology of admissible modules for $S_0(n,N)$.

Definition 2.3. *For a fixed prime p, an admissible module M (cf. [2]) for a subsemigroup S of $\mathrm{GL}(n,\mathbb{Q})$ is a finite dimensional vector space over $\bar{\mathbb{F}}_p$ for which there exists an N such that S acts on M through reduction modulo N. (In particular, the denominators in S are all prime to N.)*

Specifically, the admissible modules that we need will be irreducible $\bar{\mathbb{F}}_p[\mathrm{GL}(3,\mathbb{F}_p)]$-modules, on which subsets of $M_3(\mathbb{Z})$ consisting of matrices of determinant prime to p will act through reduction modulo p. Such modules are parameterized by triples (a,b,c) with $0 \le a-b, b-c \le p-1$ and $0 \le c \le p-2$. The module corresponding to the triple (a,b,c) will be denoted $F(a,b,c)$. (See Section 5 for details.)

In order to connect irreducible modules with Galois representations, we will need the mod p cyclotomic character ω, and the *niveau* two characters ω_2 and ω_2' [12]. Note that a power of the cyclotomic character may be written as ω^a, with a well defined modulo $p-1$ (since ω has order $p-1$). Similarly, a power of ω_2 that is not a power of ω may be written as ω_2^m with m well defined modulo p^2-1 and not a multiple of $p+1$. Given such an m, with $0 \le m < p^2-1$, we may write $m = a+bp$ with $0 \le a,b < p$ and $a \ne b$ (for instance writing m in base p). In fact, we may write $m = a+bp$ with $0 < a-b \le p$ (by adding p to a and subtracting one from b, as necessary). Note that the pair (a,b) is only well defined modulo $p-1$, since adding $p-1$ to each of a and b changes m by p^2-1, yielding the same power of ω_2.

Next, we recall the two different types of wild ramification for a representation $\rho : I_p \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ of the inertia group I_p at p having the form

$$\rho(I_p) \sim \begin{pmatrix} \omega^{a+1} & * \\ 0 & \omega^a \end{pmatrix},$$

namely *peu ramifié* and *très ramifié*, and refer the reader to [13] for definitions.

We now state a conjecture connecting certain three-dimensional Galois representations with eigenvectors in arithmetic homology groups. We state the conjecture only for certain representations; for a more general conjecture that applies to a much wider class of representations see [3, 5, 8]. Recall that a two-dimensional mod-p Galois representation is called "odd" if the eigenvalues of the action of a complex conjugation are $(1,-1)$ (which is no condition if $p=2$).

Conjecture 2.4. *Let p be a prime, and let $\bar{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(3, \bar{\mathbb{F}}_p)$ be a Galois representation that is a sum of an irreducible odd two-dimensional representation and a character. Let N be the Serre conductor of ρ and ϵ the nebentype of ρ (see [13]). Then we may choose an irreducible admissible $\bar{\mathbb{F}}_p[\mathrm{GL}(3, \mathbb{F}_p)]$-module V such that ρ is attached to a homology class in the $\mathcal{H}_{3,N}$-module $H_3(\Gamma_0(3, N), V \otimes \epsilon)$.*

If $\rho = \sigma \oplus \psi$, where $\psi = \omega^c \chi_0$ and χ_0 is unramified at p, we may describe the V that yield attached eigenclasses in terms of the restriction of σ to inertia at p. If

$$\sigma|_{I_p} \sim \begin{pmatrix} \omega^a & * \\ 0 & \omega^b \end{pmatrix},$$

we choose a, b, c modulo $p-1$ so that $0 < a-b, b-c \leq p$ and $0 \leq c < p-1$, with the restriction that if σ is très ramifié, then $a - b = p$, and let V be the irreducible module $F(a-2, b-1, c)$. We may also choose a, b, c so that $0 < c-a, a-b \leq p$ and $0 \leq b < p-1$, with $a - b = p$ if σ is très ramifié, and let $V = F(c-2, a-1, b)$.

If

$$\sigma|_{I_p} \sim \begin{pmatrix} \omega_2^{a+bp} & 0 \\ 0 & \omega_2'^{a+bp} \end{pmatrix}$$

and $0 < a - b \leq p$, we either take $0 < a-b, b-c \leq p$ and $0 \leq c < p-1$ and choose $V = F(a-2, b-1, c)$, or we take $0 < c-a, a-b \leq p$ and $0 \leq b < p-1$ and take V to be $F(c-2, a-1, b)$.

For each value of V described above, there is an $\mathcal{H}_{3,N}$-eigenclass in $H_3(\Gamma_0(3, N), V \otimes \epsilon)$ with ρ attached.

Our goal in this paper is to prove the following theorem.

Theorem 2.5. *Let $p > 3$. Then Conjecture 2.4 is true for representations ρ having squarefree Serre conductor.*

Note that generically, for a tamely ramified Galois representation, there will be two choices of the integers a and b (obtained by permuting the diagonal characters in each case). Hence, for a tamely ramified representation there will normally be four predicted weights (if σ is wildly ramified, there will only be two weights, since we cannot permute the two characters on the diagonal). It can happen that there are additional weights. For instance, if $a - b \equiv 1 \pmod{p-1}$, we may choose $a = b+1$ or $a = b+p$. Unless otherwise indicated (i.e. in the *très ramifié* case), all of these weights are predicted.

In the conjecture, the two predictions of weights arise from embedding the image of ρ into one of the two standard Levi subgroups

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

There is also an embedding of the image of ρ into the Levi subgroup

$$\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}$$

but this embedding, for ρ of the type we consider, would violate a strict parity condition, and thus has no predicted weights according to the main conjecture of [3]. Theoretical considerations lead us to believe that we might find an eigenclass attached to ρ in a weight predicted from this forbidden embedding in a homology group of different degree, namely, $H_2(\Gamma_0(3,N), V\otimes\epsilon)$. Proving the existence of this class is one possible future application of the techniques described in this paper.

3. Hecke actions on induced representations

For any set A, $\mathbb{Z}A$ denotes the abelian group of formal finite linear combinations of elements of A. If A is a semigroup or group, $\mathbb{Z}A$ is naturally a ring. Tensor products without a subscript are to be taken over $\mathbb{Z}$.

We follow Brown's notation for tensor products [6, Chap. III, p. 55], except our conventions reverse left and right. If G is a group, H is a subgroup of G, A is a right $\mathbb{Z}H$-module and B is a left $\mathbb{Z}H$-module and a right $\mathbb{Z}G$-module, then $A\otimes_{\mathbb{Z}H} B$ denotes the right $\mathbb{Z}G$-module where $ah\otimes_{\mathbb{Z}H} b = a\otimes_{\mathbb{Z}H} hb$ and $(a\otimes_{\mathbb{Z}H} b)g = a\otimes_{\mathbb{Z}H} bg$ for any $h\in H$ and $g\in G$. If $B=\mathbb{Z}G$ (with the obvious left action of H and right action of G), $A\otimes_{\mathbb{Z}H}\mathbb{Z}G$ is the induced module from H to G of A.

If M and N are two right $\mathbb{Z}G$-modules, define $M\otimes_G N$ to be the coinvariants of the $\mathbb{Z}G$-module $M\otimes N$, where $g\in G$ acts by $(m\otimes n)g = mg\otimes ng$. Thus $M\otimes_G N$ is naturally isomorphic to $M\otimes_{\mathbb{Z}G} N'$, where N' is the left $\mathbb{Z}G$-module whose underlying abelian group is N and where $g\in G$ acts by $gn = ng^{-1}$.

Let $\Gamma\subset S\subset G$ where G is a group, Γ a subgroup, S a subsemigroup and (Γ,S) a Hecke pair. Let X be a set on which G acts on the right.

Remark: In general, the S-action will not preserve the Γ-orbits in X. Here is an example: Consider $\Gamma=\Gamma_0(2,25)$ acting on $\mathbb{P}^1(\mathbb{Z})$. Let $s=\mathrm{diag}(2,1)$. Note that $(5:1)$ and $(5:6)=(5:1)\begin{bmatrix}1&1\\0&1\end{bmatrix}$ are in the same Γ-orbit. However $(5:1)s=(10:1)$ and $(5:6)s=(10:6)=(5:3)$ are in different Γ-orbits.

For any $x\in X$ and any subsemigroup T of G, write $T_x=\mathrm{Stab}_T\, x$. We define the concept of an S-sheaf W on X: For any $x\in X$, W_x is an abelian group, and for any $g\in S, x\in X$ there is given a homomorphism $\mu(g): W_x\to W_{xg}$ such that $\mu(h)\circ\mu(g)=\mu(gh)$ for all $g,h\in S$ and $\mu(1)=id_{W_x}$. Then $W=\oplus_{x\in X}W_x$ is a right $\mathbb{Z}S$-module. If F is any right $\mathbb{Z}S$-module, then $W\otimes F$ is a right $\mathbb{Z}S$-module with the action given by $(w\otimes f)g = wg\otimes fg$ for any $g\in S$.

By [6, Corollary III.5.4, p. 68], an S-sheaf W restricted to Γ is a direct sum of induced Γ-modules. For $a\in X$, write $X_a=a\Gamma$ for the orbit of a under Γ. Choose a subset $A\subset X$ such that $X=\coprod_{a\in A}X_a$. For each $a\in A$ let $W[a]=\oplus_{x\in X_a}W_x\subset W$. Then $W[a]$ is naturally isomorphic as right Γ-module to the induced module $W_a\otimes_{\mathbb{Z}\Gamma_a}\mathbb{Z}\Gamma$. Note that in general S does not preserve $W[a]$.

We abbreviate $\otimes_{\mathbb{Z}\Gamma_a}$ by $\otimes_a$. Then

$$W\approx\oplus_{a\in A}W[a]\approx\oplus_{a\in A}W_a\otimes_a\mathbb{Z}\Gamma.$$

Let F be a right S-module. Then $W\otimes F$ is an S-module and therefore $W\otimes_\Gamma F = H_0(\Gamma, W\otimes F)$ carries a natural action of the Hecke algebra $\mathcal{H}(\Gamma,S)$ which we want to compute.

If Y is a set on which a group K acts on the right, let $\mathcal{F}_c(Y/K)$ denote the set of K-invariant functions whose support is a finite number of K-orbits. If in addition a group H acts on the left on Y, commuting with the K-action, let $\mathcal{F}_c(H\backslash Y/K)$ denote the set of $H\times K$-invariant functions whose support is a finite number of $H\times K$-orbits.

Given a Hecke pair (Γ, S), the Hecke algebra $\mathcal{H} = \mathcal{H}(\Gamma, S)$ will be identified with $\mathcal{F}_c(\Gamma\backslash S/\Gamma)$, where multiplication is convolution of functions.

For any Γ-module N, let Φ denote the natural projection from N to the coinvariants $N_\Gamma = H_0(\Gamma, N)$. Let Θ be a choice of coset representatives so that $\Gamma S \Gamma = \coprod_{g\in\Theta} g\Gamma$. If $h \in \mathcal{H}$, and $z \in W \otimes F$, writing T_h for the Hecke operator corresponding to h we have:

$$\Phi(z)|T_h = \Phi\left(\sum_{g\in\Theta} h(g)zg\right).$$

This does not depend on the choice of Θ. If h is the characteristic function of $\Gamma s\Gamma = \coprod s_\alpha\Gamma$, then

$$\Phi(z)|T_h = \Phi\left(\sum_{\alpha} zs_\alpha\right)$$

since $h(g) = 0$ unless $g \in \Gamma s\Gamma$, in which case $g = s_\alpha\gamma$ for some α, γ, and then $h(s_\alpha) = 1$.

We will now determine the Hecke action in terms of the isomorphism of homology given by Shapiro's lemma. Associativity of tensor products gives a canonical isomorphism $\lambda : W[a] \otimes_\Gamma F = (W_a \otimes_a \mathbb{Z}\Gamma) \otimes_\Gamma F \approx W_a \otimes_{\Gamma_a} F$ via $w \otimes_a \gamma \otimes_\Gamma f \mapsto w \otimes_{\Gamma_a} f\gamma^{-1}$.

We seek a formula for T_h on $W[a] \otimes_\Gamma F \approx W_a \otimes_a \mathbb{Z}\Gamma \otimes_\Gamma F$, which is spanned by elements of the form $w \otimes_a 1 \otimes_\Gamma f$ with $w \in W_a$ and $f \in F$. We know

$$\Phi(w \otimes_a 1 \otimes f)|T_h = (w \otimes_a 1 \otimes_\Gamma f)|T_h = \sum_{g\in\Theta} h(g)((w \otimes_a 1)g \otimes_\Gamma fg).$$

Now $(w \otimes_a 1)g = wg$. Write $ag = b(a, g)\delta(a, g)$ with $b(a, g) \in A, \delta(a, g) \in \Gamma$. Then $w \in W_a$ implies that $wg \in W_{b(a,g)}\delta(a, g)$ so that

$$wg = wg\delta(a, g)^{-1} \otimes_{b(a,g)} \delta(a, g).$$

Thus

$$\Phi(w \otimes_a 1 \otimes f)|T_h = \sum_{g\in\Theta} h(g)wg\delta(a, g)^{-1} \otimes_{b(a,g)} \delta(a, g) \otimes_\Gamma fg.$$

Up to now, we suppressed any dependence of Θ on a, but our choice of Θ may depend on a if we wish.

If $z = \sum_a w^a \otimes_a 1 \otimes f^a$, then

$$\Phi(z)|T_h = \sum_a \sum_{g\in\Theta_a} h(g)w^a g\delta(a, g)^{-1} \otimes_{b(a,g)} \delta(a, g) \otimes_\Gamma f^a g.$$

We now choose the coset representatives Θ_a so that $ag = b$ if $g \in \Theta_a$ and $ag \in X_b$.

We obtain

$$\Phi\left(\sum_a w^a \otimes_a 1 \otimes f^a\right)|T_h = \Phi\left(\sum_a \sum_b \sum_{\substack{g\in\Theta_a\\ ag=b}} h(g)w^a g \otimes_b 1 \otimes_\Gamma f^a g\right).$$

Via the isomorphism λ, this corresponds to

$$\left(\sum_a w^a \otimes_{\Gamma_a} f^a\right)|T_h = \sum_a \sum_b \sum_{\substack{g\in\Theta_a\\ ag=b}} h(g)w^a g \otimes_{\Gamma_b} f^a g.$$

Call this Formula (1).

Define $h_{ab}(g) = h(g)$ if $ag = b$ and $= 0$ otherwise. Clearly, $h_{ab} \in \mathcal{F}_c(\Gamma_a \backslash S/\Gamma_b)$. Corresponding to h_{ab} is a Hecke operator T_{ab}. It maps Γ_a-homology to Γ_b-homology.

Theorem 3.1. *Let W be an S-sheaf on the G-set X and F a right S-module. Let A be a set of representatives of the Γ-orbits of X. Let*

$$\lambda : W \otimes_\Gamma F \to \oplus_a W_a \otimes_{\Gamma_a} F$$

be the natural isomorphism described above. Then T_h on the left is equivariant to the matrix (T_{ab}) on the right.

If F is a resolution of $\mathbb{Z}$ by projective $\mathbb{Z}[\Gamma]$-modules which are also S-modules, then λ induces the isomorphism on homology given by Shapiro's lemma and we have

$$H_q(\Gamma, W) \approx \oplus_{a\in A} H_q(\Gamma_a, W_a),$$

and again T_h on the left is equivariant to the matrix (T_{ab}) on the right.

Proof. Without loss of generality, h is the characteristic function of $\Gamma s \Gamma = \coprod s_\alpha \Gamma$. For given a, b, we choose the s_α so that if $as_\alpha \in X_b$ then $as_\alpha = b$. Then we may choose Θ_a to be the union over all double cosets of the collections of s_α's.

Then the term in Formula (1) corresponding to a, b is

$$\Psi_{ab} := \sum_{\substack{s_\alpha \\ as_\alpha \in X_b}} h(s_\alpha) w^a s_\alpha \otimes_{\Gamma_b} f^a s_\alpha.$$

We must show that

$$\Psi_{ab} = (w^a \otimes_{\Gamma_a} f^a)|T_{ab}.$$

To compute $|T_{ab}$, write $\Gamma_a S \Gamma_b = \coprod t\Gamma_b$. Then

$$(w^a \otimes_{\Gamma_a} f^a)|T_{ab} = \sum_t h_{ab}(t) w^a t \otimes_{\Gamma_b} f^a t.$$

Now $h_{ab}(t) = 0$ unless $at = b$ and $t \in \Gamma s \Gamma$. So if $h_{ab}(t) \neq 0$, we have $t = s_\alpha \gamma$ for some α, γ, and $b = at = as_\alpha\gamma = b\gamma$ (since $s_\alpha \in \Theta_a$). It follows that $\gamma \in \Gamma_b$.

In other words, $h_{ab}(t) = 0$ unless $t \in s_\alpha \Gamma_b$ for some α for which $as_\alpha = b$, and in this case $h_{ab}(t) = h(t) = h(s_\alpha)$. Therefore

$$(w^a \otimes_{\Gamma_a} f^a)|T_{ab} = \sum_{\substack{s_\alpha \\ as_\alpha = b}} \sum_{t \in s_\alpha \Gamma_b} h(t) w^a t \otimes_{\Gamma_b} f^a t = \sum_{\substack{s_\alpha \\ as_\alpha = b}} h(s_\alpha) w^a s_\alpha \otimes_{\Gamma_b} f^a s_\alpha$$

which equals Ψ_{ab}.

If F is a complex, the action of the Hecke operators on $W \otimes_\Gamma F$ and $\oplus_a W_a \otimes_{\Gamma_a} F$ commutes with the boundary maps in F. Therefore T_h on $H_q(\Gamma, W)$ and (T_{ab}) on $\oplus_{a\in A} H_q(\Gamma_a, W_a)$ are equivariant with respect to λ. □

4. Preparing to compute Hecke Operators in GL_3

In this section we determine the s_α's that we need to study reducible 3-dimensional Galois representations.

Let P_0 be the stabilizer of the line spanned by $(1, 0, \ldots, 0)$ in affine n-space, on which $\mathrm{GL}(n)$ acts on the right. Note that the elements of P_0 are characterized by the fact that all entries in the top row except for the first are zero.

We set

$$U_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ & & \cdots & \\ * & 0 & \cdots & 1 \end{pmatrix}, L_0^1 = \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & & & \\ 0 & 0 & \cdots & 1 \end{pmatrix}, L_0^2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \cdots & & & \\ 0 & * & \cdots & * \end{pmatrix}.$$

For $g \in P_0$, we define $\psi_0^1(g) \in \mathrm{GL}(1)$ and $\psi_0^2(g) \in \mathrm{GL}(n-1)$ by

$$g = \begin{pmatrix} \psi_0^1(g) & 0 \\ * & \psi_0^2(g) \end{pmatrix}.$$

We set

$$g_x = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix},$$

and we define

$$P_x = g_x^{-1} P_0 g_x,\ U_x = g_x^{-1} U_0 g_x, L_x^1 = g_x^{-1} L_0^1 g_x, L_x^2 = g_x^{-1} L_0^2 g_x.$$

For $s \in P_x$, we set $\psi_x^i(s) = \psi_0^i(g_x s g_x^{-1})$.

We have the following theorem (the steps of the proof are identical to those in [1, Theorem 7], after transposing and replacing d by $-d$):

Theorem 4.1. *Let d be a positive divisor of N, and assume* $\gcd(d, N/d) = 1$. *Then*

1. $U_d L_d^1 \cap \Gamma_0(n, N) = U_d \cap \Gamma_0(n, N)$.
2. *If $s \in P_d \cap S_0(n, N)^{\pm}$, then $\psi_d^1(s) \equiv s_{11} \pmod d$ and $\psi_d^2(s)_{11} \equiv s_{11} \pmod{N/d}$.*
3. $\psi_d^2(P_d \cap S_0(n, N)^{\pm}) \subset S_0(n-1, N/d)^{\pm}$.
4. *ψ_d^2 induces an exact sequence*

$$1 \to U_d \cap \Gamma_0(n, N) \to P_d \cap \Gamma_0(n, N) \xrightarrow{\psi_d^2} \Gamma_0(n-1, N/d)^{\pm} \to 1.$$

In order to compute Hecke operators with respect to the Hecke pair $(S_0(3, N), \Gamma_0(3, N))$, we use the coset representatives described in the next theorem.

Theorem 4.2. *We have the following coset decompositions of double cosets*

$$\Gamma_0(3, N) s \Gamma_0(3, N) = \bigcup_{g \in C} g \Gamma_0(3, N),$$

where

1. *For $s = \mathrm{diag}(1, 1, \ell)$, with ℓ prime and $(\ell, N) = 1$,*

$$C = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & \ell \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ a & \ell & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \ell & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : 0 \le a, b, c \le \ell - 1 \right\}.$$

2. *For $s = \mathrm{diag}(1, \ell, \ell)$, with ℓ prime and $(\ell, N) = 1$,*

$$C = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & \ell & 0 \\ b & 0 & \ell \end{pmatrix}, \begin{pmatrix} \ell & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & \ell \end{pmatrix}, \begin{pmatrix} \ell & 0 & 0 \\ 0 & \ell & 0 \\ 0 & 0 & 1 \end{pmatrix} : 0 \le a, b, c \le \ell - 1 \right\}.$$

Proof. One checks that the given elements are all in the double coset, and that none are in the same coset of $\Gamma_0(3, N)$. Because the cardinality of C is equal to the number of cosets of $\Gamma_0(3, N)$ in the double coset, they must form a complete set of coset representatives. ☐

The next theorem is adapted to the following situation: Using the notations of Section 3, let $G = \mathrm{GL}(3, \mathbb{Q})$, $X = \mathbb{P}^2(\mathbb{Q})$, $\Gamma = \Gamma_0(3, N)$. We assume N is squarefree, in which case, as proved in [1], the Γ-orbits of X may be represented by the set

$$A = \{(1 : d : 0) \mid d > 0, \quad d|N\}.$$

Also, any s as in Theorem 4.2 takes each Γ-orbit to itself. For each s the following theorem gives a γ such that $(1 : d : 0)s\gamma = (1 : d : 0)$. Thus s fixes each orbit set-wise, and $\{s_\alpha\} = \{s\gamma\}$ with these γ's, where we choose $a = (1 : d : 0)$. The theorem also gives the values of $\psi_0^i, i = 1, 2$ which we will need to compute the action of Γ_a on the S-sheaves we deal with in Sections 8 and 9.

Theorem 4.3. *Let N be squarefree, let ℓ be a prime not dividing N, let d be a divisor of N, and let s be a matrix of the form*

$$s = \begin{pmatrix} \ell_1 & 0 & 0 \\ a & \ell_2 & 0 \\ b & c & \ell_3 \end{pmatrix}$$

that is in one of the sets C in Theorem 4.2. Then there exists a $\gamma \in \Gamma_0(3, N)$ of the form

$$\gamma = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

such that $s\gamma \in P_d = g_d^{-1} P_0 g_d$, and for $x = g_d s\gamma g_d^{-1} \in P_0$ we have

(1) *If $\ell_1 = \ell_2$ and $a = 0$, then $x_{11} = \ell_1$ and*
$$\psi_0^2(x) = \begin{pmatrix} \ell_2 & 0 \\ c - bd & \ell_3 \end{pmatrix}.$$
(2) *If $\ell_1 = \ell$, $\ell_2 = 1$ and $a = 0$, then $x_{11} = 1$ and*
$$\psi_0^2(x) = \begin{pmatrix} \ell & 0 \\ -bd + c\ell & \ell_3 \end{pmatrix}.$$
(3) *If $\ell_1 = 1$, $\ell_2 = \ell$, and $\ell \nmid ad + 1$, then $x_{11} = 1$ and*
$$\psi_0^2(x) = \begin{pmatrix} \ell & 0 \\ -b\ell d + c(ad + 1) & \ell_3 \end{pmatrix}.$$
(4) *If $\ell_1 = 1$, $\ell_2 = \ell$, and $\ell|ad + 1$, then $x_{11} = \ell$ and*
$$\psi_0^2(x) = \begin{pmatrix} 1 & 0 \\ -bd + c\frac{ad+1}{\ell} & \ell_3 \end{pmatrix}.$$

We give the proof of this theorem in the appendix.

5. Irreducible representations

Let B_m be the Borel subgroup of $\mathrm{GL}(m)$ consisting of upper triangular matrices, and let T_m be the maximal torus of diagonal matrices. An algebraic weight with respect to the pair (B_m, T_m) is an m-tuple of integers $(a_1, \ldots, a_m)$ which represents the map $\mathrm{diag}(t_1, \ldots t_m) \mapsto t_1^{a_1} \cdots t_m^{a_m}$. This weight is dominant if $a_1 \geq a_2 \geq \cdots \geq a_m$.

A dominant weight is said to be p-restricted if $0 \leq a_i - a_{i+1} \leq p - 1$ for $1 \leq i < m$ and $0 \leq a_m \leq p - 2$. For any p-restricted weight there exists a unique (up to isomorphism) irreducible right $\mathbb{F}_p[\mathrm{GL}(m, \bar{\mathbb{F}}_p)]$-module $F'(a_1, \ldots, a_m)$ with highest weight $(a_1, \ldots, a_m)$. This module remains irreducible when restricted to $\mathbb{F}_p[\mathrm{GL}(m, \mathbb{Z}/p\mathbb{Z})]$, and all irreducible $\mathbb{F}_p[\mathrm{GL}(m, \mathbb{Z}/p\mathbb{Z})]$-modules occur this way [7, p. 412]. These modules are, in fact, absolutely irreducible [11, Corollary II.2.9], and we will write $F(a_1, \ldots, a_m) = \bar{\mathbb{F}}_p \otimes F'(a_1, \ldots, a_m)$ for the irreducible module over $\bar{\mathbb{F}}_p$.. We will relax the condition that $0 \leq a_m \leq p-2$, and allow a_m to be arbitrary, stipulating that we can adjust the entire m-tuple by adding the same multiple of $p-1$ to each entry without changing the corresponding modules. This is equivalent to tensoring with the $(p-1)$ power of the determinant, which changes the modules over $\mathrm{GL}(m, \bar{\mathbb{F}}_p)$, but not over $\mathrm{GL}(m, \mathbb{Z}/p\mathbb{Z})$.

In a similar way, all the irreducible $\mathbb{F}_p[\mathrm{GL}(1, \mathbb{Z}/p\mathbb{Z}) \times \mathrm{GL}(m-1, \mathbb{Z}/p\mathbb{Z})]$-modules are classified by m-tuples $(a_1, \ldots, a_m)$ such that $(a_2, \ldots, a_m)$ is p-restricted for $\mathrm{GL}(m-1)$ and a_1 is considered modulo $p-1$. We will denote the $\mathbb{F}_p[\mathrm{GL}(1, \mathbb{Z}/p\mathbb{Z}) \times \mathrm{GL}(m-1, \mathbb{Z}/p\mathbb{Z})]$-module corresponding to $(a_1, \ldots, a_m)$ by $M'(a_1; a_2, \ldots, a_m)$. We note that this module is just $F'(a_2, \ldots, a_m)$ as an $\mathbb{F}_p[\mathrm{GL}(1, \mathbb{Z}/p\mathbb{Z}) \times \mathrm{GL}(m-1, \mathbb{Z}/p\mathbb{Z})]$-module, with $\mathrm{GL}(1, \mathbb{Z}/p\mathbb{Z})$ acting as scalars via the a_1-power map. The corresponding module defined over $\bar{\mathbb{F}}_p$ will be denoted $M(a_1; a_2, \ldots, a_m) = \bar{\mathbb{F}}_p \otimes M'(a_1; a_2, \ldots, a_m)$.

Note that we will also consider any $\mathrm{GL}(m, \mathbb{Z}/p\mathbb{Z})$-module as a $\mathrm{GL}(m, \mathbb{Z}_p)$-module via reduction modulo p. In this way, we make any $\mathrm{GL}(m, \mathbb{Z}/p\mathbb{Z})$-module into an $S_0(m, N)$-module. Note that if N and p are relatively prime, then the image of $S_0(m, N)$ under reduction modulo p is all of $\mathrm{GL}(m, \mathbb{Z}/p\mathbb{Z})$.

Given a $\mathrm{GL}(m, \mathbb{Z}_p)$-module E, we will denote the action of $s \in \mathrm{GL}(m, \mathbb{Z}_p)$ by $e|s$. We denote by E^x the module with the same underlying abelian group as E, but with $s \in \mathrm{GL}(m, \mathbb{Z}_p)$ acting via $e|^x s = e|g_x s g_x^{-1}$. We note that if $E = F(a_1, \ldots, a_m)$ is an irreducible module, then E^x is also isomorphic to $F(a_1, \ldots, a_m)$, since g_x is an intertwining map. For a Dirichlet character χ of conductor N, we denote by E^x_χ the $S_0(n, N)$ module with the same underlying abelian group as E but with the action of s given by $e|^x_\chi s = \chi(s) e|g_x s g_x^{-1}$, where $\chi(s)$ is defined as $\chi(s_{11})$.

Theorem 5.1. *Let $n \geq 2$, let $(a_1, \ldots, a_n)$ be a p-restricted weight, and set $F = F(a_1, \ldots, a_n)$. Let χ be a Dirichlet character of conductor N, which factors as $\chi_0\chi_1$, where χ_0 has conductor d, and χ_1 has conductor N/d with $(d, N/d) = 1$. Assume p does not divide N. Then*

(1) The module of invariants $F^{U_0(\mathbb{Z}/p)}$ is isomorphic to $M(a_n; a_1, \ldots, a_{n-1})$.

(2) The module $F^{U_d \cap S_0(n,N)}$ considered as a $P_d \cap S_0(n, N)$-module is isomorphic to $(F^{U_0})^d$.

(3) The action of $P_d \cap S_0(n, N)$ on the module $F_\chi^{U_d \cap S_0(n,N)}$ is given by

$$e|^d_\chi s = \chi_0(\psi^1_d(s))(\psi^1_d(s))^{a_n} \chi_1(\psi^2_d(s)) e|(\psi^2_d(s)),$$

where the vertical bar on the right denotes the action of $\mathrm{GL}(n-1, \mathbb{Z}_p)$ *on* $F(a_1, \ldots, a_{n-1})$.

Proof. (1) Set $U = U_0(\mathbb{Z}/p)$, and $L = P_0(\mathbb{Z}/p)/U = L^1_0(\mathbb{Z}/p) \times L^2_0(\mathbb{Z}/p)$. Let A be the outer automorphism of $\mathrm{GL}(n)$ given by $A(g) = {}^t g^{-1}$. The contragredient of F, or $F^\vee$, is a $\mathrm{GL}(n, \mathbb{Z}_p)$-module with the same underlying abelian group as F, but with the action given by $f|^\vee g = f|A(g)$. We note that as $\mathrm{GL}(n, \mathbb{Z}_p)$-modules, the contragredient and the dual, $\mathrm{Hom}(F, \bar{\mathbb{F}}_p)$ are isomorphic.

As L-modules, we have that $F^U \cong ((F^\vee)^{A(U)})^\vee$. Then by [10, Proposition 5.10] (see also [9, Lemma 2.5]), we see that

$$F^U \cong (F(-a_n, \ldots, -a_1)^{A(U)})^\vee \cong M(-a_n; -a_{n-1} \ldots, -a_1)^\vee \cong M(a_n; a_1, \ldots, a_{n-1}).$$

(2) Since N and p are relatively prime, $U_d \cap S_0(n, N)$ and U_d have the same image in $\mathrm{GL}(n, \mathbb{Z}/p\mathbb{Z})$, so we need only consider F^{U_d}. We see that $F^{U_d} = ((F^{-d})^{U_0})^d$. However, as described above, $F^{-d} \cong F \cong F(a_1, \ldots, a_n)$. Hence, by part (1), we find that $F^{U_d} \cong M(a_n; a_1, \ldots, a_{n-1})^d$.

(3) We identify $F^{U_d \cap S_0(n,N)}$ with $M(a_n; a_1, \ldots, a_{n-1})^d$. Then, for $e \in F^U$, we have

$$\begin{aligned} e|_\chi^d s &= \chi(s) e | g_d s g_d^{-1} \\ &= \chi(s) e | (\psi_d^1(s) \times \psi_d^2(s)) \\ &= \chi(s) (\psi_d^1(s))^{a_n} e | \psi_d^2(s) \\ &= \chi_0(\psi_d^1(s)) \chi_1(\psi_d^2(s)) (\psi_d^1(s))^{a_n} e | \psi_d^2(s) \end{aligned}$$

since $e \in F^U \cong M(a_n; a_1, \ldots, a_{n-1})$ as a $\mathrm{GL}(1) \times \mathrm{GL}(n-1)$-module and

$$\chi(s) = \chi(s_{11}) = \chi_0(s_{11})\chi_1(s_{11}) = \chi_0(\psi_d^1(s))\chi_1(\psi_d^2(s))$$

by Theorem 4.1(2).

□

6. Points in general position

Let K be an infinite field, V an n-dimensional K-vector space, and denote the projective space by $\mathbb{P} = \mathbb{P}(V)$. Fix a basis of V. Given points $a_1, \ldots, a_r \in \mathbb{P}$, we say that the points are in general position if any subset of them spans a linear space of maximal possible dimension. We define a simplicial complex $Y^g = Y^g(K)$ as follows. The vertices of Y^g are points in $\mathbb{P}$, and are acted upon by $\mathrm{GL}_n(K)$ on the right. The p-simplices of Y^g are spanned by $(p+1)$-tuples of vertices that are in general position. We let X^g denote the chain complex of oriented chains on Y^g. The augmentation is the map $\varepsilon : X_0^g \to \mathbb{Z}$ that sends each vertex to 1. If $x_0, \ldots, x_p$ are vertices spanning a p-simplex Δ in X^g, we denote by $\vec{x} = (x_0, \ldots, x_p)$ the chain supported on Δ with coefficient 1. It is antisymmetric in the arguments. Then X_p^g is generated over $\mathbb{Z}$ by these basic chains $\vec{x}$. If $v_0, \ldots, v_p \in V - \{0\}$, and we denote by $\bar{v}_i$ the element of $\mathbb{P}$ containing v_i, we may write $(v_0, \ldots, v_p)$ for the basic chain $(\bar{v}_0, \ldots, \bar{v}_p)$.

Theorem 6.1. *If K is infinite, then X^g is an acyclic resolution of $\mathbb{Z}$ by $\mathrm{GL}(n, K)$-modules.*

Proof. Let $\vec{x}_i = (x_{i0}, \ldots, x_{ip})$ be a basic chain supported on a p-simplex for each i, and suppose that $z = \sum_i c_i \vec{x}_i$ is a cycle, i.e. a p-chain that is taken to zero by the boundary map.

Since K is infinite, we may choose a point $y \in \mathbb{P}$ such that for each i, the set $\{x_{i0}, x_{i1}, \ldots, x_{ip}, y\}$ is in general position. Let $y_i = (x_{i0}, x_{i1}, \ldots, x_{ip}, y)$.

Then

$$\begin{aligned} \partial \sum_i c_i y_i &= \sum_i c_i \partial y_i \\ &= \sum_i \sum_{j=0}^{p} (-1)^{j+1} c_i (x_{i0}, \ldots, \widehat{x}_{ij}, \ldots, x_{ip}, y) + (-1)^{p+2} \sum_i c_i (x_{i0}, \ldots, x_{ip}, \widehat{y}) \\ &= 0 \pm z, \end{aligned}$$

where the double sum is 0 since z is a cycle. Adjusting the sign, we see that every cycle is a boundary. $\square$

7. A spliced sharbly complex

In this section, K is an infinite field. We are going to splice the complex X^g with the sharbly complex. Recall the Steinberg module St_n and the sharbly complex Sh^n for an n-dimensional K-vector space V as described for example in [4]. If $v_1, \dots, v_{n+k}$ are vectors in V, denote by $[v_1, \dots, v_{n+k}]$ the basic k-sharbly. It is antisymmetric in the arguments, it doesn't change if any argument is multiplied by a non-zero element of K, and it vanishes if the arguments do not span V over K. An element $g \in \mathrm{GL}(n, K)$ acts on a basic k-sharbly by $[v_1, \dots, v_{n+k}]g = [v_1 g, \dots, v_{n+k} g]$. The $\mathbb{Z}$-span of the basic k-sharblies, subject to these relations, is by definition $Sh^n_k(V)$. The boundary map $Sh^n_{k+1}(V) \to Sh^n_k(V)$ is given by $[v_1, \dots, v_{n+k+1}] \mapsto \sum_i (-1)^i [v_1, \dots, \hat{v}_i, \dots v_{n+k+1}]$.

The module $St_n(V)$, which is free as a $\mathbb{Z}$-module, is isomorphic as a $\mathrm{GL}(n, K)$-module to the cokernel of the boundary map $Sh^n_1(V) \to Sh^n_0(V)$. For $[v_1, \dots, v_n] \in Sh^n_0(V)$, we denote the image of $[v_1, \dots, v_n]$ in the cokernel $St_n(V)$ by $\{v_1, \dots, v_n\}$.

If $K = \mathbb{Q}$, Borel-Serre duality, as improved by Brown [6, X.3.6], gives us Hecke-equivariant isomorphisms $H_i(\Gamma, St_n \otimes M) \approx H^{n(n-1)/2-i}(\Gamma, M)$ for any subgroup of finite index $\Gamma \subset \mathrm{GL}(n, \mathbb{Z})$ and any S-module M on which $(n+1)!$ acts invertibly.

We know that

$$\cdots \to Sh^n_i(V) \to Sh^n_{i-1}(V) \to \cdots \to Sh^n_0(V) \to St_n(V) \to 0$$

is an exact sequence of $\mathrm{GL}(n, K)$-modules.

Sending $(v_1, \dots, v_{n+k})$ to $[v_1, \dots, v_{n+k}]$ defines an injective map of $\mathrm{GL}(n, K)$-modules $\iota : X^g_{n+k-1} \to Sh_k$. Note that ι induces an isomorphism $X^g_{n-1} \approx Sh_0$.

We now set $n = 3$ and $V = K^3$, dropping the n and the V from the notation $Sh^n_i(V)$. Define a new complex X of $\mathrm{GL}(3, K)$-modules as follows. For $i \geq 2$, $X_i = Sh_{i-2}$ and the boundary map $X_{i+1} \to X_i$ is the same as in the sharbly complex. We define $X_0 = X^g_0$ with the same augmentation map $\varepsilon : X_0 \to \mathbb{Z}$. It remains to define X_1 and the boundary maps $X_2 \to X_1 \to X_0$.

Let $\mathbb{P}^*$ denote the set of planes in K^3. We set $X_1 = \bigoplus_{H \in \mathbb{P}^*} St_2(H)$. An element $g \in \mathrm{GL}(3, K)$ acts on $\{a, b\} \in X_1$ by sending it to $\{ag, bg\}$.

Note that X_2 is generated freely over $\mathbb{Z}$ by "generic sharblies", i.e. $[a, b, c]$ such that the determinant of the matrix with rows a, b, c is nonzero. Define the boundary map $\partial_2 : X_2 \to X_1$ by $[a, b, c] \mapsto \{a, b\} + \{b, c\} + \{c, a\}$ for any generic $[a, b, c]$. It is well-defined and is $\mathrm{GL}(3, K)$-equivariant. We define the boundary map $\partial_1 : X_1 \to X_0$ by $\{a, b\} \mapsto (b) - (a)$.

Lemma 7.1. *Let R be the submodule of X^g_1 generated by*

$$\{(a, b) + (b, c) + (c, a) \mid H \in \mathbb{P}^*,\ a, b, c \in H,\ (a, b, c) \in X^g_2(H)\}.$$

For $H \in \mathbb{P}^*$ and $a, b \in H$, let $\psi(a, b) = \{a, b\} \in St_2(H)$. Then

(a) The map $\psi : X^g_1 \to X_1$ induces an isomorphism of $\mathrm{GL}(3, K)$-modules $\phi : X^g_1/R \to X_1$.

(b) The boundary map in the generic complex $X^g_1 \to X^g_0 = X_0$ induces the $\mathrm{GL}(3, K)$-equivariant boundary map $\partial_1 : X_1 \to X_0$ after identifying X_1 with X^g_1/R via ϕ.

Proof. (a) The map ψ is clearly $\mathrm{GL}(3,K)$-equivariant and surjective. If $t \in X_1^g$, we can write $t = \sum_{H\in\mathbb{P}^*} t_H$, where t_H is supported on symbols (u,v) with $u,v \in H$. Then $\psi(t) = 0$ if and only if $\psi(t_H) = 0$ for all H. For each H, $St_2(H)$ is the cokernel of the boundary map $\beta : Sh_1(H) \to Sh_0(H)$. The image of β is generated by the basic relations $\{u,v\} + \{v,w\} + \{w,u\}$ where (u,v,w) runs over triples in H such that u,v,w generate pairwise distinct lines. Therefore the kernel of ψ is exactly R.

(b) The boundary map $\partial_1 : X_1^g \to X_0$ in X^g sends (a,b) to $(b)-(a)$. This contains R in its kernel, and induces a map $X_1/R \to X_0$ which clearly becomes ∂_1 after the identification via ϕ. It is obviously $\mathrm{GL}(3,K)$-equivariant. ☐

Lemma 7.2. *The sequence of* $\mathbb{Z}[\mathrm{GL}(3,K)]$*-modules*

$$\cdots \to X_i \to X_{i-1} \to \cdots \to X_2 \to X_1 \to X_0 \to \mathbb{Z} \to 0$$

is exact.

Proof. Since the sharbly sequence is exact and the augmentation map is surjective, the only nodes which need checking are those at X_i, $i = 0,1,2$. Denote the boundary map from $X_i \to X_{i-1}$ by ∂_i. Denote the boundary maps in X^g by ∂^g. We identify X_{k+2}^g with its image in Sh_k via ι, under which identification (1) $\partial_{k+2}^g = \partial_k|_{X_{k+2}^g}$ and (2) X_2^g and Sh_0 are isomorphic.

Node at X_0: Clearly $\varepsilon \circ \partial_1 = 0$. Let $x \in X_0$. If $\varepsilon(x) = 0$, there exists $y \in X_1^g$ with $\partial_1^g(y) = x$ because X^g is exact. Then $\partial_1(\psi y) = x$.

Node at X_1: Clearly $\partial_1 \circ \partial_2 = 0$. Let $y \in X_1$ such that $\partial_1(y) = 0$. Choose $y' \in X_1^g$ such that $y = \psi y'$. Then $\partial_1^g(y') = 0$. Hence there exists $z \in X_2^g = Sh_0 = X_2$ with $\partial_2^g(z) = y'$ because X^g is exact. Then $\partial_2(z) = y$.

Node at X_2: Clearly $\partial_2 \circ \partial_3 = 0$. Now suppose $x \in X_2$ such that $\partial_2(x) = 0$. Then $\partial_2^g(x) \in R$. We will show (*) for every $r \in R$ there exists $\widetilde{r} \in X_3 = Sh_1$ such that $r = \partial_2^g\partial_3(\widetilde{r}) \in X_1^g$. Then $\partial_2^g(x - \partial_3(\widetilde{\partial_2^g(x)})) = 0$. Therefore there exists $t \in X_3^g \subset X_3$ such that $\partial_3(t) = \partial_3^g(t) = x - \partial_3(\widetilde{\partial_2^g(x)})$. Then $\partial_3(t + \widetilde{\partial_2^g(x)}) = x$.

Proof of (*): Let $H \in \mathbb{P}^*$, $a,b,c \in H, (a,b,c) \in X_2^g(H)$. (The last condition just means that a,b,c are pairwise distinct.) It suffices to let $r = (a,b)+(b,c)+(c,a)$ and find $\widetilde{r}$. Pick $v \notin H$. Set $\widetilde{r} = [a,b,c,v]$. Then $\partial_3(\widetilde{r}) = [b,c,v]-[a,c,v]+[a,b,v]$ since $[a,b,c] = 0$ in Sh_0. Hence $\partial_2^g(\partial_3(\widetilde{r})) = r$. ☐

A straightforward attempt to generalize this lemma for $n > 3$ doesn't work. The reason the lemma works for $n = 3$ is that any tuple of pairwise distinct lines in a plane is generic.

We now specialize to the case in which $K = \mathbb{Q}$, and we let Γ be a subgroup of finite index in $\mathrm{SL}(3,\mathbb{Z})$. If M is any S-module, $X_1 \otimes M = \bigoplus_{H\in\mathbb{P}^*} St_2(H) \otimes M$. If H is a fixed plane, let P_H be its stabilizer in $\mathrm{GL}(3)$ and U_H the unipotent radical of P_H. Since $\mathrm{SL}(3,\mathbb{Z})$ acts transitively on $\mathbb{P}^*$, Γ has a finite number of orbits in $\mathbb{P}^*$, represented by the planes in a finite set, say $\mathbb{H}(\Gamma)$. It follows from [6, Corollary III.5.4] that $X_1 \otimes M$ is isomorphic as an S-module to a finite sum of induced modules $\bigoplus_{H\in\mathbb{H}(\Gamma)} \mathrm{Ind}_{P_H\cap\Gamma}^{\Gamma} St_2(H) \otimes M$, where $U_H \cap \Gamma$ acts trivially on $St_2(H)$.

8. The spectral sequence for GL_3

Fix an admissible right S-module M. Let F be a resolution of $\mathbb{Z}$ by $\mathbb{Z}[S]$-modules which are free as $\mathbb{Z}[\Gamma]$-modules. Let $\Lambda = X \otimes M$ with the diagonal S-action and form the double complex $\Lambda \otimes_\Gamma F$. To compute the homology, we use the spectral sequence from [6, VII.5.(5.3)]. All the differentials on the E^i page, $i \geq 1$ are Hecke-equivariant because they are all induced by the differential on the double complex, which is an S-module. From now on, we assume that 6 is invertible on M.

Since X is a resolution of $\mathbb{Z}$ by free $\mathbb{Z}$-modules, we see that Λ is a resolution of M. Therefore there is a weak equivalence from Λ to the chain complex consisting of M concentrated in dimension 0, so that by [6, VII.5.2], $H_*(\Gamma, \Lambda) \approx H_*(\Gamma, M)$.

Hence, in the spectral sequence, we have

$$E^1_{jq} = H_q(\Gamma, X_j \otimes M) \implies H_{j+q}(\Gamma, M).$$

For each $j \neq 1$, $X_j \otimes M$ is isomorphic to a direct sum of induced representations

$$X_j \otimes M \approx \bigoplus_{\sigma \in \widetilde{C}_j} \operatorname{Ind}_{\Gamma_\sigma}^{\Gamma} M_\sigma$$

where C_0 is the set of vertices in Y_0^g; if $j \geq 2$, C_j is the set of basic sharblies $[v_1, \ldots, v_{j+1}]$ with $v_1, \ldots, v_{j+1}$ vectors in $\mathbb{Q}^3$ that span $\mathbb{Q}^3$; $\widetilde{C}_j$ is a set of representatives for the Γ-orbits in C_j for $j \neq 1$; Γ_σ is the stabilizer of σ, ε_σ is the orientation character recording how Γ_σ acts on σ; and $M_\sigma = M \otimes \varepsilon_\sigma$.

Note that for $j \neq 1$, $\sigma \mapsto M$ is an S-sheaf on C_j whose total space is $X_j \otimes M$. Then we have by Theorem 3.1:

$$E^1_{jq} = \bigoplus_{\sigma \in \widetilde{C}_j} H_q(\Gamma_\sigma, M_\sigma)$$

if $j \neq 1$. We see easily that $H_q(\Gamma_\sigma, M_\sigma) = 0$ in the following cases:

(1) For $q > 3$ and $\sigma \in C_0$, since the virtual homological dimension of the stabilizer P of a 0-cycle is 3 and any torsion element of P has order invertible on M.
(2) For $q \geq 1$ and $\sigma \in C_j$ with $j > 1$, since the stabilizer of a basic sharbly is a finite subgroup of $\mathrm{GL}(3, \mathbb{Z})$, hence of order invertible in M (see [6, Corollary III.10.2]).

The column with $j = 1$ will be treated later.

We then have the E^1 page of our spectral sequence as follows, where $H_i(\sigma) := H_i(\Gamma_\sigma, M_\sigma)$.

$$\begin{array}{ccccc}
\bigoplus_{\sigma \in \widetilde{C}_0} H_3(\sigma) & H_3(\Gamma, X_1 \otimes M) & 0 & 0 & 0 \\
\bigoplus_{\sigma \in \widetilde{C}_0} H_2(\sigma) & H_2(\Gamma, X_1 \otimes M) & 0 & 0 & 0 \\
\bigoplus_{\sigma \in \widetilde{C}_0} H_1(\sigma) & H_1(\Gamma, X_1 \otimes M) & 0 & 0 & 0 \\
\bigoplus_{\sigma \in \widetilde{C}_0} H_0(\sigma) & H_0(\Gamma, X_1 \otimes M) & H_0(\Gamma, Sh_0 \otimes M) & H_0(\Gamma, Sh_1 \otimes M) & H_0(\Gamma, Sh_2 \otimes M)
\end{array}$$

The rows of the spectral sequence above those shown are all 0 because all the groups involved have homological dimension ≤ 3 when 6 is invertible on the coefficient modules.

It will follow from Theorem 8.1 below that any Hecke eigenclass in E^1_{03} that has an attached Galois representation which is the sum of an irreducible two-dimensional representation and a character cannot be hit by any differential in the spectral sequence and hence survives to E^∞. (This uses the fact that the Hecke eigenvalues determine the characteristic polynomials of Frobenius and hence the attached Galois representation up to semi-simplification.)

Theorem 8.1. *Assume that M is an admissible $\bar{\mathbb{F}}_p[S]$-module with $p > 3$. Then the E^2_{40} and E^1_{13} terms of the spectral sequence are finite dimensional $\bar{\mathbb{F}}_p$-vector spaces and all systems of Hecke eigenvalues appearing in them are attached to Galois representations that are sums of characters.*

Proof. Since the sharbly complex is a resolution of $St := St_3(\mathbb{Q}^3)$, and 6 acts invertibly on M, $E^2_{40} \approx H_2(\Gamma, St \otimes M)$, cf. Corollary 8 in [4]. Borel-Serre duality then gives an isomorphism of Hecke-modules $E^2_{40} \cong H^1(\Gamma, M)$, which is a finite dimensional $\bar{\mathbb{F}}_p$-vector space. By [2, Theorem 4.1.5], $H^1(\Gamma, M)$ is a sum of generalized Hecke eigenspaces, and any eigenclass appearing in $H^1(\Gamma, M)$ has an attached Galois representation that is a sum of three characters.

We now consider E^1_{13}. Recall that there is a finite set of planes $\mathbb{H}(\Gamma)$ such that

$$X_1 \otimes M \approx \bigoplus_{H \in \mathbb{H}(\Gamma)} \operatorname{Ind}^{\Gamma}_{P_H \cap \Gamma} St_2(H) \otimes M$$

where $U_H \cap \Gamma$ acts trivially on $St_2(H)$.

We define an S-sheaf on $\mathbb{P}^*$ by $H \mapsto St_2(H) \otimes M$. We then use Theorem 3.1.

By Shapiro's lemma, $E^1_{13} \approx \bigoplus_{H \in \mathbb{H}(\Gamma)} H_3(P_H \cap \Gamma, St_2(H) \otimes M)$.

Let Γ_H denote $(P_H \cap \Gamma)/(U_H \cap \Gamma)$. It is isomorphic to a congruence subgroup of $\mathrm{GL}(2, \mathbb{Z})$. We have the Hochschild-Serre spectral sequence

$$E^2_{jq} = H_j(\Gamma_H, H_q(U_H \cap \Gamma, St_2(H) \otimes M)) \implies H_{j+q}(P_H \cap \Gamma, St_2(H) \otimes M).$$

Because $U_H \cap \Gamma$ acts trivially on $St_2(H)$, we have $H_q(U_H \cap \Gamma, St_2(H) \otimes M) \approx St_2(H) \otimes C(H, q)$ where $C(H, q) = H_q(U_H \cap \Gamma, M)$ is an admissible Γ_H-module on which 6 acts invertibly. Also, the homological dimension of $U_H \cap \Gamma$ is 2. Therefore the only nonzero term in the E^2 page of the Hochschild-Serre spectral sequence when $j + q = 3$ occurs when $j = 1, q = 2$. So any packet of Hecke eigenvalues occurring in

$$H_3(P_H \cap \Gamma, St_2(H) \otimes M)$$

also occurs in

$$H_1(\Gamma_H, St_2(H) \otimes C(H, 2)).$$

By Borel-Serre duality, this is isomorphic to $H^0(\Gamma_H, C(H, 2))$ and by [2, Theorem 4.1.4], this is a sum of generalized Hecke eigenspaces, and any system of Hecke eigenvalues occurring here has as attached Galois representation a sum of two characters. At this point we also see that E^1_{13} is finite dimensional over $\bar{\mathbb{F}}_p$.

Now we use Theorem 3.1 to compute the Hecke operators on E^1_{13}, following the same outline as that used in Section 9 for E^1_{03}. In both computations, the Hecke eigenvalues occurring in the homology of Γ with coefficients in an S-sheaf on a Γ-set are determined, where the homology of a stabilizer with coefficients in a stalk of the sheaf is known to have an attached two-dimensional Galois representation τ. In section 9, τ is irreducible, while in the case at hand, τ is the sum of two characters. In both situations, the corresponding Hecke eigenclass has an attached Galois representation of the form $\tau \oplus \psi$ for some character ψ. Thus E^1_{13} is a sum of generalized Hecke eigenspaces, and any system of Hecke eigenvalues occurring here has as attached Galois representation a sum of three characters. □

9. Reducible Galois representations

We continue to assume that 6 is invertible on M and we set $\Gamma = \Gamma_0(3, N)$. We note that for $\sigma \in C_0$, the orientation character is trivial, so that $M_\sigma = M$. For any $d|N, d > 0$, we have taken $(1 : d : 0)$ (with stabilizer P_d) as a representative of its Γ-orbit in C_0. Therefore, E^1_{03} contains $H_3(P_d \cap \Gamma_0(3, N), M)$ as a direct summand. We will find the system of Hecke eigenvalues in which we are interested in a summand of this form.

Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_3(\bar{\mathbb{F}}_p)$ be a Galois representation that can be written as a direct sum $\rho = \sigma \oplus \psi$, where $\sigma : G_{\mathbb{Q}} \to \mathrm{GL}_3(\bar{\mathbb{F}}_p)$ is an odd, irreducible, two-dimensional representation, and $\psi : G_{\mathbb{Q}} \to \mathrm{GL}_1(\bar{\mathbb{F}}_p)$ is a character. Let N_1 be the Serre conductor of σ, and let d be the Serre conductor of ψ, and assume that $N = N_1 d$ is squarefree. We will show that $\rho = \sigma \oplus \psi$ is attached to a Hecke eigenclass in E^1_{03} with $\Gamma = \Gamma_0(3, N)$, and coefficient module as predicted by Conjecture 2.4.

We set $\tau = \sigma \otimes \omega^{-1}$, so that $\rho = (\tau \otimes \omega) \oplus \psi$. Assume that the predicted weight of τ in Serre's conjecture is $F(a, b)$, with $0 \le a - b \le p - 1$ and $0 \le b < p - 1$. Then if τ is ordinary, we have

$$\tau|_{I_p} \sim \begin{pmatrix} \omega^{a+1} & * \\ 0 & \omega^b \end{pmatrix}$$

and if τ is supersingular, we have

$$\tau|_{I_p} \sim \begin{pmatrix} \omega_2^{(a+1)+bp} & 0 \\ 0 & \omega_2'^{(a+1)+bp} \end{pmatrix}.$$

Note that τ has Serre conductor N_1 (the same as σ), and we may factor $\det(\tau) = \omega^{(a+b+1)}\chi_1$, where the conductor of χ_1 divides N_1. We may also factor $\psi = \omega^c \chi_0$, where χ_0 has conductor d and $0 \le c < p - 1$. We will denote by λ_ℓ the trace of $\tau(Frob_\ell)$. We note then that the trace of $\rho(Frob_\ell)$ is equal to $\ell\lambda_\ell + \chi_0(\ell)\ell^c$ and its cotrace (the coefficient of X^2 in $\det(I - \rho(Frob_\ell)X)$) is equal to $\ell(\chi_1(\ell)\ell^{a+b+2} + \chi_0(\ell)\ell^c\lambda_\ell)$. Examining Conjecture 2.4, we see that the predicted weight of ρ is $F(a, b, c)$ (where, if necessary we add $p - 1$ to both a and b so that (a, b, c) will be p-restricted, and note that by our conventions this change does not change the module $F(a, b)$). In addition, the nebentype of ρ is $\chi = \chi_0\chi_1$, and the level of ρ is $N = N_1 d$.

Denote by M the module $F(a, b, c)_\chi$. We now consider the Hochschild-Serre spectral sequence for the homology of the exact sequence

$$1 \to U_d \cap \Gamma_0(3, N) \to P_d \cap \Gamma_0(3, N) \xrightarrow{\psi_d^2} \Gamma_0(2, N/d)^{\pm} \to 1$$

with coefficients in M.

This spectral sequence degenerates at E^2 because it is only two columns thick, and we have

$$E^2_{jq} = H_j(\Gamma_0(2, N/d)^{\pm}, H_q(U_d \cap \Gamma_0(3, N), M)).$$

The desired Hecke eigenclass stems from $E^2_{12} = H_1(\Gamma_0(2, N/d)^{\pm}, H_2(U_d \cap \Gamma_0(3, N), M))$. Because 6 is invertible on M, $E^2_{jq} = 0$ for all $j \geq 2$. Also, $E^2_{03} = 0$ because $U_d \cap \Gamma_0(3, N)$ has homological dimension 2. Therefore there is a natural isomorphism $Y : H_3(P_d \cap \Gamma_0(3, N), M) \to E^{\infty}_{12} = E^2_{12}$.

Note that since $U_d \cap \Gamma_0(3, N)$ is an abelian group of rank 2,

$$H_2(U_d \cap \Gamma_0(3, N), M) = H^0(U_d \cap \Gamma_0(3, N), M) = M^{U_d \cap \Gamma_0(3,N)}.$$

By Theorem 5.1, $M^{U_d \cap \Gamma_0(n,N)} \cong M(c; a, b)^d_{\chi}$. Recall that $M(c; a, b)$ is just $F(a, b)$ as a GL_2-module, with an additional action of GL_1 through the c-power map. After unwinding the various identifications we have made, including the use of Shapiro's lemma, we find that the action of $\Gamma_0(2, N_1)^{\pm}$ on $F(a, b)_{\chi_1}$ is the standard action. Hence, we are interested in finding a certain class in $H_1(\Gamma_0(2, N/d)^{\pm}, F(a, b)_{\chi_1})$ where P_d acts on $F(a, b)_{\chi_1}$ (yielding a Hecke action on the homology) as described in Theorem 5.1.

We know by Serre's conjecture, which is now a theorem, that τ is attached to a Hecke eigenclass in $H_1(\Gamma_0(2, N_1), F(a, b)_{\chi_1})$. Since τ is irreducible, it is attached to a cusp form, and by Eichler-Shimura it is also attached to a Hecke eigenclass f in

$$H_1(\Gamma_0(2, N_1)^{\pm}, F(a, b)_{\chi_1}).$$

So for the two dimensional Hecke operator $T_\ell = T(\ell, 1)$, we have $f|_{T_\ell} = \lambda_\ell f$, where $\lambda_\ell = \mathrm{Tr}(\tau(Frob_\ell))$. We identify $H_1(\Gamma_0(2, N_1)^{\pm}, F(a, b)_{\chi_1})$ with E^2_{12} via the identifications above and set $\widetilde{f} = Y^{-1}(f) \in H_3(P_d \cap \Gamma_0(3, N), M) \subset \oplus_{\sigma \in \widetilde{C}_0} H_3(\sigma) = E^1_{03}$ in the first spectral sequence of Section 8. The calculation below shows that $H_3(P_d \cap \Gamma_0(3, N), M)$ is stable under the Hecke action on the displayed spectral sequence of the previous section and shows how the three-dimensional Hecke operators $T(\ell, 1)$ and $T(\ell, 2)$ act on $\widetilde{f}$.

For each prime $\ell \nmid pN$ and for each coset representative in $T(\ell, 1)$, we apply Theorem 4.3 to translate the coset representatives s by an element $\gamma \in \Gamma_0(3, N)$ into P_d and then let $x = g_d s \gamma g_d^{-1} \in P_0$, and obtain:

If $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_1 & \alpha_2 & \ell \end{pmatrix}$ then (by case 1 of Theorem 4.3),

$$\psi^1_d(s\gamma) = \psi^1_0(x) = 1 \text{ and } \psi^2_d(s\gamma) = \psi^2_0(x) = \begin{pmatrix} 1 & 0 \\ \alpha_2 - \alpha_1 d & \ell \end{pmatrix}.$$

If $s = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_0 & \ell & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $\ell \nmid \alpha_0 d + 1$, then (by case 3 of Theorem 4.3)

$$\psi^1_0(x) = 1 \text{ and } \psi^2_0(x) = \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}.$$

If $s = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_0 & \ell & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $\ell | \alpha_0 d + 1$, then (by case 4 of Theorem 4.3)

$$\psi^1_0(x) = \ell \text{ and } \psi^2_0(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $s = \begin{pmatrix} \ell & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then (by case 2 of Theorem 4.3)

$$\psi_0^1(x) = 1 \text{ and } \psi_0^2(x) = \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the S-sheaf on C_0 that sends every $\sigma \in C_0$ to M, whose total space is $X_0 \otimes M$. Use Theorem 3.1 and the fact that N is assumed to be squarefree. Then by Section 4, we know that s preserves the Γ-orbits of C_0 and $T(\ell, 1) = \oplus_{y \in \widetilde{C}_0} T_{yy}$ where each T_{yy} acts on $\oplus_{\sigma \in \widetilde{C}_0} H_3(\sigma)$, stabilizing each summand. Since $\widetilde{f}$ is supported on $(1 : d : 0)$, we obtain from Theorems 3.1 and 5.1(3),

$$\begin{aligned} Y(\widetilde{f}|T(\ell,1)) &= \sum_{\alpha_1,\alpha_2} f\Big|_{\chi_1} \begin{pmatrix} 1 & 0 \\ \alpha_2 - \alpha_1 d & \ell \end{pmatrix} \\ &\quad + \sum_{\alpha_0 : \ell \nmid \alpha_0 d + 1} f\Big|_{\chi_1} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} + \chi_0(\ell)\ell^c f\Big|_{\chi_1} I + f\Big|_{\chi_1} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \\ &= \ell \left(f\Big|T_\ell - f\Big|_{\chi_1} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \right) + (\ell - 1) f\Big|_{\chi_1} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} + f\Big|_{\chi_1} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} + \chi_0(\ell)\ell^c f \\ &= (\ell\lambda_\ell + \chi_0(\ell)\ell^c) Y(\widetilde{f}). \end{aligned}$$

Similarly, for $T(\ell, 2)$:
If $s = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_0 & \ell & 0 \\ \alpha_1 & 0 & \ell \end{pmatrix}$ with $\ell \nmid \alpha_0 d + 1$, then (by case 3 of Theorem 4.3)

$$\psi_0^1(x) = 1 \text{ and } \psi_0^2(x) = \begin{pmatrix} \ell & 0 \\ -\alpha_1 \ell d & \ell \end{pmatrix}.$$

If $s = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_0 & \ell & 0 \\ \alpha_1 & 0 & \ell \end{pmatrix}$ with $\ell | \alpha_0 d + 1$, then (by case 4 of Theorem 4.3)

$$\psi_0^1(x) = \ell \text{ and } \psi_0^2(x) = \begin{pmatrix} 1 & 0 \\ -\alpha_1 d & \ell \end{pmatrix}.$$

If $s = \begin{pmatrix} \ell & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha_2 & \ell \end{pmatrix}$ then (by case 2 of Theorem 4.3)

$$\psi_0^1(x) = 1 \text{ and } \psi_0^2(x) = \begin{pmatrix} \ell & 0 \\ \alpha_2 \ell & \ell \end{pmatrix}.$$

If $s = \begin{pmatrix} \ell & 0 & 0 \\ 0 & \ell & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then (by case 1 of Theorem 4.3)

$$\psi_0^1(x) = \ell \text{ and } \psi_0^2(x) = \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence,

$$\begin{aligned}
Y(\widetilde{f}|T(\ell,2)) = &\sum_{\alpha_0,\alpha_1:\ell\nmid\alpha_0 d+1} f\Big|_{\chi_1}\begin{pmatrix}\ell & 0\\ 0 & \ell\end{pmatrix} + \sum_{\alpha_1}\chi_0(\ell)\ell^c f\Big|_{\chi_1}\begin{pmatrix}1 & 0\\ -\alpha_1 d & \ell\end{pmatrix}\\
&+\sum_{\alpha_2} f\Big|_{\chi_1}\begin{pmatrix}\ell & 0\\ 0 & \ell\end{pmatrix} + \chi_0(\ell)\ell^c f\Big|_{\chi_1}\begin{pmatrix}\ell & 0\\ 0 & 1\end{pmatrix}\\
= &\,\ell^2 f\Big|_{\chi_1}\begin{pmatrix}\ell & 0\\ 0 & \ell\end{pmatrix} + \chi_0(\ell)\ell^c f\Big| T_\ell\\
= &\,(\chi_1(\ell)\ell^{a+b+2} + \chi_0(\ell)\ell^c\lambda_\ell)Y(\widetilde{f}).
\end{aligned}$$

Therefore $\widetilde{f}$ is attached to $\omega\tau \oplus \omega^c\chi_0 = \rho$, which has predicted weight $F(a,b,c)$. Since $\widetilde{f}$ is not attached to a direct sum of characters, by Theorem 8.1 it not only appears in $E^1_{0,3}$, but also survives into $E^\infty_{0,3}$, and hence appears in $H_3(\Gamma_0(3,N), F(a,b,c)_\chi)$. We have thus proved the following theorem:

Theorem 9.1. *Let $p > 3$ and let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(3, \overline{\mathbb{F}}_p)$ be a Galois representation of squarefree Serre conductor N and nebentype χ that decomposes as a sum of a character and an irreducible odd two-dimensional Galois representation τ. Then ρ is attached to a homology eigenclass in $H_3(\Gamma_0(3,N), V_\chi)$, where V is the first of the two weights predicted by conjecture 2.4.*

Proof of Theorem 2.5: We now verify that the second of the two weights predicted by Conjecture 2.4 also works. Let ${}^t\rho^{-1} \otimes \omega^2$ be the twisted contragredient of ρ. This representation will still be a sum of a two-dimensional, odd, irreducible representation and a character, and as such, will be attached to an eigenclass for its first predicted weight of Conjecture 2.4, by Theorem 9.1. Since Conjecture 2.4 is compatible with duality according to the prescription in [5, Proposition 2.8], we find that ρ is attached to an eigenclass in the dual of this weight. A simple computation shows that this dual is exactly the second predicted weight for ρ. Hence, ρ is attached to eigenclasses in both of the weights described in Conjecture 2.4, concluding the proof of Theorem 2.5.

10. Appendix: Proof of Theorem 4.3

Proof. To prove case 1, we take $\gamma = I$, since s is already in P_d. Hence $x = g_d s g_d^{-1}$, and we find that $x_{11} = \ell_1$, and $\psi_0^2(x) = \begin{pmatrix}\ell_2 & 0\\ c-bd & \ell_3\end{pmatrix}$.

For the other cases, note that a matrix is in $P_0 g_d$ if and only if it is of the form

$$\begin{pmatrix} r & rd & 0\\ * & * & *\\ * & * & *\end{pmatrix}.$$

Since we want

$$g_d s\gamma = \begin{pmatrix} A(\ell_1+ad)+C\ell_2 d & B(\ell_1+ad)+D\ell_2 d & 0\\ * & * & 0\\ * & * & *\end{pmatrix} \in P_0 g_d,$$

we see that we must have $B(\ell_1+ad)+D\ell_2 d = d(A(\ell_1+ad)+C\ell_2 d)$. Writing this as a matrix equation, we have that

$$(\ell_1+ad, d\ell_2)\begin{pmatrix} A & B \\ C & D \end{pmatrix} = r(1,d)$$

for some r. We will assume that $r>0$ (changing the signs of A, B, C, D if needed). Since the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ must have determinant 1, we get

$$(\ell_1+ad, d\ell_2) = r(1,d)\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}.$$

We now specialize to case 2, in which $\ell_1=\ell$ is prime, $\ell_2=1$ and $a=0$. In this case $r=1$ and we take

$$A = 1+\frac{kN}{d}, \quad B = kN, \quad D = \ell + Cd,$$

remembering that we must have $1 = AD - BC$. Now, using the Chinese Remainder Theorem to choose C so that

$$Cd \equiv 1 \pmod{\ell} \text{ and } Cd \equiv 1-\ell \pmod{N/d}$$

we may choose

$$k = \frac{-Cd-\ell+1}{(N/d)\ell}$$

and we have our desired

$$\gamma = \begin{pmatrix} A & kN & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We then find that $\psi_0^1(x) = 1$ and

$$\psi_0^2(x) = \begin{pmatrix} \ell & 0 \\ c\ell - bd & \ell_3 \end{pmatrix}.$$

In case 3, we proceed similarly. We have

$$(1+ad, \ell d) = r(1,d)\begin{pmatrix} D & -B \\ -C & A \end{pmatrix},$$

and again $r=1$. We set $B=kN$ and solve for $A = k(N/d)+\ell$ and $D = (ad+1)+Cd$.

Since $(N(ad+1)/d, \ell d) = 1$, integers C and k exist as needed. Also, we find $\psi_0^1(x) = 1$ and

$$\psi_0^2(x) = \begin{pmatrix} \ell & 0 \\ c(ad+1) - b\ell d & \ell_3 \end{pmatrix}.$$

Finally, for case 4, we want

$$(1+ad, \ell d) = r(1,d)\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}.$$

In this case, $r = \ell$.

Set

$$A = 1+k\frac{N}{d}, \quad B = kN, \quad D = \frac{1+ad}{\ell} + Cd.$$

Since d and $\left(\frac{N}{d}\frac{(1+ad)}{\ell}\right)$ are relatively prime, a solution exists for k and C. We find that $\psi_0^1(x) = \ell$ and

$$\psi_0^2(x) = \begin{pmatrix} 1 & 0 \\ c\left(\frac{1+ad}{\ell}\right) - bd & \ell_3 \end{pmatrix}.$$

□

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References

[1] A. Ash, Direct sums of mod p characters of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) and the homology of $GL(n,\mathbb{Z})$, to appear Comm. in Alg.

[2] A. Ash, Galois representations attached to mod p cohomology of $GL(n,\mathbb{Z})$, Duke Math. J. **65** (1992) 235-255.

[3] A. Ash, D. Doud, and D. Pollack, Galois representations with conjectural connections to arithmetic cohomology, Duke Math. J. **112** (2002) 521-579.

[4] A. Ash, P. Gunnells, M. McConnell, Resolutions of the Steinberg module for $GL(n)$, J. of Algebra *249* (2012) 380-390.

[5] A. Ash and W. Sinnott, An analogue of Serre's conjecture for Galois representations and Hecke eigenclasses in the mod p cohomology of $GL(n,\mathbb{Z})$, Duke Math. J. **105** (2000) 1-24.

[6] K. Brown, *Cohomology of Groups*, Springer, New York 1982.

[7] S. Doty and G. Walker, The composition factors of $\mathbf{F}_p[x_1, x_2, x_3]$ as a GL$(3,p)$-module, J. Algebra, **147** (1992), 411-441.

[8] F. Herzig, The weight in a Serre-type conjecture for tame n-dimensional Galois representations, Duke Math. J. **149** (2009) 37-116.

[9] F. Herzig, A Satake isomorphism in characteristic p, Compositio Math. *147* (2011) 263-283.

[10] J. Humphreys, *Modular representations of finite groups of Lie type*, Cambridge University Press, Cambridge, 2005.

[11] J. C. Jantzen, *Representations of Algebraic Groups, Second Edition*, American Mathematical Society, Providence, 2003.

[12] J.-P. Serre, Propriétés galoisiennes des points d'ordre fini des courbes elliptiques, Invent. Math. **15** (1972) 259-331.

[13] J.-P. Serre, Sur les représentations modulaires de degré 2 de Gal($\bar{\mathbb{Q}}/\mathbb{Q}$), Duke Math. J. **54** (1987) 179-230.

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