HIGHLY REDUCIBLE GALOIS REPRESENTATIONS ATTACHED TO THE HOMOLOGY OF $\mathrm{GL}(n, \mathbb{Z})$

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Abstract. Let $n \geq 1$ and $\mathbb{F}$ an algebraic closure of a finite field of characteristic $p > n+1$. Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n, \mathbb{F})$ be a Galois representation that is isomorphic to a direct sum of a collection of characters and an odd m-dimensional representation τ. We assume that $m = 2$ or m is odd, and that τ is attached to a homology class in degree $m(m-1)/2$ of a congruence subgroup of $\mathrm{GL}(m, \mathbb{Z})$ in accordance with the main conjecture of [4]. We also assume a certain compatibility of τ with the parity of the characters and that the Serre conductor of ρ is square-free. We prove that ρ is attached to a Hecke eigenclass in $H_t(\Gamma, M)$, where Γ is a subgroup of finite index in $\mathrm{SL}(n, \mathbb{Z})$, $t = n(n-1)/2$ and M is an $\mathbb{F}\Gamma$-module. The particular Γ and M are as predicted by the main conjecture of [4]. The method uses modular cosymbols, as in [1].

1. Introduction

Let $n \geq 1$ and $\mathbb{F}$ an algebraic closure of a finite field of characteristic $p > n+1$. We require $p > n+1$ in order to apply the Borel-Serre isomorphism at will. This inequality will be in force throughout this paper. By "ring" we will mean a commutative ring with identity.

By definition, a Galois representation ρ is a continuous, semisimple homomorphism of the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $\mathrm{GL}(m, \mathbb{F})$ for some positive integer m. We say that ρ is odd if the number of positive and negative eigenvalues of a complex conjugation c differ by at most one. Generalizations of Serre's conjecture [11] connect the homology of arithmetic subgroups of $\mathrm{GL}(n, \mathbb{Z})$ with odd n-dimensional Galois representations. Such a conjecture was first published in [6], extended in [4], and further improved in [7].

Let $\tau : G_{\mathbb{Q}} \to \mathrm{GL}(m, \mathbb{F})$ be an odd representation and assume that τ is attached to a homology class in degree $m(m-1)/2$ in accordance with the main conjecture of [4]. Let

$$\rho = \psi_r \oplus \cdots \oplus \psi_1 \oplus \tau \oplus \eta_1 \oplus \cdots \oplus \eta_s$$

where $\psi_i, \eta_i : G_{\mathbb{Q}} \to \mathrm{GL}(1, \mathbb{F})$ are characters. We will allow $r = 0$ or $s = 0$ and we set $n = m + r + s$. We will prove that under certain conditions on ρ, that ρ is attached to a homology class in degree $n(n-1)/2$ with weight, level, and nebentype predicted by the main conjecture of [4].

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We use the method of [1] suitably adapted. That paper proved Theorem 2.4 when $m = 1$ so henceforth we will assume that $m \geq 2$. We will recall the necessary definitions and theorems from that paper. It will be seen that the relevant proofs in that paper apply almost verbatim to the case at hand. What we must do carefully in this paper is to compute the invariants (level, nebentype and weight) for ρ and check they are compatible with the coefficient module in the group homology. We also have to be careful with the relationship between $\rho(c)$ and the action of certain matrices of negative determinant on the homology. This requires many painstaking calculations of signs, which all work out correctly, of course. If $n = 3$, $m = 2$, this paper reproves the results of [2] by a different method.

2. Attached Galois representations and arithmetic homology

At the end of this section we state our main theorem.

Definition 2.1. *Let N be a positive integer.*

1. *$S_0(n, N)^{\pm}$ is the semigroup of matrices $s \in M_n(\mathbb{Z})$ such that $\det(s)$ is relatively prime to pN and the first column of s is congruent to $(*, 0, \ldots, 0)$ modulo N.*
2. *$S_0(n, N)$ is the subsemigroup of $s \in S_0(n, N)^{\pm}$ such that $\det(s) > 0$.*
3. *$\Gamma_0(n, N)^{\pm} = S_0(n, N)^{\pm} \cap \mathrm{GL}(n, \mathbb{Z})$.*
4. *$\Gamma_0(n, N) = S_0(n, N) \cap \mathrm{GL}(n, \mathbb{Z})$.*

Let $\mathcal{H}_{n,N}$ (a Hecke algebra) be the (commutative) $\mathbb{Z}$-algebra under convolution generated by all the double cosets $T(\ell, k) = \Gamma_0(n, N)D(\ell, k)\Gamma_0(n, N)$ with

$$D_{\ell,k} = \mathrm{diag}(\underbrace{1, \cdots, 1}_{n-k}, \underbrace{\ell, \cdots, \ell}_{k}).$$

such that $\ell \nmid pN$.

A Hecke packet is an algebra homomorphism $\phi : \mathcal{H}_{n,N} \to \mathbb{F}$. If W is an $\mathcal{H}_{n,N} \otimes \mathbb{F}$-module, and $w \in W$ is a simultaneous eigenvector for all $T \in \mathcal{H}_{n,N}$, then the associated eigenvalues give a Hecke packet, called "a Hecke eigenpacket that occurs in W."

Definition 2.2. *Let ϕ be a Hecke packet, with $\phi(T(\ell, k)) = a(\ell, k)$. We say that the Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n, \mathbb{F})$ is attached to ϕ if ρ is unramified outside pN and*

$$\det(I - \rho(\mathrm{Frob}_\ell)X) = \sum_{k=0}^{n} (-1)^k \ell^{k(k-1)/2} a(\ell, k) X^k$$

for all prime $\ell \nmid pN$. We use the arithmetic Frobenius and let ω be the cyclotomic character, so that $\omega(\mathrm{Frob}_\ell) = \ell$.

If the Hecke packet comes from a Hecke eigenvector in $w \in W$, we will say ρ is attached to w and fits W.

Note that if ρ is attached to ϕ, the characteristic polynomials of $\rho(\mathrm{Frob}_\ell)$ for almost all prime ℓ are determined by ϕ and hence ρ is determined up to isomorphism, since we are assuming ρ is semisimple.

Let (Γ, S) be the Hecke pair $(\Gamma_0(n, N), S_0(n, N))$ in $\mathrm{GL}(n, \mathbb{Q})$. Let M be an $\mathbb{F}S$ module. Then there is a natural action of a double coset $T(\ell, k) \in \mathcal{H}_{n,N}$ on the homology $H_*(\Gamma, M)$ and on the cohomology $H^*(\Gamma, M)$. We then refer to $T(\ell, k)$ as a Hecke operator. This action makes $H_*(\Gamma, M)$ and $H^*(\Gamma, M)$ into $\mathcal{H}_{n,N}$-modules.

In [3] we proved that ρ fits $H_i(\Gamma, M)$ if and only if ρ fits $H^i(\Gamma, M)$. So whether we work with homology or cohomology is a matter of convenience.

The modules in which we are interested will be of the form $M_\chi := M \otimes \mathbb{F}_\chi$, where M is an irreducible $\mathbb{F}[\mathrm{GL}(n, \mathbb{F}_p)]$-module, and $\chi : S \to \mathbb{F}$ is a character. The irreducible $\mathbb{F}[\mathrm{GL}(n, \mathbb{F}_p)]$-modules are parametrized by p-restricted n-tuples $(a_1, \ldots, a_n)$. To be p-restricted means that $0 \le a_i - a_{i+i} \le p - 1$ for all $i = 1, \ldots, n-1$ and $0 \le a_n \le p - 2$. We will denote the module corresponding to this n-tuple by $F(a_1, \ldots, a_n)$. By allowing for twisting by the determinant character (which has order $p-1$), we will allow the value of a_n to be an arbitrary integer, thus allowing a single module to be described by an infinite number of different n-tuples. See Section 3 for more details concerning these modules.

Definition 2.3. *We will call a module $F(a_1, \ldots, a_n)$ a predicted weight for an odd Galois representation $\rho : G_\mathbb{Q} \to \mathrm{GL}(n, \mathbb{F})$ if it is one of the weights V predicted for ρ by* [4, Conjecture 3.1]

We now state our main theorem. The proof of Theorem 2.4 will be given in the last section of the paper.

Theorem 2.4. *Let $p > n + 1$ and let*

$$\rho = \psi_r \oplus \cdots \oplus \psi_1 \oplus \tau \oplus \eta_1 \oplus \cdots \oplus \eta_s$$

where $m \ge 1$, $\tau : G_\mathbb{Q} \to \mathrm{GL}(m, \mathbb{F})$ is an m-dimensional Galois representation, the $\psi_i, \eta_i : G_\mathbb{Q} \to \mathrm{GL}(1, \mathbb{F})$ are characters, and $n = m + r + s$. We allow $r = 0$ or $s = 0$. Let N', χ' and M' be the Serre conductor, the nebentype, and a predicted weight for τ, and let N and χ be the Serre conductor and nebentype of ρ.

We make the following assumptions:

(1) *τ is attached to a homology class in $H_{m(m-1)/2}(\Gamma_0(m, N'), M'_{\chi'})$.*
(2) *$\tau(c)$ is a diagonal matrix of the form $(e_1, \ldots, e_m)$ where $e_1 = -\psi_1(c)$, $e_m = -\eta_1(c)$, and $e_i = -e_{i+1}$ for $1 \le i < m$.*
(3) *$\psi_i(c) = -\psi_{i+1}(c)$ for $1 < i \le r$ and $\eta_i(c) = -\eta_{i+1}(c)$ for $1 \le i < s$.*
(4) *N is square-free.*
(5) *Either m is odd or $m = 2$ and τ is irreducible.*

Then ρ fits $H_{n(n-1)/2}(\Gamma_0(n, N), M_\chi)$ for some M that is a predicted weight for ρ.

We note that this theorem does not prove that all predicted weights for ρ in [4] will yield a Hecke eigenclass with ρ attached. The weight for which we prove the theorem will be derived from a specific embedding of the image of ρ into a Levi subgroup. In addition, the weight $M = F(c_1, \ldots, c_n)$ given by the theorem will have the property that $0 < c_i - c_{i+1} \le p - 1$ for $i \le r$ and $i \ge r + m$.

The alternation of the parity of the characters in Theorem 2.3 is related to the concept of "strict parity" found in [4]. For more commentary on this see the introductions to the papers [1] and [3].

3. Summary of [1]

This section recalls various notation, definitions and theorems from [1]. Omitted proofs and details may be found in that paper. *From now on we assume that N is square-free.*

For any $x \in \mathbb{Z}$, define the $n \times n$-matrix

$$g_x = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ x & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & & 0 & 1 \end{bmatrix},$$

and the $(n-1) \times n$ matrix

$$M_x = \begin{bmatrix} x & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}.$$

Then $g_x \in \mathrm{SL}(n, \mathbb{Z})$ and $M_x = M_0 g_x$.

Let P_x be the stabilizer in $\mathrm{GL}(n)$ of the row space of M_x. It is a $\mathbb{Q}$-parabolic subgroup. Then

$$P_x = g_x^{-1} P_0 g_x.$$

Also, $P_0 = L_0^1 L_0^2 U_0$ where

$$U_0 = \begin{bmatrix} 1 & * & * & \dots & * \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix},\ L_0^1 = \begin{bmatrix} * & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix},\ L_0^2 = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & * & * & \dots & * \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & * & * & \dots & * \end{bmatrix}.$$

These are algebraic groups defined over $\mathbb{Z}$.

We define $\psi_0 = \psi_0^1 \times \psi_0^2 : P_0 \to \mathrm{GL}(1) \times \mathrm{GL}(n-1)$ by

$$g = \begin{bmatrix} \psi_0^1(g) & * \\ 0 & \psi_0^2(g) \end{bmatrix}.$$

Set

$$U_x = g_x^{-1} U_0 g_x,\ L_x^1 = g_x^{-1} L_0^1 g_x,\ L_x^2 = g_x^{-1} L_0^2 g_x.$$

Then $P_x = U_x(L_x^1 \times L_x^2)$ is a Levi decomposition of P_x. For $s \in P_x$, set $\psi_x^i(s) = \psi_0^i(g_x s g_x^{-1})$ for $i = 1, 2$.

Theorem 3.1. *Let d be a positive divisor of N. Then:*
(1) $U_d L_d^1 \cap \Gamma_0(n, N) = U_d \cap \Gamma_0(n, N)$.
(2) If $s \in P_d \cap S_0(n, N)^{\pm}$, then $\psi_d^1(s) \equiv s_{11} \mod d$ and $\psi_d^2(s)_{11} \equiv s_{11} \mod N/d$.
(3) $\psi_d^2(P_d \cap S_0(n, N)^{\pm}) \subset S_0(n-1, N/d)^{\pm}$.
(4) ψ_d^2 induces an exact sequence:

$$1 \to U_d \cap \Gamma_0(n, N) \to P_d \cap \Gamma_0(n, N) \xrightarrow{\psi_d^2} \Gamma_0(n-1, N/d)^{\pm} \to 1.$$

If σ is a permutation or a permutation matrix, $(-1)^{\sigma}$ denotes the sign of σ. If $v_1, \dots, v_m$ are row vectors of the same length, $[v_1, \dots, v_m]$ denotes the matrix with those rows.

If X is an $m \times m$ matrix with rows $v_1, \dots, v_m$ and w is a row vector of length m and $1 \leq j \leq m$, we set $X_j[w]$ to be the matrix X with the j-th row replaced by

w. We denote by $M'_m(A)$ the $m \times m$ matrices with coefficients in A with nonzero rows.

Definition 3.2. *Let $m \geq 1$. Let R and A be rings. Let Γ be a subgroup of $\mathrm{GL}(m, A)$ and let Γ' be a subgroup of $\mathrm{GL}(m, A)$ containing and normalizing Γ. Suppose E is a right $R[\Gamma']$-module. Let $\theta : \Gamma' \to R^\times$ be a character that is trivial on Γ.*

An R-valued cosymbol over A for (Γ', E, θ) is an R-linear function

$$f : R[M'_m(A)] \otimes_R E \to R$$

satisfying for all $X \in M'_m(A), e \in E$:

(1) $f(X \otimes e) = f(\Lambda X \otimes e)$ for any diagonal matrix $\Lambda \in \mathrm{GL}(m, A)$;

(2) $f(X \otimes e) = (-1)^\sigma f(\sigma X \otimes e)$ for any permutation matrix σ;

(3) $f(X \otimes e) = 0$ if $\det(X) = 0$;

(4) $f(X \otimes e) = \sum_{i=0}^m f(X_i[w] \otimes e)$ for any nonzero $w \in A^m$;

(5) $f(X \otimes e) = \theta(t) f(Xt \otimes e|t)$ for all $t \in \Gamma'$.

A cosymbol is determined by its values on the tensors of the form $X \otimes e$ where $X \in M'_m(A)$ and $e \in E$. It is called a "cosymbol" (or a "modular cosymbol") because it is a linear functional on the space of $m \times m$ modular symbols as defined in [5]. We could have worked with all $m \times m$ matrices X in the definition of cosymbol, but working with $M'_m(A)$ brings out more clearly the relationship to modular symbols.

Let f be an R-valued cosymbol over A for (Γ', E, θ). Suppose (Γ, S) is a Hecke pair contained in $M_m(A)$ and that θ extends to a character of the semigroup S' generated by Γ' and S which is trivial on S. We also suppose that E has an S'-module structure extending the given action of Γ'.

If $s \in S$, we define the action of a double coset $T = \Gamma s \Gamma = \coprod_\alpha s_\alpha \Gamma$ on f by

$$T(f)(X \otimes e) = \sum_\alpha f(X s_\alpha \otimes e | s_\alpha).$$

Then $T(f)$ is a cosymbol for $(\Gamma, E, 1)$. If f is an eigenvector for T or if $\Gamma t s \Gamma = \Gamma s t \Gamma$ for all $t \in \Gamma'$ and $s \in S$, then $T(f)$ is a cosymbol for (Γ', E, θ).

Lemma 3.3. *Let $A = \mathbb{Q}$, Γ a p-torsionfree subgroup of finite index in $\mathrm{SL}(m, \mathbb{Z})$ and R a field of characteristic p or 0. Let $\nu = m(m-1)/2$. Suppose (Γ, S) is a Hecke pair in $\mathrm{GL}(m, \mathbb{Q})^+$. Let E be a right $R[S]$-module. Then there is an $\mathcal{H}(\Gamma, S)$-equivariant injective map of R-vector spaces:*

$$\beta : \{\textit{cosymbols for } (\Gamma', E, \theta)\} \to H_\nu(\Gamma, E).$$

This map is an isomorphism if $\Gamma' = \Gamma$.

Here, $\mathrm{GL}(m, \mathbb{Q})^+$ denotes the matrices with positive determinant and $\mathcal{H}(\Gamma, S)$ denotes the Hecke algebra of double cosets $\Gamma \backslash S / \Gamma$.

Fix a positive integer N relatively prime to p and d a positive divisor of N.

Let $N' = N/d$. Fix a ring R. If $\chi : (\mathbb{Z}/N)^\times \to R^\times$ is a character, we factor χ as $\chi = \chi_1 \overline{\chi}$ where χ_1 has conductor dividing d and $\overline{\chi}$ has conductor dividing N'. For $s \in S_0(m, N)^\pm$ we have the pulled-back character $\chi(\bar{s}_{11})$, where $\bar{s}_{11}$ denotes the reduction of the $(1,1)$-entry of s modulo N. We denote this pulled-back character also by χ.

Let E be an $R[\mathrm{GL}(m,\mathbb{Z}/p)]$-module. Define an action $e|^*s$ of $s \in S = S_0(m,N)^\pm$ on E by letting s act via its reduction modulo p. If $x \in \mathbb{Z}$, define the S-module $E_x(\chi)$ to be the module with underlying group E and action $e|^x_\chi s = \chi(s)e|^* g_x s g_x^{-1}$. If $x = 0$ we omit the "x" from the notation, writing $E(\chi)$ or E_χ, and $e|_\chi s = \chi(s)e|^*s$. Note that $E(\chi)$ and $E_x(\chi)$ are isomorphic S-modules, with intertwining operator $|^*g_x$.

Now let F be an $R[\mathrm{GL}(n,\mathbb{Z}/p)]$-module. Let $\overline{F} = F_{U_0(\mathbb{Z}/p)}$ be the co-invariants and denote the natural projection $F \to \overline{F}$ by an overline. Then $\overline{F}$ is a right-module for $\mathrm{GL}(1,\mathbb{Z}/p) \times \mathrm{GL}(n-1,\mathbb{Z}/p)$: Lift $g' \in \mathrm{GL}(1,\mathbb{Z}/p) \times \mathrm{GL}(n-1,\mathbb{Z}/p)$ to $g \in P_0(\mathbb{Z}/p)$. For any $f \in F$, set $\overline{f}|^*g' = \overline{f|^*g}$. This action is well-defined. It pulls back to an action of $\{1\} \times S_0(n-1,N')^\pm$ via reduction modulo p.

From (3) of Theorem 3.1, we know that $\psi_d^2(S_0(n,N)^\pm) \subset S_0(n-1,N')^\pm$.

Definition 3.4. *Let $\mu : \mathbb{Z} \setminus \{0\} \to R$ be the product of a Dirichlet character and a character of the form $t \mapsto t^b$ for some nonnegative integer b. We say that (F,χ,d) has character μ if for any $e \in F_d(\chi)$, and any element $s \in P_d \cap S_0(m,N)^\pm$, we have*

$$\overline{e|^d_\chi s} = \mu(\psi_d^1(s))\overline{e}|_{\overline{\chi}}\psi_d^2(s).$$

Let B_m be the Borel subgroup of $\mathrm{GL}(m)$ of lower triangular matrices, and T_m the maximal torus of diagonal matrices. (We use lower triangular matrices because our group actions are on the right.) An m-tuple $\underline{a} = (a_1,\dots,a_m)$ of integers denotes the algebraic weight with respect to (B_m,T_m) given by $\mathrm{diag}(t_1,\dots,t_m) \mapsto t_1^{a_1}t_2^{a_2}\cdots t_m^{a_m}$. It is dominant if $a_1 \geq a_2 \geq \cdots \geq a_m$.

A dominant weight $\underline{a}$ is called "p-restricted" if $0 \leq a_1 - a_2, \dots, a_{m-1} - a_m \leq p-1$ and $0 \leq a_m \leq p-2$. For any p-restricted weight there exists (up to isomorphism) a unique irreducible right $\mathbb{F}_p[\mathrm{GL}(m,\mathbb{Z}/p\mathbb{Z})]$-module $F(\underline{a})$ with highest weight $\underline{a}$, and all such modules occur this way. These modules are absolutely irreducible.

Similarly, all the irreducible $\mathbb{F}_p[\mathrm{GL}(1,\mathbb{Z}/p\mathbb{Z}) \times \mathrm{GL}(m-1,\mathbb{Z}/p\mathbb{Z})]$-modules are classified by m-tuples $\underline{a}$ such that $(a_2,\dots,a_m)$ is p-restricted for $\mathrm{GL}(m-1)$ and a_1 is considered modulo $p-1$. Denote the $\mathbb{F}_p[\mathrm{GL}(1,\mathbb{Z}/p\mathbb{Z}) \times \mathrm{GL}(m-1,\mathbb{Z}/p\mathbb{Z})]$-module corresponding to $\underline{a}$ by $F'(\underline{a})$. These are weights with respect to the pair $(\mathrm{GL}(1,\mathbb{Z}/p\mathbb{Z}) \times B_{m-1}, \mathrm{GL}(1,\mathbb{Z}/p\mathbb{Z}) \times T_{m-1})$.

We view any $\mathrm{GL}(m,\mathbb{Z}/p\mathbb{Z})$-module also as a $\mathrm{GL}(m,\mathbb{Z}_p)$-module, where an integral matrix acts through its reduction modulo p. Now set $R = \mathbb{F}$, an algebraic closure of $\mathbb{F}_p$.

Theorem 3.5. *Let $n \geq 2$, $\underline{a}$ be p-restricted and set $F = F(\underline{a})$.*

(1) The module of coinvariants $\overline{F} = F_{U_0(\mathbb{Z}/p)}$ is isomorphic to $F'(\underline{a})$.

(2) Let d be a positive divisor of N. Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{F}^\times$ be a character, and factor χ as $\chi = \chi_1\overline{\chi}$ where χ_1 has conductor dividing d and $\overline{\chi}$ has conductor dividing N/d. Then (F,χ,d) has character μ, where $\mu(t) = \chi_1(t)t^{a_1}$.

We now summarize the main constructions of [1] in the following theorem.

Theorem 3.6. *Let d, $F = F(a_1,\dots,a_n)$, and $\chi = \chi_1\overline{\chi}$ be as in Theorem 3.5. If $d = 1$, assume that $a_1 > a_2$. Assume that N is square-free and let $N' = N/d$. Assume that (F,χ,d) has character μ. Let θ be an $\mathbb{F}^\times$-valued character of $\Gamma_0(n-1,N')^\pm$ which is trivial on $\Gamma_0(n-1,N')$.*

Let ϕ' be an $\mathbb{F}$-valued cosymbol over $\mathbb{Q}$ for $(\Gamma_0(n-1,N'),\overline{F}_{\overline{\chi}},\theta)$. Assume that for every $\varpi \in P_d \cap S_n$,

$$\mu(\psi_d^1(\varpi)) = \theta(\psi_d^2(\varpi)).$$

Let $s \in P_d \cap S_0(m,N)^{\pm}$ such that $\psi_d^2(s) = 1$, and s normalizes $\Gamma_0(n,N)$. Let Γ' be the group generated by $\Gamma_0(n,N)$ and s. Assume there exists a character Θ of Γ' such that Θ is trivial on $\Gamma_0(n,N)$ and $\Theta(s) = \mu(\psi_d^1(s))$.

Let μ^* be the character of $G_{\mathbb{Q}}$ satisfying $\mu^*(Frob_\ell) = \mu(\ell)$.

Assume that there is a Galois representation $\rho' : G_{\mathbb{Q}} \to \mathrm{GL}(n-1,\mathbb{F})$ that is attached to ϕ'.

Then there exists a Hecke eigencosymbol Φ_d for $(\Gamma', F_\chi, \Theta^{-1})$ such that the Galois representation $\mu^*\omega^{n-1} \oplus \rho' : G_{\mathbb{Q}} \to \mathrm{GL}(n,\mathbb{F})$ is attached to Φ_d.

In [1] the cosymbol Φ_d was said to have coefficients $F_d(\chi)$ rather than $F(\chi) = F_\chi$, but these two representations of $S_0(n,N)^{\pm}$ are isomorphic since g_d is an integral unimodular matrix.

We will need the following matrix: Choose an integer x such that $dx \equiv 2 \pmod{N'}$. Let s_d^n be the following $n \times n$ matrix of determinant -1:

$$s_d^n = g_d^{-1} \begin{bmatrix} -1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix} g_d.$$

Then s_d^n has order 2 and the hypotheses of Theorem 3.6 are satisfied with $s = s_d^n$, $\Gamma' = \Gamma_0(n,N)^{\pm}$ and $\Theta = \det$.

4. Galois representations attached to cosymbols

Theorem 4.1. *Assume m is odd, and let τ be an m-dimensional Galois representation, with $\tau(c)$ diagonal, with alternating signs on the diagonal. Assume that τ is attached to an eigenclass in $H_{m(m-1)/2}(\Gamma_0(m,N), F(a_1,\dots,a_m)_\chi)$, with $F(a_1,\dots,a_m)_\chi$ as predicted by the main conjecture of [4]. If θ is given by*

$$\theta = \begin{cases} \det & \text{if } \tau(c)_{11} = -1, \\ 1 & \text{if } \tau(c)_{11} = 1, \end{cases}$$

then τ is attached to a cosymbol for $(\Gamma_0(m,N)^{\pm}, F(a_1,\dots,a_m)_\chi, \theta)$.

Proof. By Lemma 3.3, we see that τ is attached to a cosymbol ϕ for

$$(\Gamma_0(m,N), F(a_1,\dots,a_m)_\chi, 1).$$

In order to show the desired result, it suffices to determine the character θ associated to ϕ on a single element of $\Gamma_0(m,N)^{\pm}$ of determinant -1. We note that $-I \in \Gamma_0(m,N)^{\pm}$, and since m is odd, $-I$ has determinant -1. As a scalar matrix, $-I$ acts on $F(a_1,\dots,a_m)_\chi$ via the central character, as $(-1)^{a_1+\dots+a_m}\chi(-1)$. Combining Definition 3.2(1) and Definition 3.2(5), we see that we must have

$$\theta(-I) = (-1)^{a_1+\dots+a_m}\chi(-1).$$

Examining the main conjecture of [4], we find that τ has a derived m-tuple $(b_1,\dots,b_m)$, which relates to $(a_1,\dots,a_m)$ by the formula $b_i - (m-i) \equiv a_i \pmod{p-1}$. Further, $\det\tau = \omega^{b_1+\dots+b_m}\chi$. Hence,

$$\det(\tau(c)) = (-1)^{b_1+\dots+b_m}\chi(-1) = (-1)^{m(m-1)/2}(-1)^{a_1+\dots+a_m}\chi(-1).$$

Finally, one checks easily that $\tau(c)_{11} = (-1)^{(m-1)/2}\det(\tau(c))$ (because the signs on the diagonal of $\tau(c)$ alternate). Since m is odd, we see that

$$\tau(c)_{11} = (-1)^{a_1+\dots+a_m}\chi(-1)$$

gives the action of $-I$ on ϕ. The theorem follows. ∎

Theorem 4.2. *Let τ be an irreducible two-dimensional Galois representation, and assume that τ is attached to a Hecke eigenclass in $H_1(\Gamma_0(2,N), F(a_1,a_2)_\chi)$, with weight and level predicted by [4]. Then there are cosymbols ϕ for*

$$(\Gamma_0(2,N)^{\pm}, F(a_1,a_2)_\chi, 1)$$

and ϕ' for

$$(\Gamma_0(2,N)^{\pm}, F(a_1,a_2)_\chi, \det),$$

each of which has τ attached.

Proof. The representation τ is attached to two Hecke eigenclasses, one arising from a holomorphic cusp form, and the other arising from an antiholomorphic cusp form. The corresponding cosymbols will be swapped by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so their sum will be attached to τ and have $\theta = 1$, and their difference will be attached to τ and have $\theta = \det$. $\square$

Remark 4.3. We expect a similar result to hold for odd τ of any even dimension, but for $m > 2$ have been unable to prove it. For $m > 3$ in the "top" dimension $m(m+1)/2$, the cuspidal cohomology is trivial, so we think that it is likely that an m-dimensional irreducible τ attached to a homology class in the "top" dimension cannot exist if $m > 3$, although we have no way of proving this either.

Theorem 4.4. *If an odd $\tau : G_{\mathbb{Q}} \to \mathrm{GL}(n,\mathbb{F})$ is attached to an eigencosymbol ϕ for*

$$(\Gamma_0(n,N)^{\pm}, F(a_1,\ldots,a_n)_\chi, \theta)$$

then ${}^t\tau^{-1}$ is attached to an eigencosymbol for

$$(\Gamma_0(n,N)^{\pm}, (F(-a_n,\ldots,-a_1) \otimes \det{}^{-(n-1)})_{\chi^{-1}}, \theta'),$$

where $\theta' = \theta \cdot \det^{n-1}$

Proof. We note that this follows immediately from [6, Theorem 2.8] and Theorem 4.1 for n odd (using the fact that $n-1$ is even).

For arbitrary n, consider the duality map described in [6, Theorem 2.8] induced by the outer automorphism of $\mathrm{GL}(n)$ given by $\alpha : g \mapsto m^{-1}{}^tg^{-1}m$ where $m = \mathrm{diag}(N,1,\ldots,1)$. It induces an isomorphism between

$$H_{n(n-1)/2}(\Gamma_0(n,N), F(a_1,\ldots,a_n)_\chi)$$

and

$$H_{n(n-1)/2}(\Gamma_0(n,N), F(-a_n,\ldots,-a_1)_{\chi^{-1}})$$

which (because α leaves $\mathrm{diag}(-1,1,\ldots,1)$ invariant) preserves the action of $\Gamma_0(n,N)^{\pm}$. Hence, a cosymbol ϕ for

$$(\Gamma_0(n,N)^{\pm}, F(a_1,\ldots,a_n)_\chi, \theta)$$

will correspond (under the duality isomorphism) to a cosymbol $\phi^\vee$ for

$$(\Gamma_0(n,N)^{\pm}, F(-a_n,\ldots,-a_1)_{\chi^{-1}}, \theta).$$

If τ is attached to ϕ, then ${}^t\tau^{-1}$ will be attached to a twist of $\phi^\vee$ by $\det^{-(n-1)}$, which will be a cosymbol for

$$(\Gamma_0(n,N)^{\pm}, (F(-a_n,\ldots,-a_1) \otimes \det{}^{-(n-1)})_{\chi^{-1}}, \theta \cdot \det{}^{-(n-1)}).$$

The lemma follows from the fact that det has order 2 as a character of $\Gamma_0(n,N)^{\pm}$. $\square$

5. Proof of the main theorem

Lemma 5.1. *Let τ be an $(n-1)$-dimensional Galois representation of Serre conductor N' and nebentype character $\bar{\chi}$. Let χ_1 be a character with conductor equal to d, where $N = N'd$ is square-free and prime to p. Let a_1 satisfy $a_2 < a_1 \leq a_2+p-1$. Assume that τ is attached to an eigencosymbol ϕ' for $(\Gamma_0(n-1), N')^{\pm}, F(a_2, \ldots, a_n)_{\bar{\chi}}, \theta)$, with*

$$\theta = \begin{cases} 1 & \text{if } \chi_1(c)(-1)^{a_1} = 1 \\ \det & \text{if } \chi_1(c)(-1)^{a_1} = -1, \end{cases}$$

and set $\Theta = \det^a$ if $\theta = \det^a$. Let $F = F(a_1, \ldots, a_n)$ and $\chi = \chi_1\overline{\chi}$.

Then there is a Hecke eigencosymbol Φ_d for $(\Gamma_0(n, N)^{\pm}, F_\chi, \Theta)$ such that the Galois representation

$$\chi_1\omega^{a_1+(n-1)} \oplus \tau$$

is attached to Φ_d.

If $F(a_2, \ldots, a_n)$ is a predicted weight for τ, then F is a predicted weight for $\chi_1\omega^{a_1+(n-1)} \oplus \tau$.

Proof. Set $F = F(a_1, \ldots, a_n)$ and $\chi = \chi_1\bar{\chi}$. Note that by Theorem 3.5, (F, d, χ) has character $\mu(t) = \chi_1(t)t^{a_1}$. We then have that

$$\Theta(s_d^n) = \chi(-1)(-1)^{a_1} = \mu(-1) = \mu(s_d^n),$$

and since $\psi_d^1(\varpi)\det(\psi_d^2(\varpi)) = \det(\varpi) = 1$, we get

$$\mu(\psi_d^1(\varpi)) = \theta(\psi_d^2(\varpi))$$

for all $\varpi \in P_d \cap \Gamma_0(n, N)$. Hence, by Theorem 3.6, there is a Hecke eigencosymbol Φ_d for $(\Gamma_0(n, N)^{\pm}, F_\chi, \Theta)$ such that the Galois representation

$$\chi_1\omega^{a_1+(n-1)} \oplus \tau$$

is attached to Φ_d.

If $F(a_2, \ldots, a_n)$ is a predicted weight for τ, then τ has a derived n-tuple (see [4, Section 2]) congruent to $(a_2 + (n-2), a_3 + (n-3), \ldots, a_n) \bmod p-1$. Then $\chi_1\omega^{a_1+(n-1)} \oplus \tau$ has a derived n-tuple congruent to $(a_1 + (n-1), \ldots, a_n) \bmod p-1$. It follows that F is a predicted weight for $\chi_1\omega^{a_1+(n-1)} \oplus \tau$. □

Lemma 5.2. *Let τ be an $(n-1)$-dimensional Galois representation of level N' and character $\bar{\chi}$. Let χ_1 be a character with conductor d, where $N = N'd$ is square-free and prime to p, and let $a_n < a_{n-1} - 1 \leq a_n + p - 1$. Assume that τ is attached to an eigencosymbol ϕ' for*

$$(\Gamma_0(n-1), N')^{\pm}, F(a_1, \ldots, a_{n-1})_{\bar{\chi}}, \theta),$$

with

$$\theta = \begin{cases} 1 & \text{if } \chi_1(c)(-1)^{a_n} = -1 \\ \det & \text{if } \chi_1(c)(-1)^{a_n} = 1. \end{cases}$$

Set $F = F(a_1 - 1, \ldots, a_{n-1} - 1, a_n)$ and $\chi = \chi_1\bar{\chi}$.

Setting $\Theta = \det^a$ if $\theta = \det^a$, then there is a Hecke eigencosymbol Φ_d for $(\Gamma_0(n, N)^{\pm}, F_\chi, \Theta \cdot \det)$ such that the Galois representation

$$\tau \oplus \chi_1\omega^{a_n}$$

is attached to Φ_d.

If $F(a_1, \dots, a_{n-1})$ is a predicted weight for τ, then F is a predicted weight for $\tau \oplus \chi_1\omega^{a_n}$.

Proof. We note that by Theorem 4.4, ${}^t\tau^{-1}$ is attached to an eigencosymbol $\phi^\vee$ for

$$(\Gamma_0(n-1,N)^{\pm}, F(-a_{n-1}-(n-2), \dots, -a_1-(n-2))_{\bar{\chi}^{-1}}, \theta \cdot \det{}^{n-2}).$$

Set $F' = F(-a_n-(n-1), -a_{n-1}-(n-2), \cdots, -a_1-(n-2))$. Then we see by Theorem 3.5 that (F', χ^{-1}, d) has character $\mu(t) = \chi_1^{-1}(t)t^{-a_n-(n-1)}$.

In order to apply Theorem 3.6, we need to verify that

$$\mu(\psi_d^1(\varpi)) = (\theta \cdot \det{}^{n-2})(\psi_d^2(\varpi))$$

for all $\varpi \in P_d \cap S_n$.

If $\psi_d^1(\varpi) = 1$, both sides of the equation are 1. We thus assume that $\psi_d^1(\varpi) = -1 = \det(\psi_d^2(\varpi))$. Since

$$\mu(-1) = \chi_1^{-1}(-1)(-1)^{-a_n}(-1)^{n-1} = -\theta(\psi_d^2(\varpi))(-1)^{n-1} = (\theta \cdot \det{}^{n-2})(\psi_d^2(\varpi)),$$

we see that the equation holds.

Also,

$$(\Theta \cdot \det{}^{n-2})(s_d^n) = (-1)^{a+(n-2)} = \mu(-1) = \mu(\psi_d^1(s_d^n)).$$

Hence, by Theorem 3.6, we see that $\omega^{-a_n}\chi_1^{-1} \oplus {}^t\tau^{-1}$ is attached to a Hecke eigencosymbol for

$$(\Gamma_0(n,N)^{\pm}, F(-a_n-(n-1), -a_{n-1}-(n-2), \cdots, -a_1-(n-2))_{\chi^{-1}}, \Theta \cdot \det{}^{n-2}).$$

Applying Theorem 4.4, we find that $\tau \oplus \omega^{a_n}\chi_1$ is attached to a Hecke eigencosymbol for

$$(\Gamma_0(n,N)^{\pm}, F(a_1-1, \dots, a_{n-1}-1, a_n)_{\chi}, \Theta \cdot \det),$$

as desired.

If $F(a_1, \dots, a_{n-1})$ is a predicted weight for τ, then τ has a derived n-tuple congruent to $(a_1+(n-2), a_2+(n-3), \dots, a_{n-1})$ mod $p-1$. Then $\tau \oplus \chi_1\omega^{a_n}$ has a derived n-tuple congruent to $(a_1+(n-2), \dots, a_{n-1}, a_n)$ mod $p-1$. It follows that F is a predicted weight for τ. $\square$

Proof of Theorem 2.4: To prove the theorem, we will show that ρ is attached to a Hecke eigencosymbol of the correct type, for we can then invoke Lemma 3.3.

We proceed by induction on r in the case that $s = 0$. Write $\tau_i = \psi_i \oplus \cdots \oplus \psi_1 \oplus \tau$ and let $\tau_0 = \tau$. Factor each ψ_i as $\omega^{b_i}\bar{\chi}_i$, where $\bar{\chi}_i$ has conductor d_i prime to p. Assume that τ has Serre conductor d_0, and nebentype character $\bar{\chi}_0$. Define $N_i = \prod_{j=0}^{i} d_j$, and $\chi_i = \prod_{j=0}^{i} \bar{\chi}_j$. Under the hypotheses of the theorem, $\tau_i(c)$ has alternating signs on the diagonal.

Assume that τ_i is attached to a cosymbol for

$$(\Gamma_0(m+i, N_i)^{\pm}, F(b_i-(m+i-1), \dots, b_1-m, a_1, \cdots, a_m)_{\chi_i}, \theta_i)$$

where

$$\theta_i = \begin{cases} 1 & \text{if } \tau_i(c)_{11} = (-1)^{m+i-1} \\ \det & \text{if } \tau_i(c)_{11} = (-1)^{m+i}. \end{cases}$$

We note that $(-1)^{b_{i+1}}\bar{\chi}_{i+1}(c) = \psi_{i+1}(c) = -\tau_i(c)_{11}$, so we have

$$\theta_i = \begin{cases} 1 & \text{if } (-1)^{b_{i+1}-(m+i)}\bar{\chi}_{i+1}(c) = 1 \\ \det & \text{if } (-1)^{b_{i+1}-(m+i)}\bar{\chi}_{i+1}(c) = -1. \end{cases}$$

Hence, by Lemma 5.1, τ_{i+1} is attached to a cosymbol ϕ_{i+1} for

$$(\Gamma_0(m+i+1, N_{i+1})^{\pm}, F(b_{i+1}-(m+i), \ldots, b_1-m, a_1, \ldots, a_m)_{\chi_{i+1}}, \theta_{i+1})$$

where $\theta_{i+1} = 1$ if and only if $\theta_i = 1$.

We note that if the weight of ϕ_i is a predicted weight for τ_i, then the weight of ϕ_{i+1} is a predicted weight for τ_{i+1}. In addition

$$\theta_{i+1} = \begin{cases} 1 & \text{if } \tau_{i+1}(c)_{11} = (-1)^{m+(i+1)-1} \\ \det & \text{if } \tau_{i+1}(c)_{11} = (-1)^{m+(i+1)}. \end{cases}$$

since $\tau_{i+1}(c)_{11} = -\tau_i(c)_{11}$. Hence, by induction, Theorem 2.4 is true for $s = 0$.

Now assume that $r = 0$. Let $\tau_0 = \tau$ have level d_0 and character $\bar{\chi}_0$, and let $\tau_i = \tau \oplus \eta_1 \oplus \cdots \oplus \eta_i$. Factor each ψ_i as $\omega^{b_{m+i}}\bar{\chi}_i$, where $\bar{\chi}_i$ has Serre conductor d_i prime to p, and set $N_i = \prod_{i=0}^{i} d_i$, and $\chi_i = \prod_{i=o}^{i} \bar{\chi}_i$

Assume that τ_i is attached to a cosymbol ϕ_i for

$$(\Gamma_0(m+i, N_i)^{\pm}, F(a_1-i, \ldots, a_m-i, b_{m+1}-(i-1), \ldots, b_{m+i})_{\chi_i}, \theta_i)$$

where

$$\theta_i = \begin{cases} 1 & \text{if } \tau_i(c)_{11} = (-1)^{m+i-1} \\ \det & \text{if } \tau_i(c)_{11} = (-1)^{m+i}. \end{cases}$$

Using the fact that $\tau_i(c)$ has alternating signs on the diagonal, we see that $\eta_i(c) = (-1)^{m+i-1}\tau_i(c)_{11} = (-1)^{m+i-1}\tau_0(c)_{11}$. Hence,

$$\theta_i = \begin{cases} 1 & \text{if } \eta_{i+1}(c) = -1 \\ \det & \text{if } \eta_{i+1}(c) = 1. \end{cases}$$

Then, by Lemma 5.2, we find that τ_{i+1} is attached to a cosymbol ϕ_{i+1} for

$$(\Gamma_0(m+i+1, N_i)^{\pm}, F(a_1-(i+1), \cdots, a_m-(i+1), b_{m+1}-i, \cdots, b_{m+i+1})_{\chi_{i+1}}, \theta_{i+1}),$$

where $\theta_{i+1} = 1$ if and only if $\theta_i = \det$. We see that θ_{i+1} satisfies

$$\theta_{i+1} = \begin{cases} 1 & \text{if } \tau_{i+1}(c)_{11} = (-1)^{m+(i+1)-1} \\ \det & \text{if } \tau_{i+1}(c)_{11} = (-1)^{m+(i+1)}, \end{cases}$$

since $\tau_{i+1}(c)_{11} = \tau_i(c)_{11}$.

We note that if the weight of ϕ_i is predicted for τ_i, then the weight of ϕ_{i+1} is a predicted weight for τ_{i+1}. Hence, by induction, the theorem is true for $r = 0$.

The proof of the theorem is complete upon noting that the initial conditions for both inductions are satisfied by τ if τ is of odd dimension (by Theorem 4.1), if τ is irreducible of dimension two (by Theorem 4.2), or if τ results from either of the induction steps. □

Remark 5.3. Note that in order to apply the theorem, we need examples of m-dimensional representations τ that are attached to Hecke eigenclasses in degree $m(m-1)/2$ with the "correct" θ. By Theorem 4.2 and Serre's conjecture ([11, 8, 9, 10]), any odd, irreducible two-dimensional Galois representation will satisfy the condition. Examples of odd irreducible three-dimensional representations (namely symmetric squares) provably attached to eigenclasses in degree 3 homology may be found in [4, Section 4]. The same paper contains many other examples which are probably so attached. By Theorem 4.1 these will all have the correct θ. As we have said in Remark 4.3, in dimensions m higher than 3, we do not expect any irreducible Galois representations to be attached to homology eigenclasses in degree $m(m-1)/2$.

Remark 5.4. Condition (5) of Theorem 2.4 is used only to apply Theorem 4.1 or Theorem 4.2 to obtain the correct θ for the representation τ. We could relax this condition by removing the requirement that τ be odd-dimensional or irreducible and two-dimensional, and simply require that τ is attached to a cosymbol with the correct θ, and the theorem would remain true.

Remark 5.5. Note that if τ is attached to a cosymbol in a weight not predicted by [4], but with the correct θ, then the representation ρ will also be attached to an eigenclass in a weight not predicted by [4]. Computational examples of irreducible odd three-dimensional τ that appear to be attached to eigenclasses in weights not predicted by [4] are discussed in [4, Remark 3.4].

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