GALOIS REPRESENTATIONS ATTACHED TO TENSOR PRODUCTS OF ARITHMETIC COHOMOLOGY

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Abstract. We compute the action of Hecke operators on tensor products of cohomology classes of lower congruence subgroups of $\mathrm{SL}(n,\mathbb{Z})$ in trivial weight. We use this computation to prove that if each representation in a collection of Galois representations is attached to a cohomology class of a lower congruence subgroup in trivial weight, then a sum of certain twists of the representations consistent with the main conjectures of [5, 10] is also attached to such a cohomology class.

1. Introduction

Define a Galois representation to be a continuous homomorphism $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n,K)$, where $G_{\mathbb{Q}}$ is the absolute Galois group of $\mathbb{Q}$, and K is a topological field (which may be discrete).

There are conjectures connecting certain Galois representations (with K a finite field) with eigenclasses of Hecke operators acting on arithmetic cohomology [10, 5, 14]. Computational evidence for these conjectures has been given in the case $n = 3$ [5] and $n = 4$ [6, 7, 8], and special cases of these conjectures have been proven for reducible Galois representations [2, 3, 4].

Let Γ be a congruence subgroup of $\mathrm{SL}(n,\mathbb{Z})$ of level N. Define the Hecke algebra $K\mathcal{H}_\Gamma$ to be the commutative K-algebra under convolution generated by all the double cosets $T(\ell,k) = \Gamma D(\ell,k)\Gamma$ with

$$D_{\ell,k} = \mathrm{diag}(\underbrace{1,\cdots,1}_{n-k},\underbrace{\ell,\cdots,\ell}_{k}).$$

such that $\ell \nmid N$.

An algebra homomorphism $\phi : K\mathcal{H}_\Gamma \to K$ will be called a K-Hecke packet. For example, if W is a $K\mathcal{H}_\Gamma$-module, and $w \in W$ is a simultaneous eigenvector for all $T \in K\mathcal{H}_\Gamma$, then the associated eigenvalues give a K-Hecke packet, called a K-Hecke eigenpacket that "occurs" in W.

Definition 1.1. Let ϕ be a K-Hecke packet, with $\phi(T(\ell,k)) = a(\ell,k)$. We define the Hecke polynomial for ϕ at ℓ to be

$$F_{\phi,\ell}(X) = \sum_{k=0}^{n} (-1)^k \ell^{k(k-1)/2} a(\ell,k) X^k$$

for any prime $\ell \nmid N$.

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Definition 1.2. Let ϕ be a K-Hecke packet, with $\phi(T(\ell,k)) = a(\ell,k)$. We say that the Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n,K)$ is attached to ϕ if for some $M \geq 1$, ρ is unramified outside MN and

$$\det(I - \rho(\mathrm{Frob}_\ell)X) = F_{\phi,\ell}(X)$$

for all prime $\ell \nmid MN$.

If the Hecke packet comes from a Hecke eigenvector $w \in W$, where W is a K-vector space on which $K\mathcal{H}_\Gamma$ acts, we will say ρ is attached to w and fits W.

(We use the arithmetic Frobenius, so that if ω is the cyclotomic character, $\omega(\mathrm{Frob}_\ell) = \ell$.)

A theorem of Scholze [15] asserts that for any finite field $\mathbb{F}$, any $\mathbb{F}$-Hecke packet occurring in the homology or cohomology of a congruence subgroup of $\mathrm{SL}(n,\mathbb{Z})$ with $\mathbb{F}$-coefficients has a Galois representation attached. In this paper, we prove a theorem allowing us to combine Galois representations that are attached to eigenclasses of Hecke operators acting on the cohomology of lower congruence subgroups (see Definition 3.1) into a larger Galois representation that is attached to an eigenclass in the cohomology of a related lower congruence subgroup. This give information about the relations between the Galois representations and the eigenclasses provided by Scholze's result, and the details are consistent with the predictions of [5].

Our main theorem (Theorem 7.3) asserts that if we have a collection $\rho_i : G_{\mathbb{Q}} \to \mathrm{GL}(n_i, F)$ of Galois representations ($1 \leq i \leq t$), with each ρ_i attached to a cohomology eigenclass f_i on a lower congruence subgroup $\Delta_i \subset \mathrm{SL}(n_i,\mathbb{Z})$ with trivial coefficients, then we may construct a lower congruence subgroup $\Delta_0 \subset \mathrm{SL}(n,\mathbb{Z})$ with $n = n_1 + \cdots + n_t$, and a Hecke eigenclass h on Δ_0 with trivial coefficients such that

$$\rho_1 \oplus \omega^{n_1}\rho_2 \oplus \cdots \oplus \omega^{n_1+\cdots n_{t-1}}\rho_t$$

is attached to h.

Our proofs will build on the construction of tensor product cohomology classes in [1], and use Borel-Serre duality [11] (as improved by [12, p. 280]), and the sharbly and cosharbly complexes [1, 9].

It would be interesting (but probably quite difficult) to try to generalize the theorem to the case where the f_i lie in cohomology groups with nontrivial coefficients.

We note that the main theorem does not directly prove cases of the main conjecture of [5], since the congruence subgroup used in [5] will typically not be part of a lower compatible system (as defined in Definition 3.3).

2. Sharblies and cosharblies

Let $n \geqslant 1$ and let $\mathbb{Q}^n$ denote the vector space of n-dimensional column vectors with entries in $\mathbb{Q}$.

Definition 2.1. [1, 9] The *Sharbly complex* Sh_* is the complex of $\mathbb{Z}\,\mathrm{GL}(n,\mathbb{Q})$-modules defined as follows. As an abelian group, Sh_k is generated by symbols $[v_1,\ldots,v_{n+k}]$, (which we will call *basic k-sharblies*) where the v_i are nonzero vectors in $\mathbb{Q}^n$, modulo the submodule generated by the following relations:

(i) $[v_{\sigma(1)},\ldots,v_{\sigma(n+k)}] - (-1)^\sigma[v_1,\ldots,v_{n+k}]$ for all permutations σ;
(ii) $[v_1,\ldots,v_{n+k}]$ if $v_1,\ldots,v_{n+k}$ do not span all of $\mathbb{Q}^n$; and
(iii) $[v_1,\ldots,v_{n+k}] - [av_1,v_2,\ldots,v_{n+k}]$ for all $a \in \mathbb{Q}^\times$.

The boundary map $\partial\colon \mathrm{Sh}_k \to \mathrm{Sh}_{k-1}$ is given by

$$\partial([v_1, \dots, v_{n+k}]) = \sum_{i=1}^{n+k} (-1)^i [v_1, \dots, \widehat{v_i}, \dots v_{n+k}],$$

where as usual $\widehat{v_i}$ means to delete v_i.

The action of $g \in \mathrm{GL}(n, \mathbb{Q})$ on a basic sharbly is given by $g[v_1, \dots, v_{n+k}] = [gv_1, \dots, gv_{n+k}]$ and extended to all sharblies by linearity.

Of course, all these objects depend on n, which we suppress from the notation, allowing the context to determine n.

The sharbly complex gives a resolution

$$\cdots \to \mathrm{Sh}_i \to \mathrm{Sh}_{i-1} \to \cdots \to \mathrm{Sh}_1 \to \mathrm{Sh}_0 \to \mathrm{St}(n) \to 0$$

by $\mathrm{GL}(n, \mathbb{Q})$-modules of the Steinberg module St for $\mathrm{GL}(n)/\mathbb{Q}$ [9]. Therefore, if K is a field and if Γ is a congruence subgroup of $\mathrm{GL}(n, \mathbb{Z})$, the ith homology of the complex $\mathrm{Sh}_* \otimes_\Gamma K$ is isomorphic to $H_i(\Gamma, \mathrm{St}(n) \otimes K)$. The duality theorem of Borel-Serre [11], extended in [12], tells us that $H_i(\Gamma, \mathrm{St}(n) \otimes K)$ is isomorphic to $H^{n(n-1)/2-i}(\Gamma, K)$ for any field K of characteristic greater than $n+1$ or of characteristic 0.

We now fix a field K and write St instead of $\mathrm{St}(n) \otimes K$ and Sh_i instead of $\mathrm{Sh}_i \otimes K$. Let a superscript $^\vee$ denote dual of K-vector spaces. Then

$$0 \to \mathrm{St}^\vee \to \mathrm{Sh}_0^\vee \to \mathrm{Sh}_1^\vee \to \cdots$$

is a co-resolution of right modules. The elements of $\mathrm{Sh}_k^\vee$ are called k-cosharblies; they are K-linear K-valued antisymmetric functions on k-sharblies. The coboundary map in this coresolution is given by $\delta f = f \circ \partial$ (see [1, Section 1]).

The homology of $(\mathrm{Sh}_*)_\Gamma$ computes $H_*(\Gamma, \mathrm{St})$. Similarly, the homology of $(\mathrm{Sh}_*^\vee)^\Gamma$ computes $H^*(\Gamma, \mathrm{St}^\vee)$. By Kronecker duality, there is a perfect pairing $\langle \cdot, \cdot \rangle$ between $H_*(\Gamma, \mathrm{St})$ and $H^*(\Gamma, \mathrm{St}^\vee)$. The first vector space is finite dimensional by Borel-Serre duality, and hence so is the second. We call a cycle in $(\mathrm{Sh}_*)_\Gamma$ a sharbly Γ-cycle, and a cocycle in $(\mathrm{Sh}_*^\vee)^\Gamma$ a cosharbly Γ-cocycle. Given a sharbly Γ-cycle A (resp. a cosharbly Γ-cocycle f), we will denote by $\widetilde{A}$ (resp. $\widetilde{f}$) its image in the homology of Γ (resp. in the cohomology of Γ).

Let $1 \le m \le n-1$. Denote by $\{e_1, \dots, e_n\}$ the standard basis vectors of $\mathbb{Q}^n$, considered as column vectors. Let W_m be the span of the set $\{e_1, \dots e_m\}$, and let Y_m be the span of the set $\{e_{m+1}, \dots, e_n\}$ in $\mathbb{Q}^n$. Using this notation, we make the following definition.

Definition 2.2. A basic k-sharbly is said to be (i,m)-*reducible* if after permutation of its vectors, it is of the form $[w_1, \dots, w_{m+i}, y_1, \dots, y_{n+k-m-i}]$ where $w_1, \dots, w_{m+i}$ are vectors in W_m.

Given an (i,m)-reducible basic sharbly $C = [w_1, \dots, w_{m+i}, y_1, \dots, y_{n+k-m-i}]$, we may extract the sequence $A = (w_1, \dots, w_{m+i})$ which we will call the W_m-component of C. We will call the sequence $B = (y_1, \dots, y_{n+k-m-i})$ the remaining component of C.

A k-sharbly will be said to be (i,m)-*reducible* if it is a sum of (i,m)-reducible basic k-sharblies.

We note that we may consider the W_m-component A of an (i,m)-reducible basic sharbly as an i-sharbly for $\mathrm{GL}(m)$. Also, denoting projection from $\mathbb{Q}^n \to$

Y_m by a prime, we may view the projection of the remaining component $B' = (y'_1, \dots, y'_{n+k-m-i})$ as a j-sharbly for $\mathrm{GL}(n-m)$, where $j = k - i$

3. Lower congruence subgroups

Definition 3.1. A *primary lower congruence subgroup* Δ of $\mathrm{SL}(n, \mathbb{Z})$ is a subgroup of $\mathrm{SL}(n, \mathbb{Z})$ defined by

$$\Delta = \{g \in \mathrm{SL}(n, \mathbb{Z}) \mid g \bmod \mathrm{N} \ \in P(\mathbb{Z}/N)\}$$

where P is an algebraic subgroup of $\mathrm{GL}(n)$ that is either all of $\mathrm{GL}(n)$ or a parabolic subgroup which contains the upper triangular matrices, and N is a positive integer.

The *level* of Δ is the integer N.

A *lower congruence subgroup* Δ of $\mathrm{SL}(n, \mathbb{Z})$ is a finite intersection of primary lower congruence subgroups of levels $N_1, \dots, N_k$ (with possibly varying P's). The level of Δ is the least common multiple of $N_1, \dots, N_k$.

For example, the group $\Gamma_0(n, N) \subset \mathrm{SL}(n, \mathbb{Z})$ consisting of the matrices which modulo N stabilize $(\mathbb{Z}/N)e_1$ is a lower congruence subgroup.

We now give a definition from [1], modified slightly to account for the fact that the subgroups in which we are interested lie in $\mathrm{SL}(n)$, rather than $\mathrm{GL}(n)$.

Definition 3.2. Let $n = n_0 = n_1 + \cdots + n_t$, Γ_i a subgroup of $\mathrm{SL}(n_i, \mathbb{Z})$ for $i = 0, \dots, t$. Let $e_1, \dots, e_n$ be the standard basis vectors of $U = \mathbb{Q}^n$, let V_1 be the span of $\{e_1, \dots, e_{n_1}\}$, and for $1 < i \leq t$ let V_i be the span of

$$\{e_j \mid n_1 + \cdots + n_{i-1} + 1 \leq j \leq n_1 + \cdots + n_i\}.$$

Set $F_m = V_1 + V_2 + \cdots + V_m$ and let $\mathcal{F}$ denote the flag $(0) \subset F_1 \subset F_2 \subset \cdots \subset F_t = U$. We identify $\mathrm{GL}(n_i, \mathbb{Z})$ with the subgroup of $\mathrm{GL}(n, \mathbb{Z})$ which act trivially on V_j for $j \neq i$ and which stabilize V_i.

We say that the set $\{\Gamma_0, \Gamma_1, \dots, \Gamma_t\}$ is *compatible* if

(i) $\Gamma_0 \supset \Gamma_1 \times \cdots \times \Gamma_t$;

(ii) The stabilizer of $\mathcal{F}$ in Γ_0 projected onto $\mathrm{GL}(n_1, \mathbb{Z}) \times \cdots \times \mathrm{GL}(n_t, \mathbb{Z})$ and then intersected with $\mathrm{SL}(n_1, \mathbb{Z}) \times \cdots \times \mathrm{SL}(n_t, \mathbb{Z})$ lies in $\Gamma_1 \times \cdots \times \Gamma_t$;

(iii) If $v \in V_1 \cup V_2 \cup \cdots \cup V_t$ and $\gamma \in \Gamma_0$ and $\gamma v \in F_m$ for some m, then already $v \in F_m$.

We now define a specific example of a compatible set of subgroups, which we will use throughout the rest of the paper.

Definition 3.3. For $i = 1, \dots, t$ let Δ_i be a lower congruence subgroup of $\mathrm{SL}(n_i, \mathbb{Z})$ of level N_i. Let N be divisible by the least common multiple of the N_i and $n = \sum n_i$. Set $\Delta_0 = \Delta_0(N)$ equal to the set of matrices in $\mathrm{SL}(n, \mathbb{Z})$ which when written in $n_1, \dots, n_t$ block form satisfy

(1) Any block below the diagonal blocks is congruent to 0 modulo N;

(2) The i-th diagonal block modulo N is contained in the image of Δ_i modulo N.

We call the system $\Delta_0, \Delta_1, \dots, \Delta_t$ a *lower compatible system.*

Of course, $\Delta_0(N)$ depends on $\Delta_1, \dots, \Delta_t$, even though we omit them from the notation.

Proposition 3.4. *Given lower congruence subgroups Δ_i of level N_i, as above, and an integer N divisible by all the N_i, the subgroup $\Delta_0(N)$ is a lower congruence subgroup.*

Proof. For each parabolic subgroup P used in the definition of Δ_i, let N_P be the corresponding level, and let $\widehat{P}$ be the parabolic subgroup that is $(n_1, \ldots, n_t)$-block upper triangular, with $\mathrm{GL}(n_j)$ in the j-th diagonal block for $j \neq i$, and P in the ith block. Let $\widetilde{P}$ be the primary lower congruence subgroup defined by $\widehat{P}$ modulo N_P. Let Q be the parabolic subgroup of block upper triangular matrices, and let $\widetilde{Q}$ be the primary lower congruence subgroup corresponding to Q modulo N. Set $\Delta = \widetilde{Q} \cap \bigcap_{P,i} \widetilde{P}$. Then Δ is a lower congruence subgroup of level N. We will show that $\Delta = \Delta_0$.

Given $M \in \Delta$, it is clear that M satisfies condition (1) of Definition 3.3. Further, the ith diagonal block of M modulo N is in Δ_i modulo N, since reducing it further modulo N_P (which divides N) yields a matrix in $\widehat{P}(\mathbb{Z}/N_P)$ for each P defining Δ_i. Hence $\Delta \subseteq \Delta_0(N)$.

Given a matrix $M \in \Delta_0(N)$, condition (1) implies that M lies in $\widetilde{Q}$. Also, since the ith diagonal block of M lies in Δ_i modulo N, we see that it must lie in P modulo N_P (since any element of Δ_i lies in P modulo N_P and $N_P|N$) for each P used in defining Δ_i. Hence M lies in $\widetilde{P}$. Therefore, $M \in \Delta$, so $\Delta_0(N) \subseteq \Delta$. □

Lemma 3.5. *Given lower congruence subgroups Δ_i of level N_i, as above, and an integer N divisible by all the N_i, the system $\Delta_0(N), \Delta_1, \ldots, \Delta_t$ defined in Definition 3.3 is compatible if $N > 1$.*

Proof. Property (i) of a compatible system is clear. For property (ii), we note that an element of the stabilizer of $\mathcal{F}$ is block upper triangular. The diagonal blocks must all have determinant ± 1. If they all have determinant 1, they must (by part (2) of the definition of $\Delta_0(N)$) be congruent modulo N to an element of Δ_i. However, one checks easily that any matrix of determinant 1 that is congruent to an element of Δ_i modulo N actually lies in Δ_i.

For (iii), suppose $v \in V_r$, $\gamma \in \Delta_0(N)$ and $\gamma v \in F_m$. We may assume that $v \neq 0$. We must show that $r \leq m$. Suppose to the contrary that $r > m$. In block diagonal form write $\gamma = (\gamma_{ij})$ and $v = (v_i)$. Then $v_i = 0$ if $i \neq r$, $v_r \neq 0$ and $(\gamma v)_j = \gamma_{jr} v_r$. Now $\gamma v \in F_m$ implies that $\gamma_{jr} v_r = 0$ for all $j > m$. In particular $\gamma_{rr} v_r = 0$. But γ_{rr} has a nonzero determinant (as may be observed by reducing γ modulo N) and this gives the desired contradiction. □

4. Review and modification of the unstable construction

We will now review the main construction of [1]. Because we are using sharblies and cosharblies over $\mathbb{Q}$ (to allow the computation of Hecke operators), and because we have changed the definition of compatibility, we must slightly alter the construction.

By induction, we only need to consider the case of a compatible system with two blocks. Let $t = 2$, and set $n = n_1 + n_2$. Let Δ_1 be a lower congruence subgroup in $\mathrm{SL}(n_1, \mathbb{Z})$ and Δ_2 be a lower congruence subgroup in $\mathrm{SL}(n_2, \mathbb{Z})$. Fix $\Delta_0(N) = \Delta_0$ in $\mathrm{SL}(n, \mathbb{Z})$ as in Definition 3.3. Then $\Delta_0, \Delta_1, \Delta_2$ is a compatible system. Recall that V_1 is the span of the first n_1 standard basis vectors, and V_2 is the span of the remaining standard basis vectors, and let $U = \mathbb{Q}^n$.

Definition 4.1. With respect to the compatible system $\Delta_0(N), \Delta_1, \Delta_2$:

(1) A subset M of U is *pliable* if there exists $\gamma \in \Delta_0(N)$ such that γM spans V_1 over $\mathbb{Q}$. If so, we say γ *plies* M.

(2) If $M = \{m_1, \ldots, m_r\}$ is a sequence of vectors in U, set
$$\mathcal{P}(d, M) = \{S \subset \{1, \ldots, r\} | \{m_i \mid i \in S\} \text{ is pliable and } |S| = d\}.$$

For $k = i + j$, we now give a construction of a k-cosharbly for $\mathrm{GL}(n)$ from an i-cosharbly and a j-cosharbly.

Definition 4.2. Given the compatible system $\Delta_0(N), \Delta_1, \Delta_2$, assume that $N \geq 3$. Let f be a Δ_1-invariant i-cosharbly and let g be a Δ_2-invariant j-cosharbly. Then f is a function of $n_1 + i$ vectors in V_1 and g is a function of $n_2 + j$ vectors in V_2. We extend g to be 0 if any vector in its argument is 0. We denote by a prime the natural projection of U to V_2. Define $h = h_{f,g}$ by extending linearly the map $h : (\mathbb{Q}^n)^{n+i+j} \to \mathbb{Q}$ given by
$$\begin{aligned} &h(m_1, \ldots, m_{n+i+j}) \\ &\quad = \sum_S (-1)^{\sigma_S} f(\gamma_S m_{\sigma_S(1)}, \ldots, \gamma_S m_{\sigma_S(n_1+i)}) g((\gamma_S m_{\sigma_S(n_1+i+1)})', \ldots, (\gamma_S m_{\sigma_S(n)})') \end{aligned}$$
where S runs over $\mathcal{P}(n_1 + i, \{m_1, \ldots, m_{n+i+j}\})$, and for each S we choose a permutation σ_S such that $\{\sigma_S(1), \ldots, \sigma_S(n_1 + i)\} = S$ and a $\gamma_S \in \Delta_0(N)$ such that γ_S plies $\{m_{\sigma_S(1)}, \ldots, m_{\sigma_S(n_1+i)}\}$.

Lemma 4.3. *The h defined in Definition 4.2 is independent of the choices of σ_S and γ_S and is a k-cosharbly.*

Proof. Since f and g are antisymmetric, we see that the choice of a σ_S for each set S does not change the value of h.

We now show that the choices of γ_S do not change the value of h. For simplicity of notation, suppose that σ_S is the identity. Suppose γ_S and δ_S are two elements of $\Delta_0(N)$ that ply $\{m_1, \ldots, m_{n_1+i}\}$. We claim that
$$f(\gamma_S m_1, \ldots, \gamma_S m_{n_1+i}) g((\gamma_S m_{n_1+i+1})', \ldots, (\gamma_S m_n)')$$
and
$$f(\delta_S m_1, \ldots, \delta_S m_{n_1+i}) g((\delta_S m_{n_1+i+1})', \ldots, (\delta_S m_n)').$$
are equal. To see this, let $w_r = \gamma_S m_r$ for $r = 1, \ldots, n + i + j$, and let $\epsilon = \delta_S \gamma_S^{-1}$. Since γ_S^{-1} maps V_1 to the span of $\{m_i : i \in S\}$, and δ_S maps this span back to V_1, we see that ϵ stabilizes V_1. Hence, ϵ is block upper triangular. If we denote the block diagonal components of ϵ by ϵ_1 and ϵ_2, we have (by property (2) of Definition 3.3) that each ϵ_i is congruent modulo N to a matrix in Δ_i. Since $N \geq 3$ and $\det(\epsilon_i)$ is a unit in $\mathbb{Z}$, and matrices in Δ_i have determinant 1, we see that $\det(\epsilon_i) = 1$. Finally, since the Δ_i are defined solely by congruence conditions modulo divisors of N and the requirement of being in $\mathrm{SL}(n_i, \mathbb{Z})$, we deduce that ϵ_1 and ϵ_2 are in Δ_1 and Δ_2, respectively.

Now since f and g are invariant under Δ_1 and Δ_2, respectively, we see that
$$\begin{aligned} &f(\gamma_S m_1, \ldots, \gamma_S m_{n_1+i}) g((\gamma_S m_{n_1+i+1})', \ldots, (\gamma_S m_{n+i+j})') \\ &\qquad = f(w_1, \ldots, w_{n_1+i}) g(w'_{n_1+i+1}, \ldots, w'_{n+i+j}) \\ &\qquad = f(\epsilon_1 w_1, \ldots, \epsilon_1 w_{n_1+i}) g(\epsilon_2 (w_{n_1+i+1})', \ldots, \epsilon_2 (w_{n+i+j})') \\ &\qquad = f(\epsilon w_1, \ldots, \epsilon w_{n_1+i}) g((\epsilon w_{n_1+i+1})', \ldots, (\epsilon w_{n+i+j})') \\ &\qquad = f(\delta_S m_1, \ldots, \delta_S m_{n_1+i}) g((\delta_S m_{n_1+i+1})', \ldots, (\delta_S m_{n+i+j})') \end{aligned}$$
as desired.

To see that h is a cosharbly, we must also check that if $m_1, \dots, m_n$ do not span U then $h(m_1, \dots, m_n) = 0$. This follows exactly as in [1, p. 336]. $\square$

We now wish to show that if f and g are cosharbly cocycles, then $h_{f,g}$ is a cosharbly cocycle, and that if f and g are not coboundaries, then $h_{f,g}$ is not a coboundary. Clearly a Δ-invariant cosharbly cocycle f is not a coboundary if and only if there is a sharbly Δ-cycle A such that $f(A) \neq 0$. Hence, given an i-sharbly Δ_1-cycle A with $f(A)$ nonzero, and a j-sharbly Δ_2-cycle B with $g(B)$ nonzero, we wish to construct a k-sharbly Δ_0-cycle C with $h(C)$ nonzero.

Definition 4.4. Let $A = \sum_v r_v[v]$ and $B = \sum_w s_w[w]$ be respectively an i-sharbly for $\mathrm{GL}(n_1)$ and a j-sharbly for $\mathrm{GL}(n_2)$. A *lift* of B will be any formal sum $B_* = \sum_w s_w[w_*]$ where the projection of each $\mathbb{Q}^n$-vector in w_* equals the corresponding vector in w.

Given a lift B_* of B, we define the $(i+j)$-sharbly for $\mathrm{GL}(n)$

$$A \otimes B_* = \sum_{v,w} r_v s_w [v, w_*].$$

(where v can be viewed in V_1 or in U since we have embedded V_1 into U as the span of $e_1, \dots, e_{n_1}$.) We call $A \otimes B_*$ a *tensor sharbly.*

We will also set $B_\dagger = \sum_w s_w[w_\dagger]$ to be a specific lift of B, namely the one where each vector in $w_\dagger$ in (n_1, n_2)-block form has first component 0 and second component w.

Proposition 4.5. *Let $\Delta_0, \Delta_1, \Delta_2$ be a lower compatible system of groups. If A is an i-sharbly Δ_1-cycle, and B is a j-sharbly Δ_2-cycle, then the tensor sharbly $A \otimes B_\dagger$ is an $(i+j)$-sharbly Δ_0-cycle.*

Proof. This is proved in [1, p. 335]. $\square$

Proposition 4.6. *Let $\Delta_0, \Delta_1, \Delta_2$ be a lower compatible system of groups. Let f be an i-cosharbly Δ_1-cocycle and let g be a j-cosharbly Δ_2-cocycle. Then h_{fg} is an $(i+j)$-cosharbly Δ_0-cocycle.*

Proof. This follows from [1, pp. 336-338]. $\square$

Theorem 4.7. *Let A be an i-sharbly for $\mathrm{GL}(n_1)$, let B be a j-sharbly for $\mathrm{GL}(n_2)$, and let B_* be any lift of B. Suppose also that f is a Δ_1-invariant i-cosharbly and g is a Δ_2-invariant j-cosharbly. Then*

$$h_{fg}(A \otimes B_*) = f(A)g(B).$$

Proof. Let $A = \sum_v r_v[v]$, $B = \sum_w s_w[w]$, and $B_* = \sum s_w[w_*]$. Then

$$A \otimes B_* = \sum_{v,w} r_v s_w [v, w_*].$$

We compute $h([v, w_*])$ for each term of this sum. Let $\gamma \in \Delta_0(N)$ and suppose that some vector in γw_* is contained in V_1. By property (iii) of a compatible system, this implies that the corresponding vector of w^* is in V_1, which implies that the corresponding vector in w is 0. This is a contradiction (because sharblies are not allowed to have the zero vector as components), so we see that the only pliable sets of vectors among the columns of $[v, w_*]$ are the subsets of v. Hence, $\mathcal{P}(n_1 + i, v_1, \dots, w_{*,n+i+j})$ consists of the single set $S = \{1, \dots, n_1 + i\}$ (corresponding to the pliable set of vectors $\{v_1, \ldots, v_{n_1+1}\}$). Therefore, when we compute $h_{fg}([v, w_*])$, there is only one term in the sum, we may take $\sigma_S = 1$ and $\gamma_S = 1$, and we obtain $h_{fg}([v, w_*]) = f(v)g(w'_*) = f(v)g(w)$. It follows by linearity of h_{fg}, f, and g, that

$$\begin{aligned} h_{fg}(A \otimes B_*) &= h_{fg}\left(\sum_{v,w} r_v s_w [v, w_*]\right) \\ &= \sum_{v,w} r_v s_w h_{fg}([v, w_*]) \\ &= \sum_{v,w} r_v s_w f(v) g(w) \\ &= \left(\sum_{v} r_v f(v)\right)\left(\sum_{w} s_w g(w)\right) \\ &= f(A)g(B). \end{aligned}$$

□

Corollary 4.8. *In the construction of h_{fg}, assume that f and g are not coboundaries. Then h_{fg} is not a coboundary.*

Proof. Recall that a cosharbly Δ-cocycle is a coboundary if and only if it vanishes on all sharbly Δ-cycles.

Since f is not a coboundary, there exists an i-sharbly Δ_1-cycle A with $f(A) \neq 0$. Since g is not a coboundary, there exists a j-sharbly Δ_2-cycle B with $g(B) \neq 0$. By Proposition 4.5, $A \otimes B_\dagger$ is an $(i + j)$-sharbly Δ_0-cycle. By Theorem 4.7 $h_{fg}(A \otimes B_\dagger) = f(A)g(B)$ is nonzero, so h_{fg} is not a coboundary. □

5. Hecke operators

We have the sharbly complex (Sh, ∂) and the cosharbly cocomplex $(\mathrm{Sh}^\vee, \delta)$.

The group $\mathrm{GL}(n, \mathbb{Q})$ acts on Sh by $g[x_1, \ldots, x_k] = [gx_1, \ldots, gx_k]$. It acts on $\mathrm{Sh}^\vee$ by the defining formula $(hg)(A) = h(gA)$ for $g \in \mathrm{GL}(n, \mathbb{Q})$ and A a sharbly. We will also write this as

$$\langle hg, A\rangle = \langle h, gA\rangle$$

where $g \in \mathrm{GL}(n, \mathbb{Q})$, $h \in \mathrm{Sh}^\vee$ and $A \in \mathrm{Sh}$.

Let Γ be a subgroup of $\mathrm{GL}(n, \mathbb{Q})$. A Γ-sharbly is an element in the coinvariants $H_0(\Gamma, \mathrm{Sh})$. A Γ-cosharbly is an element in the invariants $H^0(\Gamma, \mathrm{Sh}^\vee)$. They are paired naturally by the same pairing $\langle \cdot, \cdot \rangle$. It is a perfect pairing.

Let $T = \Gamma s \Gamma$ where $s \in \mathrm{GL}(n, \mathbb{Q})$. Then T acts as a Hecke operator on Γ-sharblies A and Γ-cosharblies h as follows. Write $T = \coprod_\alpha \Gamma s_\alpha$. Then $h|T = \sum_\alpha h s_\alpha$ and $TA = \sum_\alpha s_\alpha A$. One easily checks as usual that these formulas give well-defined operators that do not depend on the choice of coset representatives.

We thus get an action of the Hecke operators on the coinvariant sharbly complex and the invariant cosharbly complex, and hence on the homology of these complexes. This action induces the usual Hecke operators on the (co)homology of Γ.

One reason for using lower congruence subgroups in this paper is that the Hecke representatives s_α can be taken to be the upper triangular ones we normally use for $\mathrm{SL}(n, \mathbb{Z})$.

Lemma 5.1. *Let Γ be a lower congruence subgroup of $\mathrm{SL}(n,\mathbb{Z})$. Then*

$$\Gamma D_{\ell,k}\Gamma = \coprod_{\alpha} \Gamma s_{\alpha}$$

where the s_α run over matrices M which satisfy the following conditions:
(i) M is upper triangular with $1'$s and ℓ's along the diagonal;
(ii) there are exactly k ℓ's on the diagonal of M;
(iii) the upper triangular entries of M are 0 except for those M_{ab} for $a<b$ where $M_{aa}=1$ and $M_{bb}=\ell$, in which case M_{ab} is an integer from 0 to $\ell-1$ inclusive.

Proof. The lemma is well known if $\Gamma = \mathrm{SL}(n,\mathbb{Z})$. In general, this implies that the cosets on the right are distinct. Now suppose $x \in \Gamma D_{\ell,k}\Gamma$. Then again from the level 1 case we have that $x = \delta M$ for some $\delta \in \mathrm{SL}(n,\mathbb{Z})$ and some M as described in the lemma. However, since $D_{\ell,k}$ is upper triangular (in fact it is diagonal), x satisfies the same lower congruence conditions that define Γ and since M is upper triangular, δ must satisfy all the same lower triangular congruences as well. In other words, $\delta \in \Gamma$. □

Remark. If Γ is not a lower-congruence subgroup, the conclusion of Lemma 5.1 would no longer follow. For example, if $\Gamma = \Gamma_1(N)$ for $N > 2$ it will be necessary to choose s_α's which are not upper triangular. Such s_α's would not preserve the set of (i,m)-reducible sharblies, and our method below would fail to compute the action of the Hecke operators on the cosharbly h which we will define.

We are going to state and prove a formula relating Hecke operators in three different dimensions. Some of our previous notation left the size of the matrices unstated. Now, if necessary, indicate the size of an $m \times m$ matrix by a superscript m.

Let $n = n_1 + n_2$ and let $\Gamma, \Gamma_1, \Gamma_2$ be a compatible set of lower congruence subgroups as above. We have

$$T^n_{\ell,k} = \Gamma D^n_{\ell,k}\Gamma = \coprod_{\alpha} \Gamma s^n_{k,\alpha}$$

with the $s^n_{k,\alpha}$ as described in lemma 5.1.

Let us write

$$T^{n_1}_{\ell,i} = \Gamma_1 D^{n_1}_{\ell,i}\Gamma_1 = \coprod_{\beta} \Gamma_1 s^{n_1}_{i,\beta}$$

and

$$T^{n_2}_{\ell,j} = \Gamma_2 D^{n_2}_{\ell,j}\Gamma_2 = \coprod_{\zeta} \Gamma_2 s^{n_2}_{j,\zeta}$$

Then we can enumerate the set $\{s^n_{k,\alpha}\}$ of coset representatives of $T^n_{\ell,k}$ as follows, using (n_1,n_2)-block form:

$$\bigcup \left\{ \begin{pmatrix} s^{n_1}_{i,\beta} & M \\ 0 & s^{n_2}_{j,\zeta} \end{pmatrix} \right\}$$

where the union runs over pairs of coset representatives $s^{n_1}_{i,\beta}$ of $T^{n_1}_{\ell,i}$ and coset representatives $s^{n_2}_{j,\zeta}$ of $T^{n_2}_{\ell,j}$ with $0 \le i \le n_1$, $0 \le j \le n_2$, and $i + j = k$, and over all M which cause the resulting matrix to satisfy the conditions of Lemma 5.1. For a given choice of i, j, β, ζ, the possible M will have $(n_1 - i)j$ entries which range from 0 to $\ell - 1$ inclusive, with the remaining entries being 0.

Motivated by this enumeration of the $s^n_{k,\alpha}$'s we make the following definition:

Definition 5.2. Let ϕ, ϕ_1, ϕ_2 be characters of the Hecke algebras $K[T^n_{\ell,k}]$, $K[T^{n_1}_{\ell,i}]$, and $K[T^{n_2}_{\ell,j}]$, respectively, such that

$$\phi(T^n_{\ell,k}) = \sum_{i+j=k} \ell^{(n_1-i)j}\phi_1(T^{n_1}_{\ell,i})\phi_2(T^{n_2}_{\ell,j}).$$

Then we say that ϕ is *Hecke-reducible* into (ϕ_1, ϕ_2).

Remark. Because of the enumeration of the $s^n_{k,\alpha}$'s above, and the way the cosharbly h is defined, we will see that the Hecke eigenvalues on h restricted to a certain set of (i, n_1)-reducible sharblies define a Hecke-reducible character.

Lemma 5.3. *If ϕ is Hecke-reducible into (ϕ_1, ϕ_2), then the corresponding Hecke polynomials (recall Definition 1.1) satisfy the equation*

$$F_{\phi,\ell}(X) = F_{\phi_1,\ell}(X)F_{\phi_2,\ell}(\ell^{n_1}X)$$

Proof. Write $a^n_{\ell,k} = \phi(T^n_{\ell,k})$ and similarly for the values of ϕ_1 and ϕ_2. The left hand side of the desired equation equals

$$\sum_{k=0}^{n}(-1)^k\ell^{k(k-1)/2}a^n_{\ell,k}X^k = \sum_{k=0}^{n}(-1)^k\ell^{k(k-1)/2}X^k\sum_{i+j=k}\ell^{(n_1-i)j}a^{n_1}_{\ell,i}a^{n_2}_{\ell,j}$$

whereas the right hand side equals

$$\left(\sum_{i=0}^{n_1}(-1)^i\ell^{i(i-1)/2}a^{n_1}_{\ell,i}X^i\right)\left(\sum_{j=0}^{n_2}(-1)^j\ell^{j(j-1)/2+n_1j}a^{n_2}_{\ell,j})X^j\right)$$
$$= \sum_{k=0}^{n}(-1)^kX^k\sum_{i+j=k}\ell^{i(i-1)/2+j(j-1)/2+n_1j}a^{n_1}_{\ell,i}a^{n_2}_{\ell,j}.$$

Comparing terms for a given k and a given pair (i, j) with $i+j = k$, we must prove that

$$\ell^{k(k-1)/2+(n_1-i)j}a^{n_1}_{\ell,i}a^{n_2}_{\ell,j} = \ell^{i(i-1)/2+j(j-1)/2+n_1j}a^{n_1}_{\ell,i}a^{n_2}_{\ell,ij}.$$

This is true because the exponents of ℓ on the two sides are equal. □

Corollary 5.4. *If a character ϕ on the Hecke algebra $K[\{T^n_{\ell,i}\}]$ is Hecke reducible into (ϕ_1, ϕ_2), with each ϕ_j a character of $K[\{T^{n_j}_{\ell,i}\}]$, and each ϕ_j has an attached Galois representation σ_j, then ϕ is attached to the Galois representation*

$$\rho = \sigma_1 \oplus \omega^{n_1}\sigma_2.$$

Proof. We have

$$\rho(\mathrm{Frob}_\ell) = \begin{pmatrix} \ell^{n_1}\sigma_2(\mathrm{Frob}_\ell) & 0 \\ 0 & \sigma_1(\mathrm{Frob}_\ell)\end{pmatrix}$$

so that

$$\det(I - \rho(\mathrm{Frob}_\ell)X) = \det(I - \ell^{n_1}\sigma_2(\mathrm{Frob}_\ell)X)\det(I - \sigma_1(\mathrm{Frob}_\ell)X)$$

which equals

$$\det(I - \sigma_2(\mathrm{Frob}_\ell)(\ell^{n_1}X))\det(I - \sigma_1(\mathrm{Frob}_\ell)X) = F_{\phi_2,\ell}(\ell^{n_1}X)F_{\phi_1,\ell}(X) = F_{\phi,\ell}(X).$$

Hence, ϕ is attached to ρ. □

6. The Hecke action on h_{fg}

Theorem 6.1. *Let $\Delta_1 \subset \mathrm{SL}(n_1,\mathbb{Z})$ and $\Delta_2 \subset \mathrm{SL}(n_2,\mathbb{Z})$ be lower congruence subgroups, and set $n = n_1+n_2$. Choose Δ_0 so that $\Delta_0, \Delta_1, \Delta_2$ is a lower compatible system. Let f be an i-cosharbly Δ_1-cocycle and let A be an i-sharbly Δ_1-cycle such that $f(A) \neq 0$. Let g be a j-cosharbly Δ_2-cocycle and let B be a j-sharbly Δ_2-cycle such that $g(B) \neq 0$. Recall that $\widetilde{f}$ and $\widetilde{g}$ denote the classes of f and g modulo coboundaries. Suppose that $\widetilde{f}$ and $\widetilde{g}$ are simultaneous eigenvectors of the Hecke operators, with $\widetilde{f}$ affording the character ϕ_1 and $\widetilde{g}$ affording the character ϕ_2. Define the $(i+j)$-cosharbly cocycle $h = h_{fg}$ as in Definition 4.2. Then there is a Hecke-character ϕ such that for $C = A \otimes B_\dagger$, we have*

$$\langle \widetilde{h} | T^n_{\ell,k}, C\rangle = \phi(T^n_{\ell,k})\langle \widetilde{h}, C\rangle,$$

and ϕ is Hecke-reducible into (ϕ_1, ϕ_2).

Proof. Let h be as above, and let $C = A \otimes B_\dagger$. Then C is an (i, n_1)-reducible k-sharbly Δ_0-cycle. Define the Hecke character ϕ by

$$\phi(T^n_{\ell,k}) = \sum_{i+j=k} \ell^{(n_1-i)j}\phi_1(T^{n_1}_{\ell,i})\phi_2(T^{n_2}_{\ell,j}).$$

Now let $T^n_{\ell,k} = \Delta_0 D^n_{\ell,k}\Delta_0$ be a Hecke operator. We compute

$$\langle \widetilde{h} | T^n_{\ell,k}, C\rangle = \langle \widetilde{h | T^n_{\ell,k}}, C\rangle.$$

Writing $T^n_{\ell,k} = \coprod_\alpha \Delta_0 s^n_{k,\alpha}$, with each $s^n_{k,\alpha}$ upper triangular, as in Lemma 5.1, we recall that the set $\{s^n_{k,\alpha}\}$ can be written as

$$\bigcup_{i+j=k} \left\{ \begin{pmatrix} s^{n_1}_{i,\beta} & M \\ 0 & s^{n_2}_{j,\zeta} \end{pmatrix} \right\},$$

where $\Delta_1 D^{n_1}_{\ell,i}\Delta_1 = \coprod_\beta \Delta_1 s^{n_1}_{i,\beta}$, and $\Delta_2 D^{n_2}_{\ell,j}\Delta_2 = \coprod_\zeta \Delta_2 s^{n_2}_{j,\zeta}$, and M runs through all matrices that make the displayed matrix one of the coset representatives of $T^n_{\ell,k}$. Recall that for a given pair $(s^{n_1}_{i,\beta}, s^{n_2}_{j,\zeta})$ there are ℓ^{n_1-j} possible M.

We note that since each $s^n_{k,\alpha}$ is upper triangular, it preserves the space W_{n_1}, so its action on A takes A to an i-sharbly for $\mathrm{GL}(n)$.

One computes easily that if

$$s^n_{k,\alpha} = \begin{pmatrix} s^{n_1}_{i,\beta} & * \\ 0 & s^{n_2}_{j,\zeta} \end{pmatrix},$$

then

$$s^n_{k,\alpha} C = s^{n_1}_{i,\beta} A \otimes s^n_{k,\alpha}(B_\dagger),$$

where we note that $s^n_{k,\alpha}B_\dagger$ is a lift of $s^{n_2}_{j,\zeta}B$. Applying Theorem 4.7, we find that

$$\begin{aligned}
\langle h|T^n_{\ell,k}, C\rangle &= \langle h, T^n_{\ell,k}C\rangle \\
&= \sum_\alpha \langle h, s^n_{k,\alpha}C\rangle \\
&= \sum_{i+j=k} \ell^{(n_1-i)j} \sum_\beta \sum_\zeta \langle f, s^{n_1}_{i,\beta}A\rangle\langle g, s^{n_2}_{j,\zeta}B\rangle \\
&= \sum_{i+j=k} \ell^{(n_1-i)j} \langle f, T_{\ell,i}A\rangle\langle g, T_{\ell,j}B\rangle \\
&= \sum_{i+j=k} \ell^{(n_1-i)j} \langle f|T_{\ell,i}, A\rangle\langle g|T_{\ell,j}, B\rangle \\
&= \sum_{i+j=k} \ell^{(n_1-i)j} \phi_1(T_{\ell,i})\phi_2(T_{\ell,j})\langle f, A\rangle\langle g, B\rangle \\
&= \sum_{i+j=k} \ell^{(n_1-i)j} \phi_1(T^{n_1}_{\ell,i})\phi_2(T^{n_2}_{\ell,j})\langle h, C\rangle \\
&= \langle \phi(T^n_{\ell,k})h, C\rangle.
\end{aligned}$$

The theorem follows. □

We now show that the previous theorem suffices to prove that there is a k-cosharbly cocycle that, modulo coboundaries, is an eigenvector of all the Hecke operators affording the Hecke character ϕ defined above.

Lemma 6.2. *Let K be a field, $\mathcal{H}$ a commutative K-algebra, and A a left $\mathcal{H}$-module. Define the right $\mathcal{H}$-module structure on $A^\vee$ by $\langle \mu|T, a\rangle = \langle \mu, Ta\rangle$, for all $\mu \in A^\vee, a \in A$, $T \in \mathcal{H}$.*

Let $\chi : \mathcal{H} \to K$ be a character and suppose that there are nonzero elements $\mu_0 \in A^\vee$ and $a_0 \in A$ such that $\langle \mu_0|T, a_0\rangle = \chi(T)\langle \mu_0, a_0\rangle$ for every $T \in \mathcal{H}$. Then for every x in the cyclic submodule $\mathcal{H}a_0$ of A,

$$\langle \mu_0|T, x\rangle = \chi(T)\langle \mu_0, x\rangle.$$

Proof. It suffices to prove the displayed formula when $x = T'a_0$ for some $T' \in \mathcal{H}$. Then

$$\begin{aligned}
\langle \mu_0|T, T'a_0\rangle &= \langle \mu_0|T'T, a_0\rangle \\
&= \chi(T'T)\langle \mu_0, a_0\rangle \\
&= \chi(T)\chi(T')\langle \mu_0, a_0\rangle \\
&= \chi(T)\langle \mu_0|T', a_0\rangle \\
&= \chi(T)\langle \mu_0, T'a_0\rangle.
\end{aligned}$$

□

Corollary 6.3. *Under the conditions of the lemma, if in addition A is finite-dimensional over K, then A and $A^\vee$ each contain an $\mathcal{H}$-eigenvector with system of eigenvalues given by the character χ.*

Proof. The displayed formula shows that the restriction $\mu_0|_{\mathcal{H}a_0}$ is an $\mathcal{H}$-eigenvector in $(\mathcal{H}a_0)^\vee$ with system of eigenvalues given by the character χ. The canonical surjective projection $A^\vee \to (\mathcal{H}a_0)^\vee$ is a map of $\mathcal{H}$-modules, so the second assertion follows from finite-dimensionality and Jordan canonical form. Then the first assertion follows by taking the transpose of the Jordan canonical forms. □

7. Application

Theorem 7.1. *Let K be a field of characteristic greater than n_1+n_2+1 or of characteristic 0. Let $\Delta_0, \Delta_1, \Delta_2$ be a lower compatible system of groups inside $\mathrm{SL}(n,\mathbb{Z})$, $\mathrm{SL}(n_1,\mathbb{Z})$, and $\mathrm{SL}(n_2,\mathbb{Z})$, respectively. Assume that $H^i(\Delta_1,K)$ has a Hecke eigenvector affording the Hecke character ϕ_1, and that $H^j(\Delta_2,K)$ has a Hecke eigenvector affording the Hecke character ϕ_2. Then $H^{n_1n_2+i+j}(\Delta_0,K)$ has a Hecke eigenvector affording the character ϕ that is Hecke reducible into (ϕ_1,ϕ_2).*

Remark. Note that the "top dimension" for cohomology of subgroups of $\mathrm{SL}(n,\mathbb{Z})$ is $n(n-1)/2$. In order for Theorem 7.1 to make sense, it is necessary that $n_1n_2+i+j \le (n_1+n_2)(n_1+n_2-1)/2$ whenever $i \le n_1(n_1-1)/2$ and $j \le n_2(n_2-1)/2$. Since

$$\frac{(n_1+n_2)(n_1+n_2-1)}{2} - n_1n_2 = \frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} \ge i+j$$

this will be true. We remark that this also implies that "top-dimensional" eigenvectors combine into "top-dimensional" eigenvectors.

Proof. By Borel-Serre duality,

$$\begin{aligned} H^{n_1n_2i+j}(\Delta_0,K) &\cong H_{(n1+n2)(n1+n2-1)/2-n_1n_2-i-j}(\Delta_0,\mathrm{St}) \\ &= H_{n_1(n_1-1)/2-i+n_2(n_2-1)/2-j}(\Delta_0,\mathrm{St}). \end{aligned}$$

By assumption, $H^i(\Delta_1,K) \cong H_{n_1(n_1-1)/2-i}(\Delta_1,\mathrm{St})$ has a Hecke eigenvector $\widetilde{v}$ affording the character ϕ_1. Similarly $H^j(\Delta_2,K) \cong H_{n_2(n_2-1)/2-j}(\Delta_2,\mathrm{St})$ has a Hecke eigenvector $\widetilde{w}$ affording the character ϕ_2. By duality, there is a nonzero cosharbly Δ_1-cocycle f and a nonzero Δ_2-cocycle g such that $\tilde{f}$ and $\tilde{g}$ afford the characters ϕ_1 and ϕ_2, respectively. Then Theorem 6.1 combined with Corollary 6.3 yields a cosharbly Δ_0-cocycle representing an element in $H_{n_1(n_1-1)/2-i+n_2(n_2-1)/2-j}(\Delta_0,\mathrm{St})^\vee$ affording the character $\phi=(\phi_1,\phi_2)$. Hence, there is also a nonzero sharbly Δ_0-cycle x with

$$\widetilde{x} \in H_{n_1(n_1-1)/2-i+n_2(n_2-1)/2-j}(\Delta_0,\mathrm{St})$$

affording the character ϕ. By Borel-Serre duality, using the fact that

$$\frac{(n_1+n_2)(n_1+n_2-1)}{2} - \frac{n_1(n_1-1)}{2} - \frac{n_2(n_2-1)}{2} = n_1n_2$$

we see that there is a Hecke eigenvector in $H^{n_1n_2+i+j}(\Delta_0,K)$ affording the character ϕ. □

Theorem 7.2. *Let $\mathbb{F}$ be a finite field of characteristic greater than n_1+n_2+1, and for $i=1,2$ let Δ_i be a lower congruence subgroup of $\mathrm{SL}(n_i,\mathbb{Z})$. If $\rho_i : G_\mathbb{Q} \to \mathrm{GL}(n,\mathbb{F})$ fits $H^i(\Delta_i,\mathbb{F})$, then $\rho_1 \oplus \omega^{n_1}\rho_2$ fits $H^{n_1n_2+i+j}(\Delta_0,\mathbb{F})$, where $\Delta_0,\Delta_1,\Delta_2$ is a lower compatible system.*

Proof. This follows immediately from Theorem 7.1 and Corollary 5.4. □

We can use induction to prove a theorem for arbitrarily many Galois representations fitting cohomology groups. To state the theorem, it is convenient to define the second elementary symmetric polynomial in t variables,

$$s_t(x_1,\dots,x_t)=\sum_{1\le i<j\le t} x_ix_j.$$

Theorem 7.3. *Let F be a finite field with characteristic greater than n_1+n_2+1. For $1\le i\le t$, let $\Delta_i\subseteq \mathrm{SL}(n_i,\mathbb{Z})$ be lower congruence subgroups, and assume that there is a Galois representation $\rho_i: G_{\mathbb{Q}}\to \mathrm{GL}(n_i,F)$ fitting $H^{k_i}(\Delta_i,F)$. Choose Δ_0 so that $\Delta_0,\Delta_1,\dots,\Delta_t$ is a lower compatible system. Set*

$$m_j=\sum_{i=1}^{j-1} n_i.$$

Then the Galois representation

$$\rho=\rho_1\oplus\omega^{m_2}\rho_2\oplus\cdots\oplus\omega^{m_t}\rho_t$$

fits

$$H^k(\Delta_0,F),$$

where $k=s_t(n_1,\cdots,n_t)+k_1+\cdots+k_t$.

Proof. This follows easily from Theorem 7.2 by induction, using the fact that

$$n_1(n_2+\cdots+n_t)+s_{t-1}(n_2,\dots,n_t)=s_t(n_1,n_2,\dots,n_t).$$

□

8. Examples

Throughout this section, let $\mathbb{F}$ be a finite field of sufficiently large characteristic p.

Example 8.1. Let $\rho_1: G_{\mathbb{Q}}\to \mathrm{GL}(n_1,\mathbb{F})$ and $\rho_2: G_{\mathbb{Q}}\to \mathrm{GL}(n_2,\mathbb{F})$ be Galois representations fitting $H^{n_1(n_1-1)/2}(\Delta_1,\mathbb{F})$ and $H^{n_2(n_2-1)/2}(\Delta_2,\mathbb{F})$, respectively (so both representations are attached to eigenclasses in the top-dimensional cohomology). Assume that $\Delta_0,\Delta_1,\Delta_2$ is a lower compatible system. Then $\rho_1\oplus\omega^{n_1}\rho_2$ fits $H^{n(n-1)/2}(\Delta_0,\mathbb{F})$, so it is attached to a top-dimensional cohomology class.

This example would apply to two-dimensional odd irreducible Galois representations ρ_1 and ρ_2 with Serre weight 2 and trivial nebentype that are defined over $\mathbb{F}$. By Serre's conjecture they would be attached to cohomology eigenclasses in $H^1(\Gamma_0(N_i),\mathbb{F})$ where N_i is the Serre level of ρ_i. Then $\rho_2\oplus\omega^2\rho_1$ would be attached to a cohomology class in $H^6(\Gamma_{00}(N_1,N_2),\mathbb{F})$, where $\Gamma_{00}(N_1,N_2)\subset \mathrm{SL}(4,\mathbb{Z})$ is defined so that the triple $\Gamma_{00}(N_1,N_2),\Gamma_0(N_1),\Gamma_0(N_2)$ is a lower compatible system.

Example 8.2. Let $\rho_1: G_{\mathbb{Q}}\to \mathrm{GL}(3,\mathbb{F})$ and $\rho_2: G_{\mathbb{Q}}\to \mathrm{GL}(3,\mathbb{F})$ be irreducible Galois representations, each attached to a cohomology eigenclass in $H^3(\Gamma_0(N_i),\mathbb{F})$ (computational examples of such representations may be found in [13]). Define $\Gamma_{00}(N_1,N_2)\subset \mathrm{SL}(6,\mathbb{Z})$ in such a way that the triple $\Gamma_{00}(N_1,N_2),\Gamma_0(N_1),\Gamma_0(N_2)$ is a lower compatible system. As described in [10], by Lefschetz duality, each ρ_i must also be attached to a cohomology eigenclass in $H^2(\Gamma_0(N_i),\mathbb{F})$. Hence, applying Theorem 7.2 multiple times, we find that $\rho_1\oplus\omega^3\rho_2$ is attached to eigenclasses in $H^k(\Gamma_{00}(N_1,N_2),\mathbb{F})$ for $k=3^2+3+3=15$, $k=3^2+3+2=14$, and $k=3^2+2+2=13$.

Example 8.3. In [6, 7, 8], examples are given of Hecke eigenclasses in cohomology groups of the form $H^5(\Gamma_0(4,N),\mathbb{C})$ whose systems of Hecke eigenvalues, reduced modulo p will be attached to Galois representations $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(4,\mathbb{F})$. Letting ρ_1 and ρ_2 be two such representations, with levels N_1 and N_2, and defining $\Gamma_{00}(N_1,N_2) \subset \mathrm{SL}(8,\mathbb{Z})$ so that the triple $\Gamma_{00}(N_1,N_2), \Gamma_0(N_1), \Gamma_0(N_2)$ is a lower compatible system, Theorem 7.2 demonstrates the existence of a cohomology eigenclass in

$$H^{26}(\Gamma_{00}(N_1,N_2),\mathbb{C})$$

whose system of Hecke eigenvalues, reduced modulo p, will have the eight-dimensional Galois representation $\rho_1 \oplus \omega^4\rho_2$ attached. Note that in this case, the top degree for the cohomology of congruence subgroups of SL(8) is 28.

In all of these examples, if we assume that the "predicted weights" of the original Galois representations (as defined in [5]) are all the trivial coefficient module $\mathbb{F}$, then a predicted weight of the Galois representation constructed as a direct sum can also be seen to be $\mathbb{F}$, so that the construction in Theorem 7.2 is in accordance with the weight prediction of the main conjecture of [5]. As mentioned earlier, however, the prediction made for the level in [5] does not match, because the groups $\Gamma_{00}(N_1,N_2)$ are not of the form considered in [5].

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