RELAXATION OF STRICT PARITY FOR REDUCIBLE GALOIS REPRESENTATIONS ATTACHED TO THE HOMOLOGY OF GL(3, Z)

AVNER ASH AND DARRIN DOUD

ABSTRACT. We prove the following theorem: Let $\mathbb{F}$ be an algebraic closure of a finite field of characteristic $p > 3$. Let ρ be a continuous homomorphism from the absolute Galois group of $\mathbb{Q}$ to $\mathrm{GL}(3, \mathbb{F})$ which is isomorphic to a direct sum of a character and a two-dimensional odd irreducible representation. We assume that the image of ρ is contained in the intersection of the stabilizers of the line spanned by e_2 and the plane spanned by e_1, e_3, where e_i denotes the standard basis. Such ρ will not satisfy the strict parity conditions of [4]. Under the conditions that the Serre conductor of ρ is squarefree, that the predicted weight (a, b, c) lies in the lowest alcove, and that $c \not\equiv b+1 \pmod{p-1}$, we prove that ρ is attached to a Hecke eigenclass in $H_2(\Gamma, M)$, where Γ is a subgroup of finite index in $SL(3, \mathbb{Z})$ and M is an $\mathbb{F}\Gamma$-module. The particular Γ and M are as predicted by the main conjecture of [4], minus the requirement for strict parity.

1. INTRODUCTION

In our generalization of Serre's conjecture in [4], we stipulated a parity condition stronger than mere "oddness" on a given reducible Galois representation. This was called "strict parity." As we explain further below, we incorporated this condition in our generalized conjecture on the basis of experimental evidence for GL(3). We now believe that this evidence was too partial, and that strict parity is not necessary for these Galois representations to be modular. In this paper we show that this is in fact true for a large class of reducible 3-dimensional Galois representations. The problem with our experiments was that they were restricted to the top non-vanishing homological degree.

Fix a prime $p > 3$ and fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. In this paper, a Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n, \mathbb{F})$ is a continuous, semisimple representation of the absolute Galois group of $\mathbb{Q}$, considered as a specific homomorphism. A Galois representation will be called odd if the number of positive and negative eigenvalues of complex conjugation differ by at most one (if $p > (n + 1)$, we can state this by saying that the trace of the image of complex conjugation is 0 or ± 1). Generalizations of Serre's conjecture [18] connect the homology of arithmetic subgroups of $\mathrm{GL}(n, \mathbb{Z})$ with odd Galois representations $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n, \mathbb{F})$. Such a conjecture was first published in [6], was extended in [4], and further improved in [12].

Note should be taken of the recent spectacular work of Peter Scholze [17], which is a converse to these conjectures. Namely (conditional on stabilization of the twisted trace formula) he proves that any system of Hecke eigenvalues occurring in the mod p cohomology of a congruence subgroup of $\mathrm{GL}(n, \mathbb{Z})$ has an attached Galois representation. The Serre type conjectures in the previous paragraph are still open and presumably even harder to prove than Scholze's result. However, Scholze's theorem gives additional reason to believe the conjectures to be true.

In this paper, we will examine the conjecture of [4] for Galois representations ρ which are isomorphic to $\tau \oplus \psi$, where $\tau : G_{\mathbb{Q}} \to \mathrm{GL}(2, \mathbb{F})$ is an odd representation, and $\psi : G_{\mathbb{Q}} \to \mathrm{GL}(1, \mathbb{F})$ is a character. For a reducible representation such as ρ, the weights predicted by the main conjecture of [4] depend on choosing conjugates of ρ which have image in one of the three standard $(2, 1)$-Levi subgroups:

$$
\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad
\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad \text{and} \quad
\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}.
$$

We will call the third of these standard Levi subgroups the *medial* Levi subgroup, and we will say that a conjugate of ρ is a *medial embedding* if its image is contained inside the medial Levi subgroup.

A representation ρ with image inside one of the standard Levi subgroups is said to satisfy strict parity if it can be conjugated inside the Levi subgroup in such a way that the image of complex conjugation has alternating signs on the diagonal. For $\rho = \tau \oplus \psi$ as above, with τ odd and irreducible, it is easy to see that a conjugate of ρ satisfies strict parity if it has image in either of the first two Levi subgroups. Hence, [4, Conj. 3.1] predicts certain weights for ρ, and [4] gives numerous computational examples of ρ which appear to correspond to eigenclasses in $H^3(\Gamma, M_\epsilon)$, where Γ, M, and the nebentype ϵ are determined by ρ. The conjecture for these cases is proven in [3] under the condition that the Serre conductor of ρ is squarefree. However, for ρ of the form that we consider it is easy to see that strict parity can not be satisfied by any conjugate of ρ with image in the medial Levi subgroup, so [4, Conj. 3.1] makes no prediction of a weight arising from a medial embedding. In this paper, we will show that if the Serre conductor of ρ is squarefree, $M = F(a, b, c)$ is a weight with $c \not\equiv b + 1 \pmod{p - 1}$ that would be predicted from a medial embedding except for strict parity, and M is in the lowest alcove (see Definition 2.5), then the conjecture is true, in that ρ corresponds to an element of $H^2(\Gamma, M_\epsilon)$, where Γ, M, and ϵ are as predicted by the conjecture. We expect similar results to hold for p-restricted M outside the lowest alcove, but have as yet been unable to prove a suitable form of Kostant's theorem for this case.

The strict parity condition was postulated in order to explain the relationship between reducible Galois representations and cohomology based on computations done exclusively for $H^3(\Gamma, M)$, where Γ is a congruence subgroup of $\mathrm{GL}(3, \mathbb{Z})$. Also, in [2] strict parity was found to be necessary to verify the conjecture that a Galois representation ρ into $\mathrm{GL}(n, \mathbb{Z})$ that was a sum of n characters, for any n, should be attached to a homology class in the "top" dimension $n(n - 1)/2$. We believe that we were led to invoke the strict parity condition because up until now we have only computed the homology in the top dimension. The results of this paper indicate that, had computations originally been done in H^2 as well as H^3 for $\mathrm{GL}(3, \mathbb{Z})$, the strict parity condition would never have arisen in our minds. It is noteworthy that

the strict parity condition was occasionally violated for H_3 in the computations of [4]; in that a weight that would otherwise be predicted for ρ except for the strict parity condition yielded a system of eigenvalues with ρ apparently attached. We do not have enough computational basis to guess a rule for when this happens. The point of the present paper is that we now tend to think we should scrap the strict parity condition entirely if we look at the direct sum of the cohomology in all degrees. On the other hand, there may well be some kind of strict parity condition for $\mathrm{GL}(n,\mathbb{Z})$ if we fix our attention on a single degree of cohomology.

We use the method of our previous paper [3] suitably adapted. We use the same resolution of $\mathbb{Z}$ by $\mathrm{GL}(3,\mathbb{Q})$-modules created by splicing together the sharbly resolution with a resolution involving modules induced from parabolic subgroups. Because we are now looking at H_2 rather than H_3, we need to look at stabilizers of planes, as well as lines, in $\mathrm{GL}(3)$. These preliminaries are in Section 3. For the same reason, we need to look much more carefully at the Hochschild-Serre spectral sequence for these stabilizers, taking the Hecke action into account. This is done in section 4. The main new ingredient in this paper is a generalization of a theorem of Kostant due to Polo and Tilouine [16, Corollary 3.8, pg. 128] (see also [11, Section 7] and [19]). This is found in section 5. Sections 6 and 7 contain the body of the proof of our main theorem, which is Theorem 2.6, whose background is given in section 2. In the last section of the current paper, we include several corrections for [3].

We thank Elmar Große-Klönne for answering our questions about his paper [11], and Brian Boe for answering our questions about [19].

2. Attached Galois representations and arithmetic cohomology

We use the following groups and semigroups in $\mathrm{GL}_n(\mathbb{Q})$.

Definition 2.1. *Let N be a positive integer.*

(1) $S_0(n,N)^{\pm}$ *is the semigroup of matrices $s \in M_n(\mathbb{Z})$ such that $\det(s)$ is relatively prime to pN and the first row of s is congruent to $(*, 0, \ldots, 0)$ modulo N.*

(2) $S_0(n,N)$ *is the subsemigroup of $s \in S_0(n,N)^{\pm}$ such that $\det(s) > 0$.*

(3) $\Gamma_0(n,N)^{\pm} = S_0(n,N)^{\pm} \cap \mathrm{GL}(n,\mathbb{Z})$.

(4) $\Gamma_0(n,N) = S_0(n,N) \cap \mathrm{GL}(n,\mathbb{Z})$.

Let $\mathcal{H}_{n,N}$ (a Hecke algebra) be the (commutative) $\mathbb{Z}$-algebra under convolution generated by all the double cosets $T(\ell,k) = \Gamma_0(n,N)D(\ell,k)\Gamma_0(n,N)$ with

$$D_{\ell,k} = \mathrm{diag}(\underbrace{1,\cdots,1}_{n-k},\underbrace{\ell,\cdots,\ell}_{k}).$$

such that $\ell \nmid pN$.

An algebra homomorphism $\phi : \mathcal{H}_{n,N} \to \mathbb{F}$ will be called a Hecke packet. For example, if W is an $\mathcal{H}_{n,N} \otimes \mathbb{F}$-module, and $w \in W$ is a simultaneous eigenvector for all $T \in \mathcal{H}_{n,N}$, then the associated eigenvalues give a Hecke packet, called a Hecke eigenpacket that "occurs" or "appears" in W.

Definition 2.2. *Let ϕ be a Hecke packet, with $\phi(T(\ell,k)) = a(\ell,k)$. We say that the Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n,\mathbb{F})$ is attached to ϕ if ρ is unramified*

outside pN and

$$\det(I - \rho(\mathrm{Frob}_\ell)X) = \sum_{k=0}^{n} (-1)^k \ell^{k(k-1)/2} a(\ell, k) X^k$$

for all prime $\ell \nmid pN$.

If the Hecke packet comes from a Hecke eigenvector $w \in W$, where W is an $\mathbb{F}$-vector space on which $\mathcal{H}_{n,N}$ acts, we will say ρ is attached to w and fits W.

(We use the arithmetic Frobenius, so that if ω is the cyclotomic character, $\omega(\mathrm{Frob}_\ell) = \ell$.)

If ρ is attached to ϕ, the characteristic polynomials of $\rho(\mathrm{Frob}_\ell)$ for almost all prime ℓ are determined by ϕ and hence ρ is determined up to isomorphism, since we are assuming ρ is semisimple (see Section 1).

Definition 2.3. *Let S be a subsemigroup of the matrices in $\mathrm{GL}(n, \mathbb{Q})$ with integer entries whose determinants are prime to pN. A (p, N)-admissible S-module M is an $\mathbb{F}S$-module of the form $M' \otimes_{\mathbb{F}} \mathbb{F}_\chi$, where M' is an $\mathbb{F}S$-module on which $S \cap \mathrm{GL}(n, \mathbb{Q})^+$ acts via its reduction mod p and χ is a character $\chi : S \to \mathbb{F}^\times$ which factors through the reduction of S modulo N. Here $\mathbb{F}_\chi = \mathbb{F} \otimes \chi$.*

We remark that the class of (p, N)-admissible S-modules is a subclass of the class of admissible modules as defined in [1].

For $p \nmid N$, given a character $\epsilon : G_\mathbb{Q} \to \mathbb{F}$ of conductor dividing N, we may consider ϵ as a Dirichlet character modulo N. We then consider ϵ as a character of $S_0(n, N)$ by defining, for $s \in S_0(n, N)$, $\epsilon(s) = \epsilon(s_{11})$ to be the image under ϵ of the $(1, 1)$ entry of s (which must be a unit modulo N). Hence, we may define M_ϵ as $M \otimes_{\mathbb{F}} \mathbb{F}_\epsilon$ for any $\mathbb{F}[S_0(n, N)]$-module M.

If M is a (p, N)-admissible module, there is a natural action of a double coset $T(\ell, k) \in \mathcal{H}_{n,N}$ on the homology $H_*(\Gamma, M)$ and the cohomology $H^*(\Gamma, M)$, and we then refer to $T(\ell, k)$ as a Hecke operator. This action makes $H_*(\Gamma, M)$ and $H^*(\Gamma, M)$ into $\mathcal{H}_{n,N}$-modules.

We now present a lemma that relates the fitting of a Galois representation to one space to its fitting another space.

Lemma 2.4. *Let $\Gamma = \Gamma_0(n, N)$ and M an irreducible (p, N)-admissible $S_0(n, N)$-module. Then ρ fits $H_i(\Gamma, M)$ if and only if ρ fits $H^i(\Gamma, M)$.*

Proof. If (r, V) is a representation of a group G that possesses an involution ι, let $(r^\vee, V^\vee)$ denote its contragredient. That is, $V^\vee = V$ as vector spaces and $r^\vee(g) = r(\iota(g))$. If (r, k^n) is a matrix representation with k a ring, let $(r^\vee, k^n)$ denote its contragredient. That is, $r^\vee(g) = {}^t r(g)^{-1}$. If E is an $\mathcal{H}_{n,N}$-module on which the scalar matrices act via a central character f, let $E^\vee$ denote its contragredient, that is $E = E^\vee$ as vector spaces and $\Gamma D_{\ell,k} \Gamma$ acts on $E^\vee$ as $f(\ell^{-1}) \Gamma \ell D_{\ell^{-1},k} \Gamma$ on E. It will be clear from the context which contragredient we are using.

We take the contragredient of M with respect to the involution of Γ that sends γ to $h\,{}^t\gamma^{-1}h^{-1}$ where $h = \mathrm{diag}(N, 1, \ldots, 1)$. Now [6, Prop. 2.8] says that ρ fits $H^i(\Gamma, M)$ if and only if $\rho^\vee \otimes \omega^{n-1}$ fits $H^i(\Gamma, M^\vee)$. But using [8, Theorem 3.1] and Kronecker duality, we see that $H^i(\Gamma, M^\vee) = H_i(\Gamma, M)^\vee$. As in [6, Prop. 2.8], one checks that $\rho^\vee \otimes \omega^{n-1}$ fits $H_i(\Gamma, M)^\vee$ if and only if ρ fits $H_i(\Gamma, M)$. $\square$

Recall [10] that the irreducible $\mathbb{F}[\mathrm{GL}(n, \mathbb{F}_p)]$-modules are parametrized by p-restricted n-tuples; i.e. by n-tuples $(a_1, \ldots, a_n)$ such that $0 \le a_i - a_{i+1} \le p - 1$ for

$i < n$ and $0 \leq a_n \leq p - 2$. We will denote the module corresponding to the n-tuple $(a_1, \ldots, a_n)$ by $F(a_1, \ldots, a_n)$. By allowing for twisting by the determinant character (which has order $p - 1$), we can (and will) allow the value of a_n to be an arbitrary integer, thus allowing a single module to be described by an infinite number of different n-tuples. For a Levi subgroup $L \cong \mathrm{GL}(1, \mathbb{F}_p) \times \mathrm{GL}(2, \mathbb{F}_p) \subset \mathrm{GL}(3, \mathbb{F}_p)$, we will use the notation $F(a; b, c)$ to denote the L-module $F(a) \otimes F(b, c)$.

Several of our theorems (in particular Theorems 2.6, 5.1 and 7.1) will rely on the following definition.

Definition 2.5. *An irreducible* $\mathrm{GL}(3, \mathbb{F}_p)$*-module* $F(a, b, c)$ *is in the lowest alcove if* $0 \leq a - c \leq p - 2$.

In representation theoretic terms, what we have defined as the lowest alcove corresponds to the "closure of the lowest alcove" or the closed lower alcove.

We now state our main theorem. The proof of Theorem 2.6 will be given in section 7.

Theorem 2.6. *Let* $\tau : G_{\mathbb{Q}} \to \mathrm{GL}(2, \mathbb{F})$ *and* $\psi : G_{\mathbb{Q}} \to \mathrm{GL}(1, \mathbb{F})$ *be Galois representations, with* τ *odd and irreducible, and let* $\rho = \tau \oplus \psi$. *Let* $F(a, b, c)$ *be a weight predicted by* [4, *Theorem 3.1*] *for* ρ *from a medial embedding (disregarding the strict parity condition). Let* N *be the Serre conductor of* ρ *and let* ϵ *be the nebentype of* ρ. *Assume that* N *is squarefree. If* $F(a, b, c)$ *is in the lowest alcove, and* $c \not\equiv b + 1$ (mod $p - 1$)*, then* ρ *fits* $H^2(\Gamma_0(3, N), F(a, b, c)_{\epsilon})$.

Remark 2.7. Since ρ is odd, the weights predicted for ρ from a medial embedding will necessarily fail to satisfy strict parity, and hence would not be predicted by [4, Conj 3.1]. Conjugates of the image of ρ in the other standard Levi subgroups would have different predicted weights, and if we call one of these weights M, [3] proves that ρ will fit $H^3(\Gamma_0(3, N), M_{\epsilon})$. Numerous computational examples appear in [4], for weights that satisfy strict parity, but because that paper did not compute any cohomology groups in degree 2 no computational examples of Theorem 2.6 appear. It would be interesting to perform the computations of cohomology in degree 2, to obtain examples of Theorem 2.6 to test cases when $F(a, b, c)$ is not in the lowest alcove.

3. Preliminary definitions

Let P be a $\mathbb{Q}$-parabolic subgroup of $\mathrm{GL}(3)$ with unipotent radical U and set $L = P/U$ with the natural projection $\psi : P \to L$. If A is a subset of $\mathrm{GL}(3, \mathbb{Q})$ write $A_P = A \cap P, A_U = A \cap U$, and $A_L = \psi(A_P)$. In case we use an explicit splitting of L back to P, by abuse of notation we will also use L for the image of that splitting.

Let P_0 be the stabilizer of the line spanned by $(1, 0, 0)$ in affine 3-space, on which $\mathrm{GL}(3)$ acts on the right. Then $P_0 = L_0^1 L_0^2 U_0$ where

$$U_0 = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix}, L_0^1 = \begin{pmatrix} * & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, L_0^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

For $g \in P_0$, define $\psi_0^1(g) \in \mathrm{GL}(1)$ and $\psi_0^2(g) \in \mathrm{GL}(2)$ by

$$g = \begin{pmatrix} \psi_0^1(g) & 0 \\ * & \psi_0^2(g) \end{pmatrix}.$$

For any integer x let

$$g_x = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define

$$P_x = g_x^{-1} P_0 g_x, \quad U_x = g_x^{-1} U_0 g_x, \quad L_x^1 = g_x^{-1} L_0^1 g_x, \quad L_x^2 = g_x^{-1} L_0^2 g_x.$$

For $s \in P_x$, we set $\psi_x^i(s) = \psi_0^i(g_x s g_x^{-1})$.

Theorem 3.1. *Let $\mathbb{P}^2(\mathbb{Q})$ be the projective space of lines in the vector space $\mathbb{Q}^3$ of row vectors. Assume that $N > 0$ is squarefree. Then the $\Gamma_0(3, N)$-orbits in $\mathbb{P}^2(\mathbb{Q})$ are in 1-1 correspondence with the set of positive divisors d of N where the orbit corresponding to d contains $z_d = (1 : d : 0)$. These are also the $S_0(3, N)$-orbits. Moreover,*

(1) *The stabilizer of z_d in $\mathrm{GL}(3, \mathbb{Q})$ equals $P_d(\mathbb{Q})$.*
(2) $U_d L_d^1 \cap \Gamma_0(3, N) = U_d \cap \Gamma_0(3, N).$
(3) *If $s \in P_d \cap S_0(3, N)^{\pm}$, then $\psi_d^1(s) \equiv s_{11} \pmod{d}$ and $\psi_d^2(s)_{11} \equiv s_{11} \pmod{N/d}$.*
(4) $\psi_d^2(P_d \cap S_0(3, N)^{\pm}) \subset S_0(2, N/d)^{\pm}.$
(5) ψ_d^2 *induces an exact sequence*

$$1 \to U_d \cap \Gamma_0(3, N) \to P_d \cap \Gamma_0(3, N) \xrightarrow{\psi_d^2} \Gamma_0(2, N/d)^{\pm} \to 1.$$

Proof. This follows from [3, Thm 4.1] and the discussion in the paragraph before [3, Thm. 4.3]. $\qquad\square$

Let P_0' be the stabilizer of the plane spanned by $(1, 0, 0)$ and $(0, 1, 0)$ in affine 3-space on which $\mathrm{GL}(3)$ acts on the right. Then $P_0' = M_0^2 M_0^1 U_0'$ where

$$U_0' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix}, \qquad M_0^2 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_0^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & * \end{pmatrix}.$$

For $s \in P_0'$, define $\theta_0^1(s) \in \mathrm{GL}(1)$ and $\theta_0^2(s) \in \mathrm{GL}(2)$ by

$$s = \begin{pmatrix} \theta_0^2(s) & 0 \\ * & \theta_0^1(s) \end{pmatrix}.$$

For any integer x, set

$$h_x = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define

$$P_x' = h_x^{-1} P_0' h_x, \quad U_x' = h_x^{-1} U_0' h_x, \quad M_x^1 = h_x^{-1} M_0^1 h_x, \quad M_x^2 = h_x^{-1} M_0^2 h_x.$$

For $s \in P_x'$ we set $\theta_x^i(s) = \theta_0^i(h_x s h_x^{-1}) \in \mathrm{GL}(i)$.

Theorem 3.2. *Let $N > 0$ be a squarefree integer.*

(1) *The $\Gamma_0(3, N)$-orbits of planes in $\mathbb{Q}^3$ are in 1-1 correspondence with the set of positive divisors of N, where the orbit corresponding to the divisor d contains the plane spanned by $(1, 0, d)$ and $(0, 1, 0)$. The $\Gamma_0(3, N)$-orbits are stable under the action of $S_0(3, N)$.*
(2) *The stabilizer of the plane corresponding to d is P_d'.*

(3) $U'_d M^1_d \cap \Gamma_0(3, N) = U'_d \cap \Gamma_0(3, N)$.

(4) If $s \in P'_d \cap S_0(3, N)^\pm$, then $\theta^1_d \equiv s_{11} \pmod{N/d}$ and $\theta^2_d(s)_{11} \equiv s_{11} \pmod{d}$.

(5) $\theta^2_d(P'_d \cap S_0(3, N)^\pm) \subset S_0(2, d)^\pm$.

(6) θ^2_d induces an exact sequence

$$1 \to U'_d \cap \Gamma_0(3, N) \to P'_d \cap \Gamma_0(3, N) \xrightarrow{\theta^2_d} \Gamma_0(2, d)^\pm \to 1.$$

Proof. The proof is similar to that of Theorem 3.1, and will be omitted. $\square$

4. Hecke Actions on Hochschild-Serre spectral sequences

Let $P = LU$ be a maximal $\mathbb{Q}$-parabolic subgroup of $G = GL(n)$, and let Γ be a congruence subgroup of $G(\mathbb{Z})$. Set $\Gamma_P = \Gamma \cap P(\mathbb{Q})$, $\Gamma_U = \Gamma \cap U(\mathbb{Q})$, $\Gamma_L = \psi(\Gamma) \subset L(\mathbb{Q})$. Let (Γ, S) be a Hecke pair [7, Section 1.1], and set $S_P = S \cap P(\mathbb{Q})$. Let $\mathcal{H}_P$ denote the Hecke algebra of double cosets of the Hecke pair (Γ_P, S_P).

We have a Hochschild-Serre spectral sequence:

$$E^2_{ij} = H_i(\Gamma_L, H_j(\Gamma_U, M)) \Rightarrow H_{i+j}(\Gamma_P, M).$$

We now explain why (at least under a certain hypothesis) this is $\mathcal{H}_P$-equivariant, and how to compute the action of the Hecke operators on E^2_{ij}.

Definition 4.1. *Let T be the set of matrices $t \in \mathrm{GL}(n, \mathbb{Q})$ such that all denominators of both t and t^{-1} are prime to pN and let M be a (p, N)-admissible T-module. Let $C_{\cdot}$ be the standard resolution of $\mathbb{Z}$ over $\mathrm{GL}(n, \mathbb{Q})$ [9, I.5]. Set r equal to the $\mathbb{Q}$-dimension of $U(\mathbb{Q})$. Assume that $t \in T$ normalizes $U(\mathbb{Q})$. Then we define an action of t on $C_{\cdot} \otimes_{\Gamma_U} M$ by setting*

$$(c \otimes m) \cdot t = \frac{1}{d^r} \sum_b c u_b t \otimes m u_b t \qquad (*)$$

where $d \in \mathbb{Z}$ is any integer such that the right conjugation action of t on Γ^d_U is contained in Γ_U, and $\{u_1, \ldots, u_{d^r}\}$ is a set of coset representatives for Γ^d_U inside Γ_U. As shown in Lemma 4.3 this action is well defined, and does not depend on the choice of d or the choice of coset representatives.

Remark 4.2. Note that we are not claiming that this action of the individual elements of T on the complex $C \otimes_{\Gamma_U} M$ extends to a semigroup action of T. We will see that under a certain hypothesis it does yield a semigroup action on the homology of the complex, however.

Lemma 4.3. *The action of $t \in T$ on the complex $C \otimes_{\Gamma_U} M$ satisfies the following properties.*

(1) *For a fixed d and choice of $\{u_b\}$, it is well defined.*

(2) *It does not depend on the choice of coset representatives $\{u_b\}$.*

(3) *It does not depend on the choice of d.*

(4) *It commutes with the boundary maps.*

(5) *If $t \in L(\mathbb{Z})$ then this action is the natural action on homology induced by the conjugation action of $L(\mathbb{Z})$ on $U(\mathbb{Z})$.*

Proof. We note that, since M is a (p, N)-admissible module, T acts on M via reduction mod pN.

(1) For a fixed d and a fixed choice of $\{u_b\}$, we need to show that for any $\gamma \in \Gamma_U$, $c \otimes m$ and $c\gamma \otimes m\gamma$ have the same image under the action. Note that $\gamma u_b = u_{i(b)} \gamma'_b$

where $\gamma'_b \in \Gamma^d_U$ and $i : \{1, \ldots, d^r\} \to \{1, \ldots, d^r\}$ is a permutation. Further, by our choice of d, we have that for $t \in T$, $\gamma'_b t = t\gamma''_b$ with $\gamma''_b \in \Gamma_U$. Hence

$$
\begin{aligned}
(c\gamma \otimes m\gamma) \cdot t &= \frac{1}{d^r} \sum_b c\gamma u_b t \otimes m\gamma u_b t \\
&= \frac{1}{d^r} \sum_b cu_{i(b)}\gamma'_b t \otimes mu_{i(b)}\gamma'_b t \\
&= \frac{1}{d^r} \sum_b cu_{i(b)}t\gamma''_b \otimes mu_{i(b)}t\gamma''_b \\
&= \frac{1}{d^r} \sum_b cu_{i(b)}t \otimes mu_{i(b)}t \\
&= (c \otimes m) \cdot t
\end{aligned}
$$

as desired.

(2) We will now show, that for a given d, the action does not depend on the choice of $\{u_b\}$.

Fix $t \in T$ and choose d so that $t^{-1}\Gamma^d_U t \subset \Gamma_U$. Let $\{u_b\}$ and $\{u_b\gamma_b\}$ be two collections of coset representatives, with $\gamma_b \in \Gamma^d_U$. Then $\gamma_b t = t\gamma'_b$, with $\gamma'_b \in \Gamma_U$. For any element $c \otimes m \in C_j \otimes_{\Gamma_U} M$ we have

$$
\begin{aligned}
\sum_b cu_b\gamma_b t \otimes mu_b\gamma_b t &= \sum_b cu_b t\gamma'_b \otimes mu_b t\gamma'_b \\
&= \sum_b cu_b t \otimes mu_b t.
\end{aligned}
$$

(3) We will now show that for any $d_1 | d_2$, the actions given by d_1 and d_2 match. Since for any d_1 and d_2, there is a common multiple d_3, this will prove the well-definedness of the action. Let $n_1 = d_1^r$ and $n_2 = d_2^r$. Choose coset representatives $u_1, \ldots, u_{n_1}$ of $\Gamma^{d_1}_U$ inside Γ_U. Then choose coset representatives $v_1, \ldots, v_{n_2/n_1}$ of $\Gamma^{d_2}_U$ inside $\Gamma^{d_1}_U$. One easily checks that $\{u_i v_j\}$ is a complete set of coset representatives of $\Gamma^{d_2}_U$ inside Γ_U. Note that for a fixed j, $\{u_i v_j\}$ is a complete set of coset representatives of $\Gamma^{d_1}_U$ inside Γ_U. For the purposes of this proof, we will denote by $|_d$ the action of $t \in T$ computed using the integer d.

For any $c \otimes m \in C_j \otimes_{\Gamma_U} M$, we now compute

$$
\begin{aligned}
(c \otimes m)|_{d_2} t &= \frac{1}{d_2^r} \sum_i \sum_j cu_i v_j t \otimes mu_i v_j t \\
&= \frac{1}{(d_2/d_1)^r} \sum_j \left(\frac{1}{d_1^r} \sum_i cu_i v_j t \otimes mu_i v_j t \right) \\
&= \frac{1}{(d_2/d_1)^r} \sum_j (c \otimes m)|_{d_1} t \\
&= (c \otimes m)|_{d_1} t.
\end{aligned}
$$

Hence, we have proven (3).

(4) Let $\partial : C_j \to C_{j-1}$ be a boundary map for the complex $C_{\cdot}$. We wish to show that for any $c \otimes m \in C_j \otimes_{\Gamma_U} M$ and any $t \in T$, we have

$$\frac{1}{d^r} \sum_b ((\partial c) u_b t \otimes m u_b t) = \frac{1}{d^r} \sum_b \partial (c u_b t) \otimes m u_b t.$$

This follows immediately from the fact that ∂ is a $P(\mathbb{Q})$-homomorphism.

(5) Let $t \in L(\mathbb{Z})$. Taking $d = 1$, with coset representative the identity, yields the desired result.

$\square$

Let M be a (p, N)-admissible module whose underlying $\mathrm{GL}(n, \mathbb{Z}/p\mathbb{Z})$-module is irreducible. We denote by $\mathrm{Lie}(U)$ the $\mathbb{Z}$-Lie algebra associated to the algebraic $\mathbb{Z}$-group U. By [16, Sec. 3.8, (2)], we know that there is a spectral sequence whose E^1 term is $H^*(\mathrm{Lie}(U), M)$ with the abutment $H^*(\Gamma_U, M)$. From the discussion in [16, Section 1.7] we know that $H_*(\mathrm{Lie}(U), M)$ is naturally an $L(\mathbb{Z}_{(p)})$-module on which $L(\mathbb{Z}_{(p)})$ acts via reduction modulo p (where we write $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at the prime p).

This spectral sequence degenerates at E^1, and it is functorial with respect to compatible maps on $\mathrm{Lie}(U)$ and Γ_U. In particular, it is $L(\mathbb{Z})$-equivariant. Therefore, if in some degree k, the $L(\mathbb{Z}_{(p)})$-module $H_*(\mathrm{Lie}(U), M)$ is irreducible, so is the abutment in that degree, and the spectral sequence yields an isomorphism of $L(\mathbb{Z})$-modules

$$H_*(\mathrm{Lie}(U), M) \cong H_*(\Gamma_U, M). \tag{$\dagger$}$$

In the next theorem, we show that under certain hypotheses this isomorphism is actually an isomorphism of S_P-modules, where S_P acts on $H_*(\Gamma_U, M)$ via Definition 4.1.

Theorem 4.4. *Let $T = S_P$. Assume that the reductions of S_U and of Γ_U modulo p are the same, and assume that $H_*(\mathrm{Lie}(U), M)$ is an irreducible $L(\mathbb{Z}_{(p)})$-module. Then elements of S_U act trivially on $H_*(\Gamma_U, M)$ under the action defined in Definition 4.1. The action of elements of S_P on $H_*(\Gamma_U, M)$ is equivariant under the isomorphism ($\dagger$), and thus yields a semigroup action of S_P on $H_*(\Gamma_U, M)$.*

Proof. Elements of S_U act trivially on the resolution $C_{\cdot}$ (since U is commutative), and act on M via reduction mod p. Hence, we see that they act trivially on $H_*(\Gamma_U, M)$. The equivariance under the isomorphism ($\dagger$) with the Lie algebra homology follows because the coset representatives u_b act trivially on the Lie algebra cohomology. This equivariance, combined with the fact that the action on the Lie algebra homology is a semigroup action, yields a semigroup action of S_P on $H_*(\Gamma_U, M)$. $\square$

Remark 4.5. We will see in Section 5 that under the hypotheses of Theorem 4.4, $H_*(\Gamma_U, M)$ is a (p, N)-admissible S_P-module.

Theorem 4.6. *Let M be a (p, N)-admissible S-module on which S acts modulo pN. Assume that S_U and Γ_U have the same image modulo pN. Assume that $H_*(\Gamma_U, M)$ is an irreducible $L(\mathbb{Z}_{(p)})$-module. Then the Hecke algebra $\mathcal{H}_P$ acts equivariantly on the Hochschild-Serre spectral sequence*

$$E^2_{ij} = H_i(\Gamma_L, H_j(\Gamma_U, M)) \implies H_{i+j}(\Gamma_P, M),$$

and a given packet of Hecke eigenvalues occurs in $H_k(\Gamma_P, M)$ if and only if it appears in

$$\bigoplus_{i+j=k} E_{ij}^{\infty}$$

Proof. Let $C_.$ be the standard resolution of $\mathbb{Z}$ over $\mathrm{GL}(3, \mathbb{Q})$, and let $F_.$ be the standard resolution of $\mathbb{Z}$ over $L(\mathbb{Q})$. We view F as a $P(\mathbb{Q})$-resolution of $\mathbb{Z}$ via the map $\psi : P \to L$. Hence, $U(\mathbb{Q})$ acts trivially on F.

Let (Γ, S) be a Hecke pair inside $\mathrm{GL}(3, \mathbb{Q})$, Recall that $\Gamma_P = \Gamma \cap P(\mathbb{Q})$, $\Gamma_U = \Gamma \cap U(\mathbb{Q})$, and $\Gamma_L = \psi(\Gamma_P)$. Similarly, $S_P = S \cap P(\mathbb{Q})$, $S_U = S \cap U(\mathbb{Q})$, and $S_L = \psi(S_P)$.

We now have an exact sequence $0 \to \Gamma_U \to \Gamma_P \to \Gamma_L \to 0$. Following [9, p. 171], we will study the Hochschild-Serre spectral sequence of this exact sequence as the spectral sequence associated with the double complex $F \otimes_{\Gamma_L} (C \otimes_{\Gamma_U} M)$.

There are two spectral sequences associated with this double complex. The first has

$$E_{ij}^2 = H_i(\Gamma_L, H_j(\Gamma_U, M))$$

and is just the Hochschild-Serre spectral sequence associated to the exact sequence above. The second has

$$`E_{ij}^1 = H_i(\Gamma_L, C_j \otimes_{\Gamma_U} M),$$

and the only nonzero column in the $`E^1$ page is the zeroth column. It is easy to see ([9, VII.5.5]) that

$$`E_{0j}^1 = (C_j \otimes_{\Gamma_U} M)_{\Gamma_L} = C_j \otimes_{\Gamma_P} M$$

and $`E_{0,j}^2$ is just the jth homology of $C \otimes_{\Gamma_P} M$, namely $H_j(\Gamma_P, M)$.

We note that the action of the Hecke operator $\Gamma_P s \Gamma_P$ with $s \in S_P$ on a homology class in the abutment represented by the cycle $c \otimes_{\Gamma_P} m$ is computed by writing

$$\Gamma_P s \Gamma_P = \coprod s_\alpha \Gamma_P.$$

Then $\Gamma_P s \Gamma_P$ takes the class of $c \otimes_{\Gamma_P} m$ to the class of

$$\sum_\alpha c s_\alpha \otimes_{\Gamma_P} m s_\alpha.$$

From Definition 4.1 and our hypotheses, we obtain an action by any given element $t \in S_P$ on $C_. \otimes_{\Gamma_U} M$. By Theorem 4.4, these actions compile into a semigroup action of S_P on the homology which is trivial on S_U (because S_U and Γ_U have the same image modulo p) and hence yields an S_L-action on $H_*(\Gamma_U, M)$. Since S_P acts on $H^*(\Gamma_U, M)$ via ψ, we obtain a Hecke action of $\mathcal{H}_P$ on E_{ij}^2. Because this action is derived from an action on the double complex, it commutes with all the differentials of the spectral sequence. One checks easily that this action on $\mathcal{H}_P$ is given by the formula given in the next paragraph.

We now check that this action dovetails with the action described above on the abutment. An element of E_{pq}^0 has the form $\sum_r f_r \otimes_{\Gamma_L} (c_r \otimes_{\Gamma_U} m_r)$ where r runs through a finite index set. Each term of the sum is actually Γ_P-invariant, since Γ_P acts via $\psi : \Gamma_P \to \Gamma_L$. Hence, we may actually write this element as

$$\sum_r f_r \otimes_{\Gamma_P} (c_r \otimes_{\Gamma_U} m_r),$$

Thus the action of $\Gamma_P s \Gamma_P$ takes the element above to

$$\sum_\alpha \sum_r f_r s_\alpha \otimes_{\Gamma_P} (\sum c_r \otimes_{\Gamma_U} m_r) s_\alpha = \sum_\alpha \sum_r f_r s_\alpha \otimes_{\Gamma_P} (\sum c_r \otimes_{\Gamma_U} m_r) \psi(s_\alpha).$$

A cycle in degree k of E^0 is represented by a sum of terms

$$\sum_{i+j=k} \sum_r f_r^{(j)} \otimes_{\Gamma_P} (c_r^{(i)} \otimes_{\Gamma_U} m_r)$$

where $f_r^{(j)} \in F_j$, $c_r^{(i)} \in C_i$, and $m_r \in M$. If this cycle survives to E^∞, then its image in the associated graded of the abutment is given by the image of the class of

$$\sum_r f_r^{(0)} \otimes_{\Gamma_P} (c_r^{(k)} \otimes_{\Gamma_U} m_r),$$

namely

$$\sum_r c_r^{(k)} \otimes_{\Gamma_P} m_r.$$

Therefore we see that the Hecke actions described above on the spectral sequence and its abutment are compatible. The last assertion of the statement of the theorem follows from linear algebra.

$\square$

We will use the methods of [3, Section 9] to compute the Hecke action of $\mathcal{H}_P$ on E_{ij}^2 in terms of the action of S_L on $C \otimes_{\Gamma_U} M$.

5. Kostant's theorem

Now, let M be a $\mathbb{Z}$-form of the irreducible GL(3)-module of highest weight (a, b, c), where (a, b, c) is in the lowest alcove. Let R be the ring $\mathbb{Z}_{(p)}$. Assume that $H_i(\text{Lie}(U), M(\mathbb{Z}_{(p)}))$ is an irreducible $L(\mathbb{Z}_{(p)})$-module. Then [16, Section 3.8] shows that the groups $H_i(\text{Lie}(U), M(R))$ and $H_i(U(\mathbb{Z}), M(R))$ are isomorphic. As we have seen, the natural action of $L(R)$ on the Lie algebra homology and the action of $L(R)$ defined in Definition 4.1 are equivariant with respect to this isomorphism. It follows that under these assumptions, $H_i(U(\mathbb{Z}), M(\mathbb{F}_p))$ is (p, N)-admissible. We describe this module in the cases we need in the following theorem.

Theorem 5.1. *Let $P = P_d$ and $P' = P'_d$ for d a positive divisor of N, and let $U = U_d$ and $U' = U'_d$. Let $\chi_0 : (\mathbb{Z}_d)^\times \to \mathbb{F}$ and $\chi_1 : (\mathbb{Z}/(N/d))^\times \to \mathbb{F}$ be characters and (a, b, c) a p-restricted triple in the lowest alcove. Then*

(1) *the $\psi(S_P)$-module $H_1(\Gamma_U, F(a, b, c)_{\chi_0 \chi_1}^d)$ is isomorphic to*

$$F(b + 1)_{\chi_0} \otimes F(a, c - 1)_{\chi_1}^{N/d}.$$

(2) *the $\psi(S_P)$-module $H_2(\Gamma_U, F(a, b, c)_{\chi_0 \chi_1}^d)$ is isomorphic to*

$$F(a + 2)_{\chi_0} \otimes F(b - 1, c - 1)_{\chi_1}^{N/d}.$$

(3) *the $\psi(S_{P'})$-module $H_2(\Gamma_{U'}, F(a, b, c)_{\chi_0 \chi_1}^d)$ is isomorphic to*

$$F(a + 1, b + 1)_{\chi_0}^d \otimes F(c - 2)_{\chi_1}.$$

Proof. Set $U = U_d = g_d^{-1} U_0 g_d$ and set $V = \mathrm{GL}(3, \mathbb{Z}) \cap U$. Then a short computation shows that

$$\Gamma_U = \Gamma \cap U = \left\{ g_d^{-1} \begin{pmatrix} 1 & 0 & 0 \\ (N/d)x & 1 & 0 \\ y & 0 & 1 \end{pmatrix} g_d \right\},$$

with $x, y \in \mathbb{Z}$, so that V/Γ_U is cyclic of order N/d. Since N/d is prime to p, we have, by [9, Thm III.10.4] (adapted for homology), that

$$H_*(V, M) \cong H_*(\Gamma_U, M)_{U/\Gamma_U}.$$

We see that for each k, $H_k(V, M)$ is a quotient of $H_k(\Gamma_U, M)$.

A similar computation works for $U' = U_d'$ and $V' = \mathrm{GL}(3, \mathbb{Z}) \cap U'$.

Note also that both V and Γ_U are free abelian of rank 2. Identifying V or Γ_U with $\mathbb{Z}^2$, there is a resolution of $\mathbb{Z}$ as $\mathbb{Z}[\mathbb{Z}^2]$-modules of the form

$$0 \to \mathbb{Z}[\mathbb{Z}^2] \to \mathbb{Z}[\mathbb{Z}^2] \oplus \mathbb{Z}[\mathbb{Z}^2] \to \mathbb{Z}[Z^2] \to \mathbb{Z} \to 0$$

(with different maps, depending on whether we are working with Γ_U or V). Hence, the homology of V or of Γ_U with coefficients in M is the homology of a complex

$$0 \to M \to M \oplus M \to M \to 0.$$

Since V and Γ_U act via reduction mod p on M, and have the same reduction mod p, we see that $H_0(V, M)$ and $H_0(\Gamma_U, M)$ have the same dimension, namely the dimension of $M_{\Gamma_U} = M_V$. Similarly, by Poincaré duality we have that (as vector spaces) $H_2(V, M) \cong M^V \cong M^{\Gamma_U} \cong H_2(\Gamma_U, M)$. A simple dimension counting argument then shows that $H_1(\Gamma_U, M)$ has the same dimension as $H_1(V, M)$. Hence, for $k = 0, 1, 2$, we have that $H_k(\Gamma_U, M) \cong H_k(V, M)$ as $\psi(S_P)$-modules.

We now compute $H_*(V, F(a, b, c))$ for each of the three parts of the theorem, we use the fact that the Lie algebra homology of [16] is isomorphic to the group homology as S_P-modules (Theorem 4.4).

(1) Using [16, p. 128], we find that

$$\begin{aligned} H_1(V, F(a, b, c)) &\cong H^1(V, F(-c, -b, -a))^\vee \\ &\cong F(-b-1; -c+1, -a)^\vee \\ &\cong F(b+1; a, c-1). \end{aligned}$$

(2) Using [16, p. 128], we find that

$$\begin{aligned} H_2(V, F(a, b, c)) &\cong H^2(V, F(-c, -b, -a))^\vee \\ &\cong F(-a-2; -c+1, -b+1)^\vee \\ &\cong F(a+2; b-1, c-1). \end{aligned}$$

(3) Using [16, p. 128], we find that

$$\begin{aligned} H_2(V', F(a, b, c)) &\cong H^2(V', F(-c, -b, -a))^\vee \\ &\cong F(-b-1, -a-1; -c+2)^\vee \\ &\cong F(a+1, b+1; c-2). \end{aligned}$$

Each of these modules is an irreducible (p, N)-admissible module.

All these isomorphisms are $\psi(S)$-equivariant, by Theorem 4.4 above combined with the proof of [6, Prop. 2.8]. The fact that each homology group is a (p, N)-admissible module arises from the fact that each is isomorphic to an irreducible (p, N)-admissible module. (If we were outside the lower alcove, (p, N)-admissibility

of the modules would be a problem, since they would have a filtration whose successive quotients are irreducible (p, N)-admissible modules, but would not themselves be irreducible.) Finally, using the fact that the nebentype character is trivial on $U(\mathbb{Z})$, one sees immediately, using Theorem 3.1(3), that the nebentype character acts as desired on the homology. $\qquad\square$

6. Fitting Galois representations

Now fix N squarefree, set $\Gamma = \Gamma_0(3, N)$ and $S = S_0(3, N)$.

Let $P = P_d$ for d a positive divisor of N. Let $\chi_0 : (\mathbb{Z}/d)^\times \to \mathbb{F}^\times$ and $\chi_1 : (\mathbb{Z}/(N/d))^\times \to \mathbb{F}^\times$ be characters and $\chi = \chi_0\chi_1$. We use the notation of [3] for the irreducible modules $F(a, b, c)^d_\chi$ of $G(\mathbb{F})$ and $F(c; a, b)^d_{\chi_0\chi_1}$, determined by a dominant weight (a, b, c) (the superscript d denotes twisting by g_d and the subscript denotes tensoring with the indicated nebentype character). We also write $F(a, b)_\chi$ for the (p, N)-admissible irreducible $S_0(2, N)^\pm$-module of highest weight (a, b) and nebentype χ.

Let ω denote the mod p cyclotomic character, $\omega : G_\mathbb{Q} \to \mathbb{F}^\times$.

Recall from Sections 3 and 9 of [3] that under the hypothesis of the following theorem, there is a natural map $\mathcal{H}_{3,N} \to \mathcal{H}_{P_d}$, so that it makes sense to speak of Hecke eigenpackets in the homology of $\Gamma_0(3, N) \cap P_d$ and Galois representations fitting that homology.

Theorem 6.1. *Let $P = P_d$ for d a positive divisor of the squarefree positive integer N, let $U = U_d$ and let $L = P/U$. Let $\chi_0 : (\mathbb{Z}/d)^\times \to \mathbb{F}^\times$ and $\chi_1 : (\mathbb{Z}/(N/d))^\times \to \mathbb{F}^\times$ be characters.*

Suppose the action of elements of S_L on $H_j(\Gamma_U, M)$ compile to make it an S_L-module which is isomorphic to $F(\gamma)_{\chi_0} \otimes F(\alpha, \beta)^{N/d}_{\chi_1}$ and that the two-dimensional irreducible Galois representation τ fits $H_1(\Gamma_0(2, N/d)^\pm, F(\alpha, \beta)_{\chi_1})$. Then $\omega\tau \oplus \omega^\gamma \chi_0$ fits $H_{1+j}(\Gamma_0(3, N) \cap P_d, M)$.

Proof. The homology of Γ_L vanishes in degrees > 1 because we are assuming $p > 3$ and Γ_L is isomorphic to the subgroup $\Gamma_0(2, N/d)^\pm$ of $\mathrm{GL}(2, \mathbb{Z})$ by Theorem 3.1(5). Thus the Hochschild-Serre spectral sequence for the exact sequence

$$0 \to \Gamma_U \to \Gamma_P \to \Gamma_L \to 0$$

is two columns thick and degenerates at E^2. So any packet of Hecke eigenvalues occurring in $E^2_{1j} = H_1(\Gamma_L, H_j(\Gamma_U, M))$ will also occur in $H_{1+j}(\Gamma_P, M)$.

The hypotheses of the theorem imply that τ fits $H_1(\Gamma_L, H_j(\Gamma_U, M))$ (considered as an $\mathcal{H}_L$-module). Hence, the calculation in [3, Section 9] implies that $\tau\omega \oplus \omega^\gamma \chi_0$ fits $H_1(\Gamma_L, H_j(\Gamma_U, M))$, considered as an $\mathcal{H}_P$ (or an $\mathcal{H}_{3,N}$) module. $\qquad\square$

Next, we state a theorem analogous to Theorem 6.1, for the groups P'_d.

Theorem 6.2. *Let $P' = P'_d$ for d a positive divisor of the squarefree positive integer N, let $U' = U'_d$ and let $L' = P'/U'$. Let $\chi_0 : (\mathbb{Z}/d)^\times \to \mathbb{F}^\times$ and $\chi_1 : (\mathbb{Z}/(N/d))^\times \to \mathbb{F}^\times$ be characters.*

Assume that the actions of elements of $S_{L'}$ on $H_j(\Gamma_{U'}, M)$ compile to make it an $S_{L'}$-module which is isomorphic to $F(\alpha, \beta)^{N/d}_{\chi_0} \otimes F(\gamma)_{\chi_1}$. Suppose that the two-dimensional Galois representation τ is attached to a class in

$$H^i(\Gamma_0(2, d)^\pm, F(\alpha, \beta)_{\chi_0}).$$

Then this class, interpreted as an element of $H^i(\Gamma_{L'}, H_j(\Gamma_{U'}, M))$, has $\tau \oplus \omega^{\gamma+2}\chi_1$ attached.

Proof. The proof of this theorem is a result of calculations similar to those in [3, Section 9].   □

The next item is not directly a review from [3] but it is very close to Theorem 8.1 of that paper and its proof.

Let H be a plane in $\mathbb{Q}^3$ and now let P_H be the stabilizer of H in GL(3). Set $St(H)$ to be the Steinberg module of H tensored with $\mathbb{F}$. It is a P_H-module, where P_H acts through $\psi : P_H \to \mathrm{GL}(H)$.

Definition 6.3. *Let S denote a subsemigroup of $\mathrm{GL}(n, \mathbb{Q})$ and X, Y two $\mathbb{F}S$-modules. Write $X \asymp Y$ if X and Y have finite length filtrations whose associated graded modules are isomorphic and on each of which S acts through its reduction modulo some positive integer.*

In the situation of this definition, putting the commuting Hecke operators into Jordan Canonical Form shows that a given Galois representation fits X if and only if it fits Y.

Following the notation of [3, 9], we write $\mathrm{Ind}_H^G M = M \otimes_{\mathbb{Z}H} \mathbb{Z}G$, for H a subgroup of G and M a right H-module.

Theorem 6.4. *Let $M = F(a, b, c)_\chi$ where (a, b, c) is in the lowest alcove and χ is a character of squarefree conductor N, and let $X(H, M) = \mathrm{Ind}_{\Gamma_{P_H}}^{\Gamma} St(H) \otimes_{\mathbb{F}} M$. Suppose that the three-dimensional Galois representation ρ fits $H_2(\Gamma, X(H, M))$. Then ρ is isomorphic either to a sum of three characters or to a sum $\omega^c \xi \oplus \sigma$ with σ irreducible and two-dimensional and ξ a character unramified at p.*

Proof. P_H is conjugate by an element g of Γ to P_d' for some positive divisor d of N. Hence, without loss of generality, we may assume that $P_H = P_d'$. Let H_d be the plane stabilized by P_d'. Then the conjugation $Ad(g)$ takes the S-module $X(H, M)$ isomorphically to the S-module $X(H_d, M \circ Ad(g))$. We see that $M \circ Ad(g)$ is isomorphic to M as $\mathrm{GL}(3, \mathbb{F}_p)$-module, and since $g \in \Gamma$, the nebentype character is still χ. We may then factor χ into $\chi_0\chi_1$, where χ_0 has conductor d and χ_1 has conductor N/d.

By Shapiro's Lemma, $H_2(\Gamma, X(H, M))$ is isomorphic to $H_2(\Gamma_{P_H}, St(H) \otimes M)$. The homology of $\Gamma_{L'}$ vanishes in degrees larger than 1. Therefore the Hochschild-Serre Spectral sequence for the short exact sequence

$$0 \to \Gamma_{U_H} \to \Gamma_{P_H} \to \Gamma_{L'} \to 0$$

degenerates at E_2, and we have

$$H_2(\Gamma_{P_H}, St(H) \otimes M) \asymp H_0(\Gamma_{L'}, H_2(\Gamma_{U_H}, St(H) \otimes M)) \oplus H_1(\Gamma_{L'}, H_1(\Gamma_{U_H}, St(H) \otimes M)).$$

Since Γ_{U_H} acts trivially on $St(H)$, we have,

$$H_0(\Gamma_{L'}, H_2(\Gamma_{U_H}, St(H) \otimes M)) \cong H_0(\Gamma_{L'}, St(H) \otimes H_2(\Gamma_{U_H}, M))$$

and then by Borel-Serre duality,

$$H_0(\Gamma_{L'}, H_2(\Gamma_{U_H}, St(H) \otimes M)) \cong H^1(\Gamma_{L'}, H_2(\Gamma_{U_H}, M)).$$

Now, by Theorem 5.1, $H_2(\Gamma_{U_H}, F(a, b, c)_\chi) \cong F(a + 1, b + 1; c - 2)_\chi$, so by Theorem 6.2, any eigenpacket in the first summand must have attached a Galois representation of the form $\tau \oplus \omega^c \chi_1$, as desired

We now examine the second summand.

$$H_1(\Gamma_{L'}, H_1(\Gamma_{U_H}, X(H, M))) \approx H_1(\Gamma_{L'}, St(H) \otimes H_1(\Gamma_{U_H}, M))$$
$$\approx H^0(\Gamma_{L'}, H_1(\Gamma_{U_H}, M)).$$

For any (p, N)-admissible module W, any Hecke eigenpacket in $H^0(\Gamma_0(2, d)^{\pm}, W)$ has an attached two-dimensional Galois representation which is a sum of two characters, by [1, Thm. 4.1.4]. Therefore, if ρ fits the part of the homology arising by Shapiro's lemma from a Hecke eigenpacket occurring in $H^0(\Gamma_{L'}, H_1(\Gamma_{U_H}, M))$, it follows from Theorem 6.2 that ρ is the sum of three characters. $\square$

7. THE SPLICED COMPLEX

We use exactly the same spliced complex X as in [3] to serve as a resolution of $\mathbb{Z}$ by $GL(3, \mathbb{Q})$-modules. We describe the complex and the associated spectral sequence briefly here, but refer the reader to [3] for full details.

For $i > 1$, we set $X_i = \mathrm{Sh}_{i-2}$ to be the $(i-2)$-sharblies for $GL(3, \mathbb{Q})$, and the boundary $X_{i+1} \to X_i$ to be the standard boundary map in the sharbly complex [5].

$X_1 = \oplus_H St_2(H)$ is the sum over all planes in $\mathbb{Q}^3$ of the respective Steinberg module. X_0 is generated freely by the points of the projective plane over $\mathbb{Q}$. The boundary maps $X_2 \to X_1$ and $X_1 \to X_0$, as well as the augmentation map $X_0 \to \mathbb{Z}$ are fully described in [3, Section 7], and other than their existence, their definitions will not play a role in this paper.

We now construct a spectral sequence from the complex X, by choosing a resolution F of $\mathbb{Z}$ by $\mathbb{Z}[S]$-modules that are free as $\mathbb{Z}[\Gamma]$-modules. We fix a (p, N)-admissible right S-module M, and let $\Lambda = X \otimes M$ with the diagonal S-action. We then construct the double complex $\Lambda \otimes_{\Gamma} F$. To compute the homology $H_*(\Gamma, \Lambda)$, we use the spectral sequence described in [9, VII.5.3]:

$$E_{ij}^1 = H_j(\Gamma, \Lambda_i) \implies H_{i+j}(\Gamma, \Lambda) \cong H_{i+j}(\Gamma, M),$$

where the last isomorphism comes from the fact that Λ is a resolution of M. This spectral sequence with its differentials is equivariant for the Hecke action.

We then have the E^1 page of our spectral sequence as follows, where $H_i(\sigma) := H_i(\Gamma_\sigma, M_\sigma)$. Here Γ_σ denotes the stabilizer of the basis element σ and M_σ denotes M twisted with the orientation character of Γ_σ on σ. See [3] for more details and explanation of the 0's in the diagram below.

$\bigoplus_{\sigma \in X_0/\Gamma} H_3(\sigma)$	$H_3(\Gamma, X_1 \otimes M)$	0	0	0
$\bigoplus_{\sigma \in X_0/\Gamma} H_2(\sigma)$	$H_2(\Gamma, X_1 \otimes M)$	0	0	0
$\bigoplus_{\sigma \in X_0/\Gamma} H_1(\sigma)$	$H_1(\Gamma, X_1 \otimes M)$	0	0	0
$\bigoplus_{\sigma \in X_0/\Gamma} H_0(\sigma)$	$H_0(\Gamma, X_1 \otimes M)$	$H_0(\Gamma, Sh_0 \otimes M)$	$H_0(\Gamma, Sh_1 \otimes M)$	$H_0(\Gamma, Sh_2 \otimes M)$

Let $\rho = \tau \oplus \psi$ in medial form. It will follow from the following theorem that E_{02}^1 contains a Hecke eigenclass that has ρ for its attached Galois representation.

Moreover, any Hecke eigenclass in E^1_{02} that has ρ for its attached Galois representation either survives to E^∞ and therefore gives rise to an eigenclass in $H_2(\Gamma, M)$, or is killed by an eigenclass in E^2_{30} that also has ρ attached. In the first case this is immediate, and in the second case it follows from Borel-Serre duality and the fact that the sharbly complex is a resolution of the Steinberg module. In either case, we will conclude that ρ fits $H_2(\Gamma, M_\epsilon)$ and $H^2(\Gamma, M_\epsilon)$, where the level N, nebentype ϵ and weight M are what would be predicted by [4, Conj 3.1], if the strict parity requirement were dropped. Recall that by Lemma 2.4, ρ fits H_2 if and only if it fits H^2.

Theorem 7.1. *Assume $p > 3$. Let $\tau : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F})$ be odd and irreducible and let $\psi : G_\mathbb{Q} \to \mathrm{GL}_1(\mathbb{F})$ be a character. Let $\rho = \tau \oplus \psi$ (medially embedded), N be the Serre conductor of ρ, χ be the factor of $\det(\rho)$ unramified at p, and $M = F(a, b, c)_\chi$, where $F(a, b, c)$ is a weight predicted by [4] for ρ (without taking strict parity into account).*

Assume that (a, b, c) is in the lowest alcove, that $c \not\equiv b + 1$ modulo $p - 1$, and that N is squarefree.

Let $\Gamma = \Gamma_0(3, N)$ and consider the spectral sequence constructed from the complex X.

(i) The E^1_{12} term of the spectral sequence is a finite-dimensional $\mathbb{F}$-vector space not fitted by ρ.

(ii) E^2_{30} is isomorphic to $H^2(\Gamma, M)$.

(iii) ρ fits E^1_{02}.

Proof. Let the Serre conductor of τ be N_1 and the Serre conductor of ψ be d. Then $N = N_1 d$. Since $F(a, b, c)$ is a predicted weight for ρ as formulated in [4, Def. 2.23] (in which strict parity is not mentioned), it follows that $F(a+1, c)$ is a Serre weight for τ and $\psi = \omega^{b+1}\chi_0$ for some character χ_0 of conductor d.

Since the sharbly complex is a resolution of the Steinberg module St for $\mathrm{GL}(3, \mathbb{Q})$, and 6 acts invertibly on M, $E^2_{30} \cong H_1(\Gamma, St \otimes M)$, cf. Corollary 8 in [5]. Borel-Serre duality then gives an isomorphism of Hecke-modules $E^2_{30} \cong H^2(\Gamma, M)$. This proves (ii).

We now consider E^1_{12}. Recall from the study of X in [3] that there is a finite set of planes $\mathbb{H}(\Gamma)$ in $\mathbb{Q}^3$ such that

$$X_1 \otimes M \cong \bigoplus_{H \in \mathbb{H}(\Gamma)} \mathrm{Ind}^\Gamma_{\Gamma_{P_H}} St_2(H) \otimes M$$

where P_H is the stabilizer of H and acts on $St_2(H)$ via its quotient L^2_H. In fact $\mathbb{H}(\Gamma)$ may be parametrized by the set of positive divisors d of N in such a way that $P_H = P'_d$ for some d. Because $\mathbb{H}(\Gamma)$ is finite, an application of Shapiro's lemma plus the Hochschild-Serre spectral sequence and Borel-Serre duality shows that $E^1_{1,2}$ is finite-dimensional over $\mathbb{F}$. (Compare the proof of Theorem 6.4.)

By Theorem 6.4, any Galois representation π fitting E^1_{12} is either a sum of three characters or a sum of $\omega^c\xi$ and an irreducible two-dimensional representation, where ξ is a character unramified at p. Since $\omega^c \neq \omega^{b+1}$, and ρ is not the sum of three characters, π cannot equal ρ. This proves (i).

Now we prove (iii): We may replace M by $M^* = F(a, b, c)^d_{\chi_0\chi_1}$ with which it is isomorphic, since d is prime to p. Then $E^1_{02} = H_2(\Gamma, X_0 \otimes M^*) = \oplus H_2(\sigma)$. From the definition of X in [3], there is a σ such that $H_2(\sigma) = H_2(\Gamma_{P_d}, M^*)$ is a direct

summand of E_{02}^1. We will find the system of eigenvalues in which we are interested in this summand.

Since $F(a+1,c)$ is a Serre weight for τ, $F(a,c-1)$ is a Serre weight for $\omega^{-1}\tau$. By Serre's conjecture (which is now a theorem [13, 14, 15]), $\omega^{-1}\tau$ is attached to a Hecke eigenclass in $H_1(\Gamma_0(2,N_1), F(a,c-1)_{\chi_1})$. Because $\omega^{-1}\tau$ is absolutely irreducible, this class is cuspidal, so the same Hecke eigenpacket (by the Eichler-Shimura theorem) occurs in

$$H_1(\Gamma_0(2,N_1)^\pm, F(a,c-1)_{\chi_1}).$$

By Theorem 5.1, the Γ_L-module $H_1(\Gamma_U, M)$ is isomorphic to

$$F(b+1)_{\chi_0} \otimes F(a,c-1)_{\chi_1}^{N/d}.$$

We finish by invoking Theorem 6.1.

$\square$

Proof of Theorem 2.6: First suppose ρ fits E_{30}^2. By Theorem 7.1(ii) and Lemma 2.4, ρ fits $H_2(\Gamma, M)$ and we are finished.

So assume ρ does not fit E_{30}^2. Since $E_{12}^1 \oplus E_{30}^2$ is finite dimensional over $\mathbb{F}$, it is a sum of generalized Hecke eigenspaces. By Theorem 7.1(i), ρ fits none of these eigenspaces. Then Theorem 7.1(iii) implies that some Hecke eigenclass in E_{02}^1 has ρ attached, and will survive to E^∞. Hence ρ fits the abutment $H^2(\Gamma, M) \cong E_{30}^2$, which is a contradiction. $\square$

Remark 7.2. One can work out in a similar way that not only does ρ fit E_{02}^2 and E_{30}^2 but also E_{11}^1. We do not know anything about the multiplicities of the corresponding Hecke eigenspaces, or generalized eigenspaces. We suspect that the differentials $d : E_{30}^2 \to E_{11}^2$ and/or $d : E_{30}^3 \to E_{02}^3$ will often be nonzero on some of the eigenspaces fitting medially embedded ρ, and that the vanishing of these differentials may account for some of the anomalies reported in the calculations of [4, Remark 3.4]. Explicit computations of $H_2(\Gamma, F(a,b,c))$ would be interesting, and studying the multiplicity of eigenspaces in $H_2(\Gamma, F(a,b,c))$ could provide insight into which terms of the spectral sequence are fitted by ρ, and when the eigenspaces corresponding to ρ survive to E^∞ to show up in the homology.

8. COMMENTS ON [3]

We note the presence of two errors in [3], and give the necessary corrections to the proofs there.

First, in the beginning of [3, Section 5], B_m should be the lower triangular matrices. Then [3, Theorem 5.1], which is correct with weights relative to the upper triangular Borel, becomes incorrect; using the lower triangular Borel, the weight in the theorem should now be $F(a; b, c)$, rather than $F(c; a, b)$.

In section 9, $H_2(U_d \cap \Gamma_0(3,N), M)$ is incorrectly identified with

$$H^0(U_d \cap \Gamma_0(3,N), M).$$

The two spaces are, in fact, isomorphic as vector spaces (by Poincaré duality), but not as Γ_L-modules. Rather,

$$H_2(U_d \cap \Gamma_0(3,N), F(a,b,c)_\chi) \cong F(a+2; b-1, c-1)_\chi^d$$

as $\psi(S_P)$-modules. This can be proved by viewing $H_2(U_d \cap \Gamma_0(3,N), F(a,b,c))$ as the homology of the real torus with fundamental group $U_d \cap \Gamma_0(3,N)$ with local

coefficient system determined by $F(a, b, c)$. If f is the fundamental class of the torus, then elements in $H_2(U_d \cap \Gamma_0(3, N), F(a, b, c))$ can be identified with cycles of the form $f \otimes m$ where $m \in F(a, b, c)^{U_d}$. By the preceding paragraph, this gives us the fact that $H_2(U_d \cap \Gamma_0(3, N), F(a, b, c))$ is isomorphic to $F(a; b, c)$ as an $L(\mathbb{Z})$-module. To determine its further structure as $L(\mathbb{Z}_{(p)})$-module as given in Definition 4.1, we perform the calculation given by the displayed formula in that definition and compute that the highest weight vector in $H_2(U_d \cap \Gamma_0(3, N), F(a, b, c))$ (which has highest weight (a, b, c) with respect to the diagonal matrices in $L(\mathbb{Z})$) is twisted by a certain character on the diagonal matrices in $L(\mathbb{Z}_{(p)})$. This twist is by the character $\mathrm{diag}(t, u, v) \mapsto t^2/(uv)$, and we find that $H_2(U(\mathbb{Z}), F(a, b, c))$ is isomorphic to $F(a + 2; b - 1, c - 1)$.

In order to compensate for this change, we apply the subsequent computations in [3, Section 9] to the twisted contragredient representation ${}^t\rho^{-1} \otimes \omega^2$ to prove that it corresponds to a Hecke eigenclass in the desired cohomology space. The result for ρ then follows by duality.

REFERENCES

[1] A. Ash, Galois representations attached to mod p cohomology of $\mathrm{GL}(n, \mathbb{Z})$, Duke Math. J. **65** (1992), 235-255.

[2] A. Ash, Direct sums of mod p characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the homology of $\mathrm{GL}(n, \mathbb{Z})$, Comm. Algebra **41** (2013), 1751-1775.

[3] A. Ash and D. Doud, Reducible Galois representations and the homology of $\mathrm{GL}(3, \mathbb{Z})$, Int. Math. Res. Not. **2014** (2014), 1379-1409.

[4] A. Ash, D. Doud, and D. Pollack, Galois representations with conjectural connections to arithmetic cohomology, Duke Math. J. **112** (2002), 521-579.

[5] A. Ash, P. Gunnells, M. McConnell, Resolutions of the Steinberg module for $\mathrm{GL}(n)$, J. of Algebra *249* (2012), 380-390.

[6] A. Ash and W. Sinnott, An analogue of Serre's conjecture for Galois representations and Hecke eigenclasses in the mod p cohomology of $\mathrm{GL}(n, \mathbb{Z})$, Duke Math. J. **105** (2000), 1-24.

[7] A. Ash and G. Stevens, Cohomology of arithmetic groups and congruences between systems of Hecke eigenvalues, J. Reine Angew. Math. **365** (1986), 192-220.

[8] A. Ash and P. Tiep, Modular representations of $\mathrm{GL}(3, \mathbf{F}_p)$, symmetric squares, and mod-p cohomology of $\mathrm{GL}(3, \mathbf{Z})$, J. Algebra, **222** (1999), 376–399.

[9] K. Brown, *Cohomology of Groups*, Springer, New York 1982.

[10] S. Doty and G. Walker, The composition factors of $\mathbf{F}_p[x_1, x_2, x_3]$ as a $\mathrm{GL}(3, p)$-module, J. Algebra, **147** (1992), 411-441.

[11] E. Große-Klönne, On the universal module of p-adic spherical Hecke algebras, Amer. J. Math. **136** (2014), 599-652.

[12] F. Herzig, The weight in a Serre-type conjecture for tame n-dimensional Galois representations, Duke Math. J. **149** (2009), 37-116.

[13] C. Khare, Serre's modularity conjecture: The level one case, Duke Math. J. **134** (2006), 557-589.

[14] C. Khare and J.-P. Wintenberger, Serre's modularity conjecture (I), Inv. Math., **178** (2009), 485-504.

[15] C. Khare and J.-P. Wintenberger, Serre's modularity conjecture (II), Inv. Math., **178** (2009), 505-586.

[16] Patrick Polo and Jacques Tilouine, Bernstein-Gelfand-Gelfand complexes and cohomology of nilpotent groups over $\mathbb{Z}_{(p)}$ for representations with p-small weights, Astérisque, **280** (2002), 97–135.

[17] Peter Scholze, On torsion in the cohomology of locally symmetric varieties, preprint, arXiv:1306.2070v1 (2013).

[18] J.-P. Serre, Sur les représentations modulaires de degré 2 de $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, Duke Math. J. **54** (1987), 179-230.

[19] University of Georgia VIGRE Algebra Group, *On Kostant's theorem for Lie algebra cohomology*, Representation Theory, Contemp. Math., vol. 478, Amer. Math. Soc., Providence, RI, 2009, University of Georgia VIGRE Algebra Group: Irfan Bagci, Brian D. Boe, Leonard Chastkofsky, Benjamin Connell, Bobbe J. Cooper, Mee Seong Im, Tyler Kelly, Jonathan R. Kujawa, Wenjing Li, Daniel K. Nakano, Kenyon J. Platt, Emilie Wiesner, Caroline B. Wright and Benjamin Wyser, pp. 39-60.

Boston College, Chestnut Hill, MA 02467
E-mail address: Avner.Ash@bc.edu

Brigham Young University, Provo, UT 84602
E-mail address: doud@math.byu.edu