p-ADIC PROPERTIES OF COEFFICIENTS OF WEAKLY HOLOMORPHIC MODULAR FORMS

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Abstract. We examine the Fourier coefficients of modular forms in a canonical basis for the spaces of weakly holomorphic modular forms of weights 4, 6, 8, 10, and 14, and show that these coefficients are often highly divisible by the primes 2, 3, and 5.

1. Introduction

Let $\Delta(\tau)$ be the unique normalized cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$ given by

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n.$$

Here, as is traditional, τ is a complex number in the upper half plane, $q = e^{2\pi i\tau}$, and $\tau(n)$ is Ramanujan's tau-function. Let

$$j(\tau) = q^{-1} + 744 + 196884q + \ldots = q^{-1} + 744 + \sum_{n\geq 1} c(n)q^n$$

be the standard modular j-function, a weakly holomorphic modular form of weight 0 for $\mathrm{SL}_2(\mathbb{Z})$.

The arithmetic properties of the Fourier coefficients $\tau(n)$ of Δ and $c(n)$ of j have been studied extensively. For instance, Ramanujan [12] proved that

$$\tau(2n) \equiv 0 \pmod 2, \quad \tau(3n) \equiv 0 \pmod 3, \quad \tau(5n) \equiv 0 \pmod 5,$$

and Lehner [9, 10] proved that

$$c(2^a3^b5^c7^dn) \equiv 0 \pmod{2^{3a+8}3^{2b+3}5^{c+1}7^d}.$$

In fact, Lehner's results are much more general; he proved similar congruences for the coefficients of many weakly holomorphic modular functions on $\Gamma_0(p)$. In this paper we will prove such divisibility results for the Fourier coefficients of a large class of weakly holomorphic modular forms of positive integer weight on $\mathrm{SL}_2(\mathbb{Z})$.

Recall that a *weakly holomorphic modular form* of weight $k \in 2\mathbb{Z}$ satisfies the modular equation

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in some finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ and is holomorphic on the upper half plane, but may have poles at any or all of the cusps of Γ. We denote the space of holomorphic modular forms by $M_k(N)$ and denote by $M_k^!(N)$ the space of weakly holomorphic modular forms on $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N\}$. When $N = 1$ we write simply M_k

Date: October 29, 2009.

and $M_k^!$, and note that any form $f \in M_k^!$ is uniquely determined by its Fourier expansion $\sum_{n \geq n_0} a(n)q^n$, where $n_0 \geq 0$ if $f \in M_k$.

Because $\mathrm{SL}_2(\mathbb{Z})$ has only one cusp (at ∞), there is a very nice canonical basis for the space $M_k^!$, indexed by the order of the pole at ∞. To define this basis (as in [5]) write $k = 12\ell + k'$, where $\ell \in \mathbb{Z}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$, so that if $\ell \geq 0$, the space of cusp forms of weight k has dimension ℓ. For every integer $m \geq -\ell$, there exists a unique weakly holomorphic modular form $f_{k,m} \in M_k^!$ with a q-expansion of the form

$$f_{k,m}(\tau) = q^{-m} + O(q^{\ell+1});$$

together these $f_{k,m}$ form a basis for $M_k^!$. It is straightforward to see that any modular form $f = \sum a(n)q^n$ in $M_k^!$, if its first few Fourier coefficients are known, may easily be written in terms of these basis elements as

$$f = \sum_{n_0 \leq n \leq \ell} a(n) f_{k,-n}.$$

The basis elements $f_{k,m}$ may be directly constructed from Δ, j, and the Eisenstein series $E_{k'}$, where we let $E_0 = 1$. From the standard valence formula, we know that

$$\operatorname{ord}_\infty(f) \leq \ell \tag{1}$$

for all $f \in M_k^!$. It is also clear that the form $\Delta^\ell E_{k'}$ has order ℓ at ∞, so it must be $f_{k,-\ell}$; it is unique because the difference of any two such forms will have a Fourier expansion that is $O(q^{\ell+1})$, which must be zero by (1). We can then construct the $f_{k,m}$ iteratively by multiplying $f_{k,m-1}$ by $j(\tau) \in M_0^!$ to get a form of weight k with Fourier expansion beginning q^{-m}, and then subtracting appropriate integer multiples of $f_{k,i}$ (where $-\ell < i < m$) to eliminate the q^{-i} terms in the Fourier expansion. This construction shows that

$$f_{k,m} = \Delta^\ell E_{k'} F_{k,D}(j),$$

where $F_{k,D}(x)$ is a monic polynomial in x of degree $D = \ell + m$ with integer coefficients.

We define the Fourier coefficients $a_k(m,n)$ of these basis elements by

$$f_{k,m}(\tau) = q^{-m} + \sum_n a_k(m,n) q^n,$$

noting that $a_k(m,n) = 0$ when $m < -\ell$ or $n \leq \ell$ or when m or n are not integers. Since each of $E_{k'}, \Delta, j$, and $F_{k,m}$ has integer coefficients, it follows that $a_k(m,n) \in \mathbb{Z}$.

These basis elements are studied extensively in [5]; for instance, for basis elements which are not cusp forms, the zeros in the fundamental domain are all shown to lie on the unit circle. Additionally, the following theorem is proved.

Theorem 1.1 ([5], Theorem 2)**.** *For any even integer k, the basis elements $f_{k,m}$ satisfy the generating function*

$$\sum_{m \geq -\ell} f_{k,m}(z) q^m = \frac{f_{k,-\ell}(z) f_{2-k,1+\ell}(\tau)}{j(\tau) - j(z)}.$$

Replacing k with $2-k$ and switching τ and z, we find the following beautiful duality of Fourier coefficients.

Corollary 1.2 ([5], Corollary 1)**.** *For any even integer k and any integers m, n, the equality*

$$a_k(m,n) = -a_{2-k}(n,m)$$

holds for the Fourier coefficients of the modular forms $f_{k,m}$ and $f_{2-k,n}$.

We note that these bases for $M_k^!$ closely parallel those of half integral weight defined by Zagier in his work on traces of singular moduli [17]. For the weights 1/2 and 3/2, Zagier constructed modular forms on $\Gamma_0(4)$ satisfying a plus space condition and with Fourier expansions

$$\begin{aligned} f_d &= q^{-d} + O(q) \in M_{1/2}^! \quad (0 \le d \equiv 0, 1 \pmod 4), \\ g_D &= q^{-D} + O(1) \in M_{3/2}^! \quad (0 \le D \equiv 0, 3 \pmod 4). \end{aligned}$$

These modular forms have a generating function and satisfy a duality theorem very similar to Theorem 1.1 and Corollary 1.2, and bases with similar properties for all half integral weights are constructed in [4].

The Fourier coefficients of Zagier's f_d and g_D can be interpreted as traces and twisted traces of singular moduli, and have been widely studied. For instance, Ahlgren and Ono [1] proved many congruences for these traces (and their associated half integral weight Fourier coefficients) modulo p^s, and gave an elementary argument that if p splits in $\mathbb{Q}(\sqrt{-d})$, then $\mathrm{Tr}(p^2 d)$ is congruent to $0 \pmod p$. Edixhoven [6] extended their observation, proving that if $\left(\frac{-d}{p}\right) = 1$, then $\mathrm{Tr}(p^{2n} d) \equiv 0 \pmod{p^n}$. An elementary proof of this result using Hecke operators was given by the second author in [8], and Boylan [3] exactly computed $\mathrm{Tr}(2^{2n} d)$, obtaining even stronger congruences and divisibility results for $p = 2$.

With these congruences for Fourier coefficients of forms of half integral weight as a model, it is natural to ask whether similar divisibility results exist for the Fourier coefficients $a_k(m,n)$ of the integral weight basis elements $f_{k,m}$. A result similar to Edixhoven's appears in [5], where it is proved that for the weights $k = 4, 6, 8, 10$, and 14, if $p^r | n$ and $p \nmid m$, it is true that $p^{(k-1)r} | a_k(m,n)$. Thus, if $(m,n) = 1$, we have $n^{k-1} | a_k(m,n)$. This theorem seems to be sharp if $p > 7$. However, looking at the divisibility of these Fourier coefficients by smaller primes, it appears that the $a_k(m,n)$ are divisible by higher powers of these primes. For instance, certain coefficients of

$$f_{4,1}(\tau) = E_4(\tau)(j(\tau) - 984) = q^{-1} + \sum_{n \ge 1} a_4(1,n) q^n$$

factor in the following way.

$$\begin{aligned} a_4(1,2) &= 2^{10} \cdot 5 \cdot 13327 \\ a_4(1,4) &= 2^{13} \cdot 3^2 \cdot 13 \cdot 113 \cdot 2543 \\ a_4(1,5) &= 2^3 \cdot 5^4 \cdot 19 \cdot 9931 \cdot 7639 \\ a_4(1,8) &= 2^{16} \cdot 3 \cdot 5^2 \cdot 293 \cdot 15918317 \\ a_4(1,10) &= 2^{11} \cdot 3^6 \cdot 5^4 \cdot 2184176461 \\ a_4(1,20) &= 2^{14} \cdot 5^4 \cdot 29243 \cdot 235531684534847 \\ a_4(1,25) &= 2^2 \cdot 3^2 \cdot 5^7 \cdot 11491 \cdot 102481 \cdot 4609259 \cdot 4679867 \end{aligned}$$

While Theorem 3 in [5] predicts that if $p^r \| n$, then $p^{3r} | a_4(1,n)$, we see that in fact there are extra powers of 2 and 5 dividing these coefficients. For instance, we expect to see that $2^3 | a_4(1,2)$ and $5^3 | a_4(1,5)$, while in fact the stronger divisibility results $2^{10} | a_4(1,2)$ and

$5^4|a_4(1,5)$ are actually true. Further computation reveals similar divisibility by small primes for other modular forms of these weights.

In this paper, we will prove the following theorem making these divisibility results more explicit. For an integer N, let $v_p(N)$ be the p-adic valuation of N, or the largest integer s such that $p^s|N$.

Theorem 1.3. *Let $k \in \{4,6,8,10,14\}$ and let $p \in \{2,3,5\}$. Then for*

$$\epsilon_{k,p} = \begin{cases} 7 & \text{if } p=2,\ k=4,6,14, \\ 8 & \text{if } p=2,\ k=8,10, \\ 2 & \text{if } p=3,\ k=4,10, \\ 3 & \text{if } p=3,\ k=6,8,14, \\ 1 & \text{if } p=5, \end{cases}$$

we have for all $m,n>0$,

$$v_p(a_k(m,n)) \geq \begin{cases} \epsilon_{k,p} & \text{if } v_p(m) > v_p(n), \\ (v_p(n)-v_p(m))(k-1)+\epsilon_{k,p} & \text{if } v_p(n) > v_p(m). \end{cases}$$

We note that the theorem makes no prediction about divisibility if $v_p(m) = v_p(n)$. If $(m,30)=1$, the theorem implies that for $a,b,c>0$,

$$a_k(m, 2^a 3^b 5^c n) \equiv 0 \pmod{2^{(k-1)a+7} 3^{(k-1)b+2} 5^{(k-1)c+1}}.$$

Because any weakly holomorphic modular form of weight $k \in \{4,6,8,10,14\}$ is a linear combination of the $f_{k,m}(\tau)$ and the Eisenstein series $E_k(\tau)$, we can use this result to derive divisibility results for many coefficients of more general weakly holomorphic modular forms. For instance, if

$$f(\tau) = c_{-2}q^{-2} + c_{-1}q^{-1} + \sum_{n>0} c_n q^n = c_{-2} f_{4,2}(\tau) + c_{-1} f_{4,1}(\tau)$$

is any modular form of weight 4 with integer coefficients, constant coefficient 0, and a pole of order 2 at the cusp, we can deduce easily that for $3|n$, $v_3(c_n) \geq 3v_3(n)+2$.

2. Definitions

In this section, we define various modular forms and operators that will be used throughout the paper.

2.1. **Modular forms derived from Eisenstein series.**

We define the standard normalized Eisenstein series of even weight $k \geq 4$ as

$$E_k(\tau) = 1 + A_k \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $A_k = -2k/B_k$ and B_k is the kth Bernoulli number. In addition, we will need the forms

$$S_{k,p}(\tau) = \frac{E_k(\tau) - E_k(p\tau)}{A_k}, \qquad \text{and} \qquad T_{k,p}(\tau) = \frac{p^k E_k(p\tau) - E_k(\tau)}{p^k - 1}.$$

For $k \in \{4, 6, 8, 10, 14\}$ and any p, it is well known that A_k is an integer and it is easily seen that $S_{k,p}$ has integral Fourier coefficients. For $k \in \{4, 6\}$ and $p = 2$ and for $k = 4$ and $p = 3$, one checks easily that $T_{k,p}$ has integer coefficients. Often, when p is clear from context, we will write S_k and T_k in place of $S_{k,p}$ and $T_{k,p}$. Note that $S_{k,p}$ and $T_{k,p}$ are modular forms of weight k and level p, with $S_{k,p}$ vanishing at ∞ and $T_{k,p}$ vanishing at 0.

In weight 2, we will need the Eisenstein series $E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$, which, although not modular itself, is used to produce the modular form

$$\frac{E_2(\tau) - pE_2(p\tau)}{1-p}$$

of weight 2 and level p [14, p. 88].

2.2. **Newforms.** In level 2, we use the following notation for the normalized newforms of weight 8 and 10:

$$\Xi_8(\tau) = (\eta(\tau)\eta(2\tau))^8 \qquad \text{and} \qquad \Xi_{10}(\tau) = S_{4,2}(\tau)T_{6,2}(\tau).$$

In level 3, we use the following notation for the normalized newform of weight 6:

$$\Omega_6(\tau) = (\eta(\tau)\eta(3\tau))^6.$$

In level 5, we use the following notation for the normalized newforms of weights 4 and 6:

$$\Lambda_4(\tau) = (\eta(\tau)\eta(5\tau))^4 \qquad \text{and} \qquad \Lambda_6(\tau) = \frac{5E_2(5\tau) - E_2(\tau)}{4}\Lambda_4(\tau).$$

These newforms can all be found in William Stein's online tables of modular forms [15], and we check there that the sign of the Fricke involution is 1 for Ξ_8 and Λ_4 and -1 for Ξ_{10}, Ω_6, and Λ_6.

2.3. **Weakly holomorphic forms of weight 0.** Following Apostol [2, pg. 87], for $p \in \{2, 3, 5\}$ we define $\lambda = \lambda_p = 24/(p-1)$,

$$\Phi(\tau) = \Phi_p(\tau) = \left(\frac{\eta(p\tau)}{\eta(\tau)}\right)^{\lambda},$$

and $\psi(\tau) = 1/\Phi(\tau)$. Although λ, Φ, and ψ depend on p, we often omit this dependence from the notation, since it will be clear from context.

We recall from [2] that both ψ and Φ are in $M_0^!(p)$, both have integer Fourier coefficients, Φ has a zero at ∞ and a pole at 0, ψ has a pole at ∞ and a zero at 0, and

$$\psi\left(\frac{-1}{p\tau}\right) = p^{\lambda/2}\Phi(\tau).$$

2.4. **Operators on modular forms.** For p a prime, and $f(\tau) = \sum_{n \geq n_0} a_n q^n$ a weakly holomorphic modular form, we define the U_p operator by

$$(f|U_p)(\tau) = \sum_{n \geq n_0/p} a_{pn} q^n.$$

An alternative definition of $f|U_p$ that we will find useful is [2, Thm. 4.5]

$$(f|U_p)(\tau) = \frac{1}{p}\sum_{j=0}^{p-1} f\left(\frac{\tau + j}{p}\right).$$

If f has weight k and level N, then $f|U_p$ is again a weakly holomorphic modular form of weight k and level N (if $p|N$) or level pN (if $p \nmid N$) [11, Prop. 2.22].

We also define the operators V_p by $(f|V_p)(\tau) = f(p\tau)$ and the Hecke operators T_p by $(f|T_p) = (f|U_p) + p^{k-1}(f|V_p)$. Note that V_p takes modular forms of weight k and level N to forms of weight k and level Np, while T_p preserves both weight and level [11, Prop. 2.2, Thm. 4.5].

3. Reduction to the case where $p|m$, $(n,p)=1$

In this section, we will prove that if Theorem 1.3 is true for the special case of Fourier coefficients $a_k(m,n)$ with $p|m$ and $(n,p)=1$, then it is true in all cases. We begin by citing a result of Duke and Jenkins relating certain Fourier coefficients, from which the divisibility result of [5] cited in the introduction follows immediately.

Proposition 3.1 ([5], Lemma 1)**.** *Let p be a prime and $k \in \{4,6,8,10,14\}$. Then for $m,n,s \in \mathbb{Z}$, with $m,n,s > 0$ and p a prime,*

$$a_k(m, np^s) = p^{s(k-1)}\left(a_k(mp^s, n) - a_k(mp^{s-1}, n/p)\right) + a_k(m/p, np^{s-1}).$$

Applying induction to this proposition, we obtain the following.

Corollary 3.2. *Let $(m,p) = (n,p) = 1$, $r,s \geq 0$, $k \in \{4,6,8,10,14\}$. Then for $0 \leq t \leq \min(r, s-1)$,*

$$a_k(mp^r, np^s) = a_k(mp^{r-t-1}, np^{s-t-1}) + \sum_{j=0}^{t} p^{(s-j)(k-1)} a_k(mp^{r+s-2j}, n).$$

Applying this corollary, we obtain the following reduction.

Theorem 3.3. *Let $p \in \{2,3,5\}$ and let $k \in \{4,6,8,10,12\}$. Assume that $v_p(a_k(a,b)) \geq \epsilon_{k,p}$ for all $a,b > 0$ having $p|a$ and $p \nmid b$. Let $m,n > 0$ be relatively prime to p and $r,s \geq 0$. Then if $r > s$,*

$$v_p(a_k(mp^r, np^s)) \geq \epsilon_{k,p},$$

and if $r < s$,

$$v_p(a_k(mp^r, np^s)) \geq (s-r)(k-1) + \epsilon_{k,p}.$$

Proof. For $r > s$, we apply Corollary 3.2 with $t = s-1$ to $a_k(mp^r, np^s)$, and note that each term obtained is divisible by $p^{\epsilon_{k,p}}$.

For $r < s$, we apply Corollary 3.2 with $t = r$. In this case, we note that the term outside the sum vanishes, and each term inside the sum is divisible by $p^{(s-r)(k-1)+\epsilon_{k,p}}$. □

Hence, we see that in order to prove Theorem 1.3, we need only prove it for $a_k(m,n)$ when $p|m$ and $p \nmid n$.

4. Poles at zero of weakly holomorphic modular forms

In this section, we derive a formula which will allow us to compute the Fourier expansion at 0 of $f|U_p$, where f is a weakly holomorphic modular form for which we know the Fourier expansion at ∞. This will allow us to use the expansion at 0 of $f|U_p$ to obtain information about the Fourier coefficients in the expansion of f at ∞.

Lemma 4.1. *Let f be a meromorphic modular form of weight k on $SL_2(\mathbb{Z})$, and let $f_p = f|U_p$. Then*

$$p\tau^{-k} f_p(-1/\tau) = -f(\tau/p) + pf_p(\tau) + p^k f(p\tau).$$

Note the similarity of the lemma to Theorem 4.6 of [2].

Proof. For an integer j with $1 \leq j \leq p-1$, we will denote by j' the unique integer with $-(p-1) \leq j \leq -1$ such that $jj' \equiv 1 \pmod p$, and we will write $b_j = (jj'-1)/p$. We note that

$$\begin{aligned}
p\tau^{-k} f_p(-1/\tau) &= p\tau^{-k} \frac{1}{p} \sum_{j=0}^{p-1} f\left(\frac{-1/\tau + j}{p}\right) \\
&= \tau^{-k} \sum_{j=0}^{p-1} f\left(\frac{j\tau - 1}{p\tau}\right) \\
&= \tau^{-k} \sum_{j=1}^{p-1} f\left(\begin{pmatrix} j & b \\ p & j' \end{pmatrix} \begin{pmatrix} 1 & -j' \\ 0 & p \end{pmatrix} \tau\right) + \tau^{-k} f(-1/(p\tau)) \\
&= \tau^{-k} \sum_{j=1}^{p-1} \left(p\left(\frac{\tau - j'}{p}\right) + j'\right)^k f\left(\frac{\tau - j'}{p}\right) + \tau^{-k}(p\tau)^k f(p\tau)) \\
&= -f(\tau/p) + \sum_{j=0}^{\infty} f\left(\frac{\tau + j}{p}\right) + p^k f(p\tau) \\
&= -f(\tau/p) + pf_p(\tau) + p^k f(p\tau).
\end{aligned}$$

□

Corollary 4.2. *Let f be a meromorphic modular form of weight k on Γ, and let $f_p = f|U_p$. Then*

$$p(p\tau)^{-k} f_p(-1/pt) = -f(\tau) + pf_p(p\tau) + p^k f(p^2\tau).$$

Further, $p(p\tau)^{-k} f_p(-1/pt)$ is modular of weight k and level p.

Proof. The equality follows immediately by replacing τ by $p\tau$ in Lemma 4.1. The statement about the weight and level follows from the fact that $pf_p(p\tau) + p^k f(p^2\tau) = p((f|T_p)|V_p)$, along with the fact that T_p preserves weight and level, while V_p raises levels by a factor of p. □

5. Integral bases for spaces of modular forms

As seen in the previous section, modular forms on $\Gamma_0(p)$ can be used to study associated modular forms on $\mathrm{SL}_2(\mathbb{Z})$. Therefore, for $p = 2, 3, 5$ we now construct integral bases for the spaces $M_k(p)$, or bases having the property that the modular forms in $M_k(p)$ with integer coefficients are exactly the integer linear combinations of the basis elements. These bases allow us to study divisibility properties of Fourier coefficients by studying the first several coefficients of a given form.

Lemma 5.1. *Let $p \in \{2,3,5\}$, let $k \geq 0$ be even, and let $d = \dim M_k(p)$. Then there is a basis $\{B_{n,k,p} : 0 \leq n < d\}$ of $M_k(p)$ such that each $B_{n,k,p} = q^n + O(q^d)$ and each $B_{n,k,p}$ has integer coefficients.*

Proof. To construct the basis, for each weight k we will find a modular form f in $M_k(p)$ with integer coefficients and leading coefficient 1 that vanishes with order $d-1$ at ∞. We can then multiply f by $\psi(\tau)$ and subtract off an appropriate integer multiple of f to get a form with q-expansion beginning $q^{d-2} + O(q^d)$. Repeating this process of multiplying by ψ and subtracting earlier basis elements, we generate a basis for $M_k(p)$ of modular forms with integer coefficients and the desired Fourier expansions.

To construct these modular forms vanishing to order $d-1$, we note that standard dimension formulas [14, Prop. 6.1] yield the following values for $d = \dim(M_k(p)$:

$$\dim(M_k(2)) = \left\lfloor \frac{k}{4} \right\rfloor + 1, \quad \dim(M_k(3)) = \left\lfloor \frac{k}{3} \right\rfloor + 1, \quad \text{and} \quad \dim(M_k(5)) = 2\left\lfloor \frac{k}{4} \right\rfloor + 1.$$

In level 2, we find that for weight 0, the constant function 1 is the desired modular form, and for weight 2, the form $2E_2(2\tau) - E_2(\tau)$ works. We then note that for $k \geq 0$, $\dim(M_{k+4}(2)) = \dim(M_k(2)) + 1$, so the appropriate form of weight $k+4$ can be obtained from the form of weight k by multiplying by $S_{4,2}(\tau)$, which is of weight 4 and has a Fourier expansion at ∞ beginning $q + O(q^2)$.

For level 3, in weights $0, 2, 4$ we have $1, \frac{1}{2}(3E_2(3\tau) - E_2(\tau)), S_{4,3}$ respectively. Additionally, $\dim(M_{k+6}(3)) = \dim(M_k(3)) + 2$, so multiplying by a form of weight 6 with Fourier expansion beginning with q^2 suffices to construct the needed forms of higher weight; the form

$$\frac{\eta^{18}(3\tau)}{\eta^6(\tau)} = \Phi_3(\tau)\Omega_6(\tau) = q^2 + 6q^3 + \cdots$$

works.

For level 5, $\dim(M_{k+4}(5)) = \dim(M_k(5)) + 2$, and the weight 4 form $\eta^{10}(5\tau)\eta^{-2}(\tau) = \Phi_5(\tau)\Lambda_4(\tau)$, along with the constant function 1 and the weight 2 form $\frac{1}{4}(5E_2(5\tau) - E_2(\tau))$, suffice to construct the basis in all weights. $\square$

Note that the set $\{B_{n,k,p}\}$ does in fact form an integral basis of $M_k(p)$, since any form

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in M_k(p)$$

with integer coefficients can clearly be written as the integer linear combination

$$f(\tau) = \sum_{n=0}^{d-1} a_n B_{k,n,p}(\tau).$$

The importance of this basis for our purposes is that it allows us to check the divisibility of all coefficients of a modular form by checking only finitely many coefficients.

Lemma 5.2. *Let $k \geq 0$ be even and let $p \in \{2,3,5\}$. Let $d = \dim M_k(p)$. If $F(\tau) \in M_k(p)$ has integer coefficients and its first d coefficients are divisible by p^s, then $F(\tau) \equiv 0 \pmod{p^s}$.*

Proof. For $F(\tau) = \sum_{n=0}^{\infty} a_n q^n$, we have $F(\tau) = \sum_{n=0}^{d-1} a_n B_{k,n,p}(\tau)$. If each a_n with $0 \leq n \leq d-1$ is divisible by p^s, then clearly every coefficient of $F(\tau)$ is divisible by p^s. $\square$

6. Weakly holomorphic modular forms of negative weight with minimal poles

Because of the duality of coefficients of basis elements of the weights k and $2-k$ (Corollary 1.2), for a fixed n we can study the coefficients $a_k(mp^s, n)$ of the forms f_{k,mp^s} for all positive values of m and s simply by studying the Fourier coefficients of the single negative weight form $f_{2-k,n}|U_p$. To facilitate our study of these forms, we now examine negative weight weakly holomorphic modular forms of level p which are holomorphic except for a pole at exactly one of 0 and ∞. Since there are no holomorphic modular forms of negative weight, we see that in fact such a form must have a pole of order at least one at one of 0 and ∞. In fact, using a valence formula, we will see that in many cases, such a modular form must have a pole of order greater than one at one of the two cusps.

We begin by bounding the order of the pole.

Proposition 6.1. *Let $p \in \{2,3,5\}$, and let $k<0$ be even. Let $f \in M_k^!(p)$ and suppose that f is holomorphic at 0. Denote by $\operatorname{ord}_\infty(f)$ the order of vanishing of f at ∞ (so if f has a pole, this is negative). Then*

$$\operatorname{ord}_\infty(f) \leq k\left(\frac{\nu_2(p)}{4}+\frac{\nu_3(p)}{3}\right),$$

where ν_2 and ν_3 are given by [7, p. 535] *(see also* [13, p. 25]*). If equality holds, then f must be nonvanishing in the upper half plane and at* 0.

We note that $\nu_2(2)=1$, $\nu_2(3)=0$ and $\nu_2(5)=2$, while $\nu_3(2)=0$, $\nu_3(3)=1$ and $\nu_3(5)=0$. In addition, we note that reversing the roles of 0 and ∞ yields an analogous bound for modular functions holomorphic at ∞ with a pole at 0.

Proof. This follows immediately from the valence formula [7, (3.9)], by noticing that for a weakly holomorphic modular form whose only pole is at ∞, all the orders of vanishing at any point except ∞ must be nonnegative. For equality, it is clear that the order of vanishing at any non-infinite point must be 0. ☐

For $k<0$ and even and p prime, we now define $\theta_{k,p}$ to be a weakly holomorphic modular form of weight k and level p that is holomorphic on the upper half plane and at 0, with a pole of minimal possible order at ∞ and leading coefficient 1. We define $\alpha_{k,p}$ in a similar way, but require that it be holomorphic at ∞ and with minimal possible pole at 0.

Note that $\theta_{k,p}$ and $\alpha_{k,p}$ are well-defined, since given two such objects, their difference must be a holomorphic modular form of negative weight, hence 0. In addition, for the k and p that we study, $(p\tau)^{-k}\theta_{k,p}(-1/p\tau)$ can easily be shown (using properties of the Fricke involution) to be a multiple of $\alpha_{k,p}$. It will be convenient to write this relation as $\theta_{k,p}(-1/p\tau)=\mu_{k,p}\tau^k\alpha_{k,p}(\tau)$. We also note that often, $\alpha_{k,p}\equiv 1 \pmod{p^{\xi_{k,p}}}$ for some value of $\xi_{k,p}$. In Tables 1, 2, and 3 we define $\theta_{k,p}$ and $\alpha_{k,p}$ for $k\in\{-2,-4,-6,-8,-12\}$ and $p\in\{2,3,5\}$, and give the values of $\mu_{k,p}$ and $\xi_{k,p}$.

We remark that in all cases except $p=5$, $k=-2$ or -6, the minimum order of the pole predicted by the valence formula was achieved. For $p=5$, $k=-2$, the valence formula predicts that $\theta_{k,p}$ should have a pole of order at least 1 (since $k(\nu_2(5)/4+\nu_3(5)/3)=-1$). If equality were achieved, then $\theta_{k,p}$ would have to be nonvanishing on the upper half plane and at 0, so that $1/\theta_{k,p}$ would be a holomorphic modular form of weight 2 and level 5, vanishing

k	$\theta_{k,2}$	$\alpha_{k,2}$	$\mu_{k,2}$	$\xi_{k,2}$
-2	$\frac{\Xi_{10}(\tau)}{\Delta(2\tau)}$	$\frac{\Xi_{10}(\tau)}{\Delta(\tau)}$	-2^5	3
-4	$\frac{\Xi_8(\tau)}{\Delta(2\tau)}$	$\frac{\Xi_8(\tau)}{\Delta(\tau)}$	2^4	4
-6	$\frac{T_6(\tau)}{\Delta(2\tau)}$	$\frac{S_6(\tau)}{\Delta(\tau)}$	-2^9	3
-8	$\frac{T_4(\tau)}{\Delta(2\tau)}$	$\frac{S_4(\tau)}{\Delta(\tau)}$	2^8	5
-12	$\frac{\Delta(\tau)}{\Delta^2(2\tau)}$	$\frac{\Delta(2\tau)}{\Delta^2(\tau)}$	2^{12}	4

TABLE 1. Values of $\theta_{k,p}$, $\alpha_{k,p}$, $\mu_{k,p}$ and $\xi_{k,p}$ for $p = 2$.

k	$\theta_{k,3}$	$\alpha_{k,3}$	$\mu_{k,3}$	$\xi_{k,3}$
-2	$\frac{S_4(\tau)\Omega_6(\tau)}{\Delta(3\tau)}$	$\frac{T_4(\tau)\Omega_6(\tau)}{\Delta(\tau)}$	-3^2	1
-4	$\frac{S_4(\tau)T_4(\tau)}{\Delta(3\tau)}$	$\frac{S_4(\tau)T_4(\tau)}{\Delta(\tau)}$	3^4	1
-6	$\frac{\Omega_6(\tau)}{\Delta(3\tau)}$	$\frac{\Omega_6(\tau)}{\Delta(\tau)}$	-3^3	2
-8	$\frac{T_4(\tau)}{\Delta(3\tau)}$	$\frac{S_4(\tau)}{\Delta(\tau)}$	3^5	1
-12	$\frac{\Phi(\tau)\Delta(\tau)}{\Delta^2(3\tau)}$	$\frac{\psi(\tau)\Delta(3\tau)}{\Delta^2(\tau)}$	3^6	2

TABLE 2. Values of $\theta_{k,p}$, $\alpha_{k,p}$, $\mu_{k,p}$ and $\xi_{k,p}$ for $p = 3$.

k	$\theta_{k,5}$	$\alpha_{k,5}$	$\mu_{k,5}$	$\xi_{k,5}$
-2	$\frac{\Lambda_4(\tau)\Lambda_6(\tau)\Phi(\tau)}{\Delta(5\tau)}$	$\frac{\Lambda_4(\tau)\Lambda_6(\tau)\psi(\tau)}{\Delta(\tau)}$	-5^2	0
-4	$\frac{\Lambda_4(\tau)^2\Phi(\tau)}{\Delta(5\tau)}$	$\frac{\Lambda_4^2(\tau)\psi(\tau)}{\Delta(\tau)}$	5	1
-6	$\frac{\Lambda_6(\tau)}{\Delta(5\tau)}$	$\frac{\Lambda_6(\tau)}{\Delta(\tau)}$	-5^3	0
-8	$\frac{\Lambda_4(\tau)}{\Delta(5\tau)}$	$\frac{\Lambda_4(\tau)}{\Delta(\tau)}$	5^2	1
-12	$\frac{\Phi(\tau)\Lambda_4^3(\tau)}{\Delta^2(5\tau)}$	$\frac{\psi(\tau)\Lambda_4^3(\tau)}{\Delta^2(\tau)}$	5^3	1

TABLE 3. Values of $\theta_{k,p}$, $\alpha_{k,p}$, $\mu_{k,p}$ and $\xi_{k,p}$ for $p = 5$.

at ∞. Since no such form exists, we see that $\theta_{-2,5}$ must have at least a double pole. Similarly, for $p = 5$ and $k = -6$ the bound arising from the valence formula indicates that $\theta_{-6,5}$ must have a pole of order at least 3. However, if $\theta_{-6,5}$ had a pole of order 3, then its reciprocal would be a nonzero holomorphic modular form of weight 6, level 5 that vanishes at ∞ with order 3. The basis computations in the previous section show that such an object cannot exist. Hence, the order of the pole of $\theta_{-6,5}$ must be at least 4.

The congruences on $\alpha_{k,p}$ follow from the following lemma.

Lemma 6.2. *Let $\alpha(\tau) = \sum_{n=0}^{\infty} a_n q^n$ be a weight k weakly holomorphic modular form of level p, with k even and negative. Suppose that α has integer coefficients, is holomorphic at ∞ with $a_0 = 1$, and has a pole of order m at 0. If $p^r | a_i$ for all $1 \leq i \leq m$ and there is some $F(\tau) \in M_{-k}(p)$ having leading coefficient 1 and integer coefficients with $F(\tau) \equiv 1 \pmod{p^r}$, then $\alpha(\tau) \equiv 1 \pmod{p^r}$.*

Proof. Note that $\alpha(\tau)F(\tau) - 1 = \sum_{n=1}^{\infty} b_n q^n$ is weakly holomorphic of weight 0 and is holomorphic at ∞, with a pole of order at most m at 0 and vanishing at ∞. As such, it must be an integer linear combination of $\Phi(\tau), \dots, \Phi^m(\tau)$. Since $F(\tau) \equiv 1 \pmod{p^r}$, we see that each $b_n \equiv a_n$, so that each b_n is divisible by p^r. If we write

$$\alpha(\tau)F(\tau) - 1 = \sum_{n=1}^{\infty} b_n q^n = \sum_{n=1}^{m} c_n \Phi^n(\tau)$$

it is clear that $c_1 = b_1$, and that $c_2 \equiv b_2 \pmod{b_1}$, so that $c_2 \equiv b_2 \equiv 0 \pmod{p^r}$. Inductively, we find that each $c_n \equiv 0 \pmod{p^r}$, so that $\alpha(\tau)F(\tau) - 1 \equiv 0 \pmod{p^r}$. Since $F(\tau) \equiv 1 \pmod{p^r}$, we see that $\alpha(\tau) \equiv 1 \pmod{p^r}$. $\square$

We apply this lemma to each of the $\alpha_{k,p}$ with $k \in \{-2, -4, -6, -8, -12\}$. For $k = -4, -6, -8$, we use the Eisenstein series of weight $-k$ for $F(\tau)$. For $k = -12$, we set $F(\tau) = E_4(\tau)^3$. For $k = -2$ and $p = 2, 3$, we let $F(\tau)$ be the unique monic form of weight 2, level p, which is given by [14, p. 88]

$$\frac{E_2(\tau) - pE_2(p\tau)}{1 - p}$$

and is congruent to 1 mod 8 in level 2, and congruent to 1 mod 3 in level 3. In each case, one checks easily that $F(\tau) \equiv 1 \pmod{p^{\xi_{k,p}}}$, and that the first m coefficients of $\alpha_{k,p}$ are divisible by $p^{\xi_{k,p}}$. Note that there is no congruence on the coefficients of $\alpha_{-2,5}$ modulo 5.

7. Reduction to finitely many coefficients

To prove Theorem 1.3, it remains to show that for any integer $s \geq 1$, the coefficient $a_k(mp^s, n)$ of the form f_{k,mp^s} is sufficiently divisible by p when $p \nmid n$. To accomplish this, we define

$$g(\tau) = f_{k,mp^s}(\tau) - f_{k,mp^{s-1}}(p\tau) = \sum_{n=1}^{\infty} \gamma_n q^n.$$

Note that for $p \nmid n$, $\gamma_n = a_k(mp^s, n)$. We will prove that all of the γ_n are divisible by $p^{\epsilon_{k,p}}$. In this section, we show that it is actually sufficient to prove that finitely many of the γ_n satisfy the desired divisibility.

Lemma 7.1. *Let $p \in \{2, 3, 5, 7, 13\}$, $k \in \{4, 6, 8, 10, 14\}$, $\lambda = 24/(p-1)$, and $m > 0$. Let $g(\tau) = f_{k,mp^s} - f_{k,mp^{s-1}}(p\tau)$ with $s > 0$. Then we have that*

$$g(\tau) \equiv F(\tau) \pmod{p^{\lambda/2}}$$

for some holomorphic modular form $F \in M_k(p)$, where F has integer coefficients and vanishes at ∞.

Proof. We immediately notice that $g(\tau) \in M_k^!(p)$ and vanishes at ∞. We are then interested in its behavior at the cusp 0 of $\Gamma_0(p)$.

We have $g(t) = f_{k,mp^s}(\tau) - f_{k,mp^{s-1}}(p\tau)$. Hence,

$$\begin{aligned}
\tau^{-k} g(-1/\tau) &= \tau^{-k} \left(f_{k,mp^s}(-1/\tau) - f_{k,mp^{s-1}}(-p/\tau)\right) \\
&= \tau^{-k} \left(\tau^k f_{k,mp^s}(\tau) - \left(\frac{\tau}{p}\right)^k f_{k,mp^{s-1}}(\tau/p)\right) \\
&= f_{k,mp^s}(\tau) - \frac{1}{p^k} f_{k,mp^{s-1}}(\tau/p).
\end{aligned}$$

Replacing τ by $p\tau$, we obtain

$$\begin{aligned}(p\tau)^{-k}g(-1/(p\tau)) &= f_{k,mp^s}(p\tau) - \frac{1}{p^k}f_{k,mp^{s-1}}(\tau).\\ &= q^{-mp^{s+1}} - \frac{1}{p^k}q^{mp^{s-1}} + O(q)\\ &= \sum_{i=1}^{mp^{s+1}} B_i\psi(\tau)^i E_k(\tau) + h(\tau)\end{aligned}$$

where $p^k B_i \in \mathbb{Z}$ for all i, and where $h(\tau)$ is modular on $\Gamma_0(p)$ and holomorphic at both 0 and ∞.

We now replace τ by $-1/p\tau$, to obtain

$$\begin{aligned}\tau^k g(\tau) &= \sum_{i=1}^{mp^{s+1}} B_i\psi(-1/p\tau)^i E_k(-1/p\tau) + h(-1/p\tau)\\ &= \sum_{i=1}^{mp^{s+1}} B_i(p^{\lambda/2}\Phi(\tau))^i(p\tau)^k E_k(p\tau) + h(-1/p\tau)\\ &= \sum_{i=1}^{mp^{s+1}} (p^k B_i)(p^{\lambda/2}\Phi(\tau))^i\tau^k E_k(p\tau) + h(-1/p\tau)\end{aligned}$$

Dividing by τ^k, writing $A_i = p^k B_i \in \mathbb{Z}$, and setting $F(\tau) = \tau^{-k}h(-1/p\tau)$, we see that

$$g(\tau) = p^{\lambda/2}\sum_{i=1}^{mp^{s+1}} A_i(p^{\lambda/2})^{i-1}\Phi(\tau)^i E_k(p\tau) + F(\tau).$$

We note that $F(\tau)$ has the following properties: it is modular of weight k on $\Gamma_0(p)$ and has integer coefficients (since both $g(\tau)$ and the sum have these properties), and it is holomorphic at both cusps of $\Gamma_0(p)$ (since $h(\tau)$ has this property). Further, since both $g(\tau)$ and the sum vanish at ∞, it is clear that $F(\tau)$ vanishes at ∞. □

Now, for a given $p \in \{2,3,5\}$, since $\lambda/2 > \epsilon_{k,p}$, we can study the divisibility of the coefficients of $g(\tau)$ by $p^{\epsilon_{k,p}}$ by studying the coefficients of $F(\tau)$. Since $F(\tau) \in M_k(p)$, we can apply Lemma 5.2 to it to obtain the following corollary.

Corollary 7.2. *Let $k \in \{4,6,8,10,14\}$, $p \in \{2,3,5\}$, $m,s > 0$. Let d be the dimension of $M_k(p)$ If*

$$g(\tau) = f_{k,mp^s}(\tau) - f_{k,mp^{s-1}}(p\tau) = \sum_{n=1}^{\infty}\gamma_n q^n,$$

and the γ_n with $1 \le n \le d-1$ are all divisible by $p^{\epsilon_{k,p}}$, then all of the γ_n are divisible by $p^{\epsilon_{k,p}}$. In particular, if $p \nmid n$, we have $a_k(mp^s,n) = \gamma_n$ and thus $v_p(a_k(mp^s,n) \ge \epsilon_{k,p}$.

Proof. By Lemma 7.1, we have that $g(\tau) \equiv F(\tau) \pmod{p^{\lambda/2}}$, for some $F(\tau) \in M_k(p)$ that vanishes at infinity. Write

$$F(\tau) = \sum_{n=1}^{\infty} d_n q^n.$$

Since $\lambda/2 > \epsilon_{k,p}$, the assumptions of the corollary show that $p^{\epsilon_{k,p}} | d_n$ for all $1 \leq n < d$. Lemma 5.2 then shows that $p^{\epsilon_{k,p}} | d_n$ for all n, and hence that $p^{\epsilon_{k,p}} | \gamma_n$ for all n, as desired. $\square$

Corollary 7.2 allows us to test divisibility of all the γ_n (for a given m) by testing finitely many of them. We now prove a theorem which uses duality to allow us to test divisibility of all the γ_n (for arbitrary $m > 0$) by studying a small number of modular forms.

Theorem 7.3. *Let $p \in \{2, 3, 5\}$, $k \in \{4, 6, 8, 10, 14\}$, and let d be the dimension of $M_k(p)$. If*

$$(f_{2-k,j} | U_p)(\tau) \equiv a_{2-k}(j, 0) \pmod{p^{\epsilon_{k,p}}}$$

for each j with $1 \leq j \leq d-1$ and $p \nmid j$, and

$$(f_{2-k,j} | U_p)(\tau) - f_{2-k,j/p}(\tau) \equiv a_{2-k}(j, 0) - a_{2-k}(j/p, 0) \pmod{p^{\epsilon_{k,p}}}$$

for each j with $1 \leq j \leq d-1$ and $p | j$, then $p^{\epsilon_{k,p}} | a_k(mp^s, n)$ for all $s > 0$ and $m, n > 0$ prime to p.

Proof. By Corollary 7.2, we need to show that for any $m, s > 0$, $p^{\epsilon_{k,p}}$ divides each $\gamma_n = a_k(mp^s, n)$ for $1 \leq n \leq d-1$ with $p \nmid n$ and each $\gamma_n = a_k(mp^s, n) - a_k(mp^{s-1}, n/p)$ for $1 \leq n \leq d-1$ with $p | n$. Using duality (Corollary 1.2), we see that this is equivalent to checking that for $1 \leq n < d$, $p^{\epsilon_{k,p}}$ divides each $a_{2-k}(n, mp^s)$ when $p \nmid n$ and each $a_{2-k}(n, mp^s) - a_{2-k}(n/p, mp^{s-1})$ when $p | n$. As m and s run through all positive integers, the $a_{2-k}(n, mp^s)$ are just the nonconstant coefficients of $f_{2-k,n} | U_p$, and the $a_{2-k}(n, mp^s) - a_{2-k}(n/p, mp^{s-1})$ are the nonconstant coefficients of $(f_{2-k,n} | U_p) - f_{2-k,n/p}$. $\square$

To conclude the proof of Theorem 1.3, we must verify that the modular forms in Theorem 7.3 satisfy the congruences described there. In the next section, we describe how to prove these congruences by computing finitely many coefficients of each form, and we summarize the results of these computations.

8. Conclusion

In examining the case $p = 2$ (as well as some weights for $p = 3$ or $p = 5$), we use the following lemma to reduce the congruences we desire to the computation of a single coefficient of a modular form. Since the modular forms in question are easily computable, Theorem 1.3 follows.

Lemma 8.1. *Let $p \in \{2, 3, 5\}$ and $k \in \{4, 6, 8, 10, 14\}$. Let $f(\tau) \in M^!_{2-k}(p)$ be holomorphic at ∞. Suppose that $f(\tau)$ and $p^{k-2}\left(p(p\tau)^{k-2} f(-1/p\tau)\right)$ have integer Fourier coefficients, and that $v_p(\mu_{2-k,p}) + 1 - k + \lambda_p/2 \geq \epsilon_{k,2}$. Let K be the constant Fourier coefficient of $f(\tau)$. If $v_p(K) \geq \epsilon_{k,p} - \xi_{2-k,p}$, then $f(\tau) \equiv K \pmod{p^{\epsilon_{k,p}}}$.*

Proof. The assumptions of the lemma indicate that $(p\tau)^{k-2} f(-1/p\tau)$ has its only pole at ∞; hence, it may be written as a linear combination

$$(p\tau)^{k-2} f(-1/p\tau) = p^{1-k} \sum_{i=0}^{\infty} A_i \psi^i(\tau) \theta_{2-k}(\tau),$$

where the $A_i \in \mathbb{Z}$. Replacing τ by $-1/p\tau$, and dividing by τ^{2-k}, we find that

$$f(\tau) = \sum_{i=0}^{\infty} A_i \mu_{2-k,p} p^{1-k+i\lambda_p/2} \Phi^i(\tau) \alpha_{2-k}(\tau).$$

Since $v_p(\mu_{2-k,p}) + 1 - k + \lambda_p/2 \geq \epsilon_{k,p}$ we see that for $i > 0$, the coefficient of $\Phi^i \alpha_{2-k}$ is divisible by $p^{\epsilon_{k,p}}$. Further, comparing constant coefficients, we see that $K = A_0 \mu_{2-k,p} p^{1-k}$. Hence, we have that

$$f(\tau) \equiv K\alpha_k(\tau) \pmod{p^{\epsilon_{k,p}}}.$$

Under the assumption that $v_p(K) \geq \epsilon_{k,p} - \xi_{2-k,p}$, and using that $\alpha_{2-k} \equiv 1 \pmod{p^{\xi_{2-k,p}}}$, we see that

$$f(\tau) \equiv K \pmod{p^{\epsilon_{k,p}}},$$

as desired. □

We note that all of the modular forms considered in Theorem 7.3 satisfy the holomorphicity condition of this theorem, and Corollary 4.2 guarantees that they satisfy the integrality conditions. The inequality $v_p(\mu_{2-k,p}) + 1 - k + \lambda_p/2 \geq \epsilon_{k,p}$ holds for $p = 2$ (all weights), for $p = 3$ in weights $k = 4, 6, 10$, and for $p = 5$ in weight $k = 4$. Hence, by calculating a single coefficient of each form, we prove the desired divisibility properties for these cases. In the cases where the theorem applies, we have calculated the necessary constant coefficients and verified Theorem 1.3.

As an example, for $p = 2$, $k = 14$, we compute the constant coefficients of $f_{-12,1}|U_2$, $f_{-12,2}|U_2 - f_{-12,1}$, and $f_{-12,3}|U_2$ to be, respectively, 24, 196608, and 38263776. Since each of these is divisible by 2^3 and $\xi_{-12,2} = 4$, we see that each of the modular forms is congruent to a constant modulo 2^7, proving Theorem 1.3 for $p = 2$ in weight 14.

We remark that this lemma does not immediately work in general for $p = 3$ and $p = 5$ because we do not always have the inequality $v_p(\mu_{2-k,p}) + 1 - k + \lambda_p/2 \geq \epsilon_{k,p}$. One could prove a similar theorem, involving checking the divisibility of additional coefficients, but we find it simpler to perform explicit computations, as described below.

8.1. **Computational proof of the remaining cases.** For $p = 3$, $k = 8, 14$ and for $p = 5$, $k > 4$ we now describe the calculations needed to prove the congruences in Theorem 1.3 and give a single illustrative example.

Let f be one of the modular forms of weight $2 - k$ described in Theorem 7.3. We wish to show that f satisfies the congruence stated in Theorem 7.3. We begin by using Lemma 4.1 to compute the Fourier expansion of $p(p\tau)^{k-2} f(-1/p\tau)$ to high precision (say to $O(q^{500})$). This allows us to easily determine a linear combination

$$p(p\tau)^{k-2} f(-1/p\tau) = \sum_{i=0}^{N} A_i p^{2-k} \psi^i(\tau) \theta_{2-k}(\tau),$$

with $A_i \in \mathbb{Z}$. Replacing τ by $-1/p\tau$ and dividing by $p\tau^{2-k}$, we obtain a linear combination

$$f(\tau) = \sum_{i=0}^{N} A_i \mu_{2-k,p} p^{\lambda i/2+1-k} \Phi^i(\tau) \alpha_k(\tau) = \sum_{i=0}^{N} B_i \Phi^i(\tau) \alpha_k(\tau).$$

In each case, we find that for $i > 0$, $v_p(B_i) \geq \epsilon_{k,p}$, so that

$$f(\tau) \equiv B_0 \alpha_k(\tau) \pmod{p^{\epsilon_{k,p}}}.$$

We then check that $v_p(B_0) \geq \epsilon_{k,p} - \xi_{2-k,p}$, thereby proving that

$$f(\tau) \equiv B_0 \pmod{p^{\epsilon_{k,p}}},$$

as desired.

As an example (the simplest) for $p = 3$, $k = 8$, we must show that $f_{-6,1}|U_3$ is congruent to a constant modulo 3^3. Following the procedure above, we find that

$$(f_{-6,1}|U_3)(\tau) = \sum_{i=0}^{7} B_i \Phi^i(\tau)\alpha_{-6}(\tau)$$

with

$$B_0 = -480, \quad B_1 = -10451430, \quad B_2 = -8628476076,$$
$$B_3 = -1922380466418, \quad B_4 = -177993370102248, \quad B_5 = -7892493396961545,$$
$$B_6 = -166771816996665690, \quad B_7 = -1350851717672992089.$$

One checks easily that each B_i with $i > 0$ is divisible by $3^{\epsilon_{8,3}} = 3^3$ (indeed, even by 3^5) and that $3|B_0$. Since $\xi_{-6,3} = 2$, we see that $f_{-6,1}|U_3 \equiv -480 \pmod{3^3}$.

Note that there is nothing difficult about the computations; they are nothing more than basic arithmetic with q-series. They are somewhat daunting to write down; for instance, if $p = 5$ and $k = 14$, proving that $f_{-12,6}$ is congruent to a constant modulo 5 involves working with a linear combination of 144 terms of the form $\Phi^i\alpha_{-12,5}$, many of which have coefficients over 290 digits long. All of the computations were done using GP/PARI [16], and the scripts used are available from the authors upon request.

These computations have been performed for $p = 3$, $k = 8, 10, 14$ and $p = 5$, $k = 4, 6, 8, 10, 14$, and complete the proof that Theorem 1.3 is true.

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