SUMS OF TWO IRREDUCIBLE GALOIS REPRESENTATIONS AND THE HOMOLOGY OF $\mathrm{GL}_n(\mathbb{Z})$

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Abstract. Given two irreducible Galois representations with relatively prime conductors, each attached to a Hecke eigenclass in an appropriate homology group, we prove that their direct sum is also attached to a Hecke eigenclass in a homology group, and that if the two Galois representations have weight, level, and nebentype predicted by a Serre-type conjecture of the authors and David Pollack, then so does the direct sum. Our methods utilize a study of $\Gamma_0(n, N)$-orbits of flags of subspaces of $\mathbb{Q}^n$, reducibility results for Galois representations attached to cohomology of parabolic subgroups of GL_n, and a spectral sequence derived from the Tits building. In addition, we use the spectral sequence to prove two results about the degrees of homology to which irreducible Galois representations can be attached.

1. Introduction

The Serre-type conjecture of [6], refined in [13], asserts that an odd Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is attached to a Hecke eigenclass in the cohomology of a congruence subgroup Γ of $\mathrm{SL}_n(\mathbb{Z})$ (see section 2 for terminology and definitions). Using the duality between homology and cohomology, this conjecture can be stated in terms of group homology. The proof of the case when $n > 2$ and ρ is irreducible seems to be well beyond any current techniques.

In [4, 5], we prove the conjecture in its homological form (given certain conditions) for Galois representations $\rho = \rho_1 \oplus \rho_2$, where ρ_1 and ρ_2 are irreducible n-dimensional Galois representations with squarefree conductor, and assuming that the conjecture holds for ρ_1 and ρ_2.

In this paper, we extend the results of [4, 5] to apply to $\rho = \rho_1 \oplus \rho_2$, where the dimensions of ρ_1 and ρ_2 may be different. For reasons which will be apparent, we slightly enlarge the group with respect to which we take homology from the subgroup $\Gamma_0(n, N)$ of $\mathrm{SL}_n(\mathbb{Z})$ to the subgroup $\Gamma_0^{\pm}(n, N)$ of $\mathrm{GL}_n(\mathbb{Z})$. The latter group contains matrices of determinant -1 and contains $\Gamma_0(n, N)$ with index 2.

Our main theorem (Theorem 12.1) says that for p sufficiently large, given two odd irreducible Galois representations ρ_1 and ρ_2 of dimensions n_1 and n_2 such that ρ_1 and $\omega^{-n_1}\rho_2$ are attached to Hecke eigenclasses in the homology of $\Gamma_0^{\pm}(n_i, N_i)$ for two relatively prime squarefree N_i and some coefficient modules M_i, then $\rho_1 \oplus \rho_2$ is attached to a Hecke eigenclass in the homology

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of $\Gamma_0^{\pm}(n_1+n_2, N_1N_2)$ with some coefficient module M. For the exact relationship between M_1 M_2 and M, see the full statement of Theorem 12.1. If the levels N_i and the coefficient modules M_i are those predicted by the main conjecture of [6] for ρ_1 and $\omega^{-n_1}\rho_2$, then the level N_1N_2 and the coefficient module M are as predicted in [6] for $\rho_1 \oplus \rho_2$.

With our current techniques, we are unable to handle Galois representations ρ which are sums of three or more irreducible constituents. The reason for this comes from the spectral sequence we use to prove the main theorem. When ρ is the sum of two or more irreducible constituents, each of which is attached to a homology class, we can show that ρ shows up in a certain term of the first page of the spectral sequence. If ρ survives to the infinity page, it is attached to homology in a certain degree. There are two ways that it can fail to survive, however. First, in the passage to the second page, it can map to a nonzero element. In this case, when ρ is a sum of two irreducible constituents we are able to show that ρ is attached to a homology class in a different degree. Second, as we move from the first page to the infinity page, it can be in the image of some other term of the spectral sequence. Because of the construction of the spectral sequence, this cannot happen when ρ is a sum of two irreducible constituents, since all eigenclasses that could map to it must have attached Galois representations that are sums of three or more irreducible constituents. If ρ were to have three or more irreducible constituents, there would be other ways for a class attached to ρ in the first page of the spectral sequence to fail to survive to the infinity page, and we are unable to account for this.

A complete understanding of how reducible mod p Galois representations are attached to the homology of a congruence subgroup Γ would require a thorough understanding of the mod p homology of the Borel-Serre boundary of the locally symmetric space X for Γ and its relations to the mod p homology of X. Such an understanding is well beyond the limits of any current knowledge of the subject. In our work we use spectral sequences of various kinds to get partial information which suffices for the theorems we can prove. At one point of the argument we make crucial use of Scholze's results [15] that imply that a mod p Hecke eigenclass in the homology of Γ always has a mod p Galois representation attached to it.

To prove the desired theorem, we utilize three new results of independent interest. The first is that a Galois representation attached to the homology of a parabolic subgroup of type $(m_1, \dots, m_k)$ has at least k irreducible components. Although this may seem "obvious," the only proof we found is rather intricate. A large part of this paper is devoted to this proof. We need this result to show that certain differentials in a spectral sequence vanish.

Since this takes up so much space in our paper, it is worth explaining roughly what makes it hard to prove. Let Q be a parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$ with unipotent radical U. Suppose we have a Hecke eigenclass in the homology of a $\Gamma_Q = \Gamma \cap Q$ with coefficients in an admissible module M, with attached Galois representation σ. We first need to show that σ is attached to a class z in the Levi quotient Γ_L of Γ_Q. (Γ_L is isomorphic to a direct product of arithmetic subgroups Δ_i for GL_m's with $m < n$.)

To do this we have to analyze the Hecke action on the Lyndon-Hochschild-Serre spectral sequence attached to the exact sequence whose quotient is Γ_L. Then we have to show that the homology of $\Gamma \cap U$ with coefficients in M is an admissible Γ_L-module, in particular that Γ_L acts on it via reduction modulo p. This is not at all obvious. After we have z, we have to use a Hecke-equivariant version of the Künneth formula to show that σ is the direct sum of constituents each of which is attached to the homology of Δ_i. To do this, we have to reduce to the case where the coefficient module M is one-dimensional. (This is similar to what Scholze has to do in his paper [15].) In our context this is quite arduous.

The second new tool is a new spectral sequence which is easier to work with than the spectral sequence used in [4, 5]. In addition to our main theorem, the new spectral sequence also yields two interesting results on the degrees of homology to which irreducible Galois representations can be attached; Theorem 10.2 states that an irreducible degree n Galois representation with squarefree conductor cannot be attached to homology in degree below $n-1$, and Theorem 10.3 proves that an irreducible n-dimensional Galois representation with squarefree conductor that is attached to homology in degree k must also be attached to homology in degree $(n+2)(n-1)/2-k$.

The third new tool in our proofs is the main result of [2], which says that certain classes in the E_2 page of a Lyndon-Hochschild-Serre spectral sequence related to our Galois representations must persist to the infinity page. The main theorems of [4, 5] proved this persistence in special cases, but knowing this persistence more generally allows us to prove the much better Theorem 12.1 of this paper.

The outline of the paper is as follows. In section 2 we define some basic terminology. In section 3 we describe the $\Gamma_0^{\pm}(n,N)$-orbits of flags of subspaces in $\mathbb{Q}^n$ and define canonical representative parabolic subgroups. In our earlier work we did this for flags whose stabilizers are maximal parabolic subgroups. Now we need this for all parabolic subgroups.

In section 4 we define a Hecke pair which acts as a parabolic version of the principal congruence Hecke pair. In section 5 we compare this Hecke pair to others, and show that they are compatible. Sections 6 and 7 generalize results of [4] to $\Gamma_0^{\pm}(n,N)$, leading up to section 8, in which we finish the proof of the first new result mentioned above. Note that the results of Section 6 are necessary in order to use Scholze's theorem that attaches Galois representations to homology with admissible representations as coefficients in the proof of Theorem 12.1.

In section 9 we describe the new spectral sequence that we use to prove our main result. In section 10 we prove some theorems concerning irreducible Galois representations and homology classes that follow easily from the new spectral sequence. In section 11 we state the main result of [2], which is used in the proof of our main theorem. In section 12 we prove the main theorem and give some examples and consequences of it.

As a sample here, the main theorem implies the following new result (see Example 12.5):

Theorem 1.1. *Let $p > 16$ be prime, and let*

$$\rho_1 : G_{\mathbb{Q}} \to \mathrm{GL}_5(\mathbb{F}_p) \quad \textit{and} \quad \rho_2 : G_{\mathbb{Q}} \to \mathrm{GL}_3(\mathbb{F}_p)$$

be odd irreducible Galois representations such that $\rho_1 \oplus \rho_2$ is odd. Assume that

(1) ρ_1 has level N_1, nebentype ϵ_1, and one of its predicted weights is $M_1 = F(a_1 + 3, \dots, a_5 + 3)$, and ρ_2 has level N_2, nebentype ϵ_2, and one of its predicted weights is $M_2 = F(a_6, a_7, a_8)$;

(2) $(a_1, \dots, a_8)$ is chosen so that $0 \le a_5 - a_6 \le p - 1$ and N_1N_2 is squarefree;

(3) ρ_1 is attached to a Hecke eigenclass in

$$H_7(\Gamma_0(5, N_1), (M_1)_{\epsilon_1})$$

and ρ_2 is attached to a Hecke eigenclass in

$$H_2(\Gamma_0(3, N_2), (M_2)_{\epsilon_2}).$$

Let $M = F(a_1, \dots, a_n)$ and $\epsilon = \epsilon_1\epsilon_2$. Then $\rho_1 \oplus \rho_2$ is attached to a Hecke eigenclass in either

$$H_{24}(\Gamma_0(8, N_1N_2), M_\epsilon)$$

or

$$H_{10}(\Gamma_0(8, N_1N_2), M_\epsilon)$$

or both. Moreover, N_1N_2, ϵ, and M are the level, nebentype, and a predicted weight for $\rho_1 \oplus \rho_2$.

All modules in this paper will be right modules unless otherwise noted. Throughout this paper we fix a rational prime number p. We write $\mathbb{F}$ for a fixed algebraic closure of $\mathbb{F}_p$.

2. Galois representations and homology

A *Galois representation* is a continuous homomorphism $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{F})$. We say that a Galois representation ρ is odd, if the image of complex conjugation under ρ is conjugate to an upper triangular matrix with alternating ones and minus ones on the diagonal.

Definition 2.1. Let $n > 1$, and let N be a positive integer prime to p.

(1) $S_0^{\pm}(n, N)$ consists of the set of all $n \times n$ matrices with integer entries and nonzero determinant prime to pN whose first row is congruent to $(*, 0, \dots, 0)$ modulo N.
(2) $S_0(n, N)$ consists of elements of $S_0^{\pm}(n, N)$ with positive determinant.
(3) $\Gamma_0^{\pm}(n, N) = S_0^{\pm}(n, N) \cap \mathrm{GL}(n, \mathbb{Z})$.
(4) $\Gamma_0(n, N) = S_0(n, N) \cap \mathrm{SL}(n, \mathbb{Z})$.

Let (Γ, S) be equal to either $(\Gamma_0(n, N), S_0(n, N))$ or $(\Gamma_0^{\pm}(n, N), S_0^{\pm}(n, N))$. Then (Γ, S) is a Hecke pair (see [1]), and we denote its $\mathbb{F}_p$-Hecke algebra by $\mathcal{H}(\Gamma, S)$. In each case, the Hecke algebra is commutative, and contains the double cosets of matrices

$$s(\ell, n, k) = \mathrm{diag}(\underbrace{1, \dots, 1}_{n-k}, \underbrace{\ell, \dots, \ell}_{k}),$$

for $0 \le k \le n$ as ℓ runs through all primes not dividing pN.

The algebra $\mathcal{H}(\Gamma, S)$ acts on the homology and cohomology of Γ with coefficients in any $\mathbb{F}_p[S]$-module M. When the double coset of $s(\ell, n, k)$ acts on homology or cohomology we call it a Hecke operator and denote it by $T_n(\ell, k)$.

Definition 2.2. Let V be any $\mathcal{H}(\Gamma, S)$-module, and suppose that $v \in V$ is a simultaneous eigenvector of all the $T_n(\ell, k)$ for $\ell \nmid pN$, with eigenvalues $a(\ell, k) \in \mathbb{F}$. Suppose that $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n, \mathbb{F})$ is a Galois representation unramified outside pN. We say that ρ is *attached* to v if, for all $\ell \nmid pN$,

$$\det(I - \rho(\mathrm{Fr}_\ell)X) = \sum_{k=0}^{n} (-1)^k \ell^{k(k-1)/2} a(\ell, k) X^k.$$

The *level* N of a Galois representation ρ is the prime to p part of the Artin conductor of ρ. We denote the cyclotomic character modulo p by ω, and note that we can factor the determinant of ρ as $\det \rho = \omega^k \epsilon$, where ϵ is a character modulo N. We call ϵ the *nebentype* of ρ.

We parameterize the irreducible $\mathrm{GL}_n(\mathbb{F}_p)$-modules as in [12].

Definition 2.3. We say that an n-tuple of integers $(a_1, \ldots, a_n)$ is *p-restricted* if

$$0 \le a_i - a_{i+1} \le p - 1, \qquad \text{for } 1 \le i \le n - 1,$$

and

$$0 \le a_n < p - 1.$$

Proposition 2.4. [12] *The set of isomorphism classes of irreducible* $\mathrm{GL}_n(\mathbb{F}_p)$*-modules is in one-to-one correspondence with the collection of all* p*-restricted* n*-tuples.*

We denote the irreducible module corresponding to the n-tuple $(a_1, \ldots, a_n)$ by $F(a_1, \ldots, a_n)$.

Given an odd Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{F})$, [6] describes a collection of irreducible $\mathrm{GL}_n(\mathbb{F}_p)$-modules (which are called *weights*) related to ρ, and predicts that if N is the level of ρ, ϵ the nebentype, and M any one of these predicted weights, then ρ is attached to a Hecke eigenclass in

$$H_*(\Gamma_0(n, N), M_\epsilon),$$

where $M_\epsilon = M \otimes \mathbb{F}_\epsilon$, and $\mathbb{F}_\epsilon$ denotes the one-dimensional space on which $s \in S_0(n, N)$ acts via multiplication by $\epsilon(s_{11})$, where s_{11} denotes the $(1, 1)$ entry of s.

Definition 2.5. If a level N, a nebentype ϵ or a weight M are predicted by Conjecture 3.1 of [6] for a Galois representation ρ, we call them a *predicted level*, *predicted nebentype* and *predicted weight* for ρ.

We use this definition also when ρ is attached to a homology class for $\Gamma_0^{\pm}(n, N)$ rather than $\Gamma_0(n, N)$.

3. Representative flags of parabolic subgroups

In this section, we determine representatives for the $\Gamma_0(n, N)$-orbits of flags of subspaces of the vector space of row vectors $\mathbb{Q}^n$. These representatives will be used in defining our spectral sequence in section 9.

The next two theorems (Theorems 3.1 and 3.3) are proved in [4].

Theorem 3.1. [4, Theorem 5.1] *Let $0 < k < n$ and let N be a positive squarefree integer. Then the $\Gamma_0(n, N)$-orbits of k-dimensional subspaces of $\mathbb{Q}^n$ are in one-to-one correspondence with the set of positive divisors of N, where the orbit corresponding to the divisor d contains the k-dimensional subspace spanned by*

$$e_1 + de_{k+1}, e_2, e_3, e_4, \ldots, e_k,$$

where e_i denotes the standard basis element of $\mathbb{Q}^n$ with a 1 in the ith column, and 0's elsewhere.

Right multiplication by elements of $S_0^{\pm}(n, N)$ preserves the $\Gamma_0(n, N)$-orbits.

Note that the last sentence in this theorem implies that the $\Gamma_0(n, N)$-orbits are equal to the $\Gamma_0^{\pm}(n, N)$-orbits.

We will let W_d^k be the representative subspace of $\mathbb{Q}^n$ described in the theorem. We define P_d^k to be the maximal parabolic subgroup of $\mathrm{GL}(n, \mathbb{Q})$ that is the stabilizer of W_d^k. Typically, we will omit the k if it is understood from context. If we define the matrix g_d to be the $n \times n$ identity matrix with the $(1, k+1)$-entry replaced by d, $P_0 = g_d P_d g_d^{-1}$ is the stabilizer of the subspace $\mathrm{Span}(e_1, \ldots, e_k)$.

The subgroup P_0 consists of block matrices

$$g = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

where A is an invertible $k \times k$ matrix, C is an invertible $(n-k) \times (n-k)$ matrix, B is an arbitrary $(n-k) \times k$ matrix, and 0 represents a block of zeros. We now define two homomorphisms from P_d to smaller matrix groups.

Definition 3.2. For a block matrix

$$g = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \in P_0,$$

we define $\psi_0^1(M) = A$ and $\psi_0^2(M) = B$. For a matrix $s \in P_d$, $g_d s g_d^{-1} \in P_0$ and for $i = 1, 2$, we define

$$\psi_d^i(s) = \psi_0^i(g_d s g_d^{-1}).$$

Clearly $\psi_0^1 : P_0 \to \mathrm{GL}(k, \mathbb{Q})$, $\psi_0^2 : P_0 \to \mathrm{GL}(n-k, \mathbb{Q})$, $\psi_d^1 : P_d \to \mathrm{GL}(k, \mathbb{Q})$ and $\psi_d^2 : P_d \to \mathrm{GL}(n-k, \mathbb{Q})$ are surjective group homomorphisms.

We have the following facts about the maps ψ_d^i.

Theorem 3.3. [4, Theorem 5.2] *Let d be a positive divisor of N and assume that $(d, N/d) = 1$.*

(1) If $s \in P_d \cap S_0^{\pm}(n, N)$, then $\psi_d^1(s)_{11} \equiv s_{11} \pmod{d}$ and $\psi_d^2(s)_{11} \equiv s_{11} \pmod{N/d}$.

(2) $\psi_d^1(P_d \cap S_0^{\pm}(n,N)) \subset S_0^{\pm}(k,d)$.
(3) $\psi_d^2(P_d \cap S_0^{\pm}(n,N)) \subset S_0^{\pm}(n-k,N/d)$.
(4) There is an exact sequence of groups

$$1 \to U_d \cap \Gamma_0^{\pm}(n,N) \to P_d \cap \Gamma_0^{\pm}(n,N) \xrightarrow{\psi_d^1 \times \psi_d^2} \Gamma_0^{\pm}(k,d) \times \Gamma_0^{\pm}(n-k,N/d) \to 1.$$

We now wish to extend our investigation to $\Gamma_0(n,N)$-orbits of arbitrary (not necessarily maximal) parabolic subgroups. We make the following definitions.

Definition 3.4. A *flag* in $\mathbb{Q}^n$ is a sequence

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots V_{t-1} \subsetneq V_t = \mathbb{Q}^n,$$

where each V_i is a subspace of $\mathbb{Q}^n$. We say that this flag has *length* t, and define a *basis for the flag* to be an ordered basis of $\mathbb{Q}^n$ such that for each i, the first $\dim(V_i)$ vectors in the basis are a basis for V_i. The stabilizer of this flag is a parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$, also said to be of length t.

Thus a standard parabolic subgroup of length t has t blocks down the diagonal, and a maximal parabolic subgroup has length 2. To facilitate inductive proofs of some results, we will consider the full group $\mathrm{GL}_n(\mathbb{Q})$ to be a parabolic subgroup Q stabilizing the trivial flag $0 = V_0 \subsetneq V_1 = \mathbb{Q}^n$ of length 1, which is consistent with the terminology in [9].

Note that multiplication of a flag of subspaces on the right by an element of $\mathrm{GL}_n(\mathbb{Q})$ yields another flag of subspaces with the same dimensions. Hence, given a collection $0 = k_0 < k_1 < \cdots < k_t = n$ of dimensions, we see that $\mathrm{GL}_n(\mathbb{Q})$ acts on the right on the collection of all flags $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_t = \mathbb{Q}^n$ in which each V_i has dimension k_i.

Theorem 3.5. *Let N be a squarefree positive integer, and let $n \geq 1$. Given a list of integers $0 = k_0 < k_1 < \cdots < k_{t-1} < k_t = n$, there is a one-to-one correspondence between the set of $\Gamma_0(n,N)$-orbits of flags of subspaces*

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{t-1} \subsetneq V_t = \mathbb{Q}^n,$$

in which each V_i has dimension k_i and the set of ordered factorizations $N = d_1 d_2 \cdots d_t$.

The orbit corresponding to the ordered factorization $N = d_1 d_2 \cdots d_t$ contains the flag with basis

$$\begin{gathered} e_1 + d_1 e_{k_1+1}, e_2, \ldots, e_{k_1}, \\ e_{k_1+1} + d_2 e_{k_2+1}, \ldots, e_{k_2}, \\ \vdots \\ e_{k_{t-2}+1} + d_{t-1} e_{k_{t-1}+1}, \cdots, e_{k_{t-1}}, \\ e_{k_{t-1}+1}, \cdots, e_{k_t}, \end{gathered}$$

where each V_i is the span of the first k_i vectors in this basis.

Remark 3.6. In an ordered factorization $N = d_1 \ldots d_t$, any or all of the d_i may be equal to 1. Sometimes we will drop the adjective "ordered".

Proof. We begin by proving that each orbit contains a flag of the desired form. We will then establish that distinct flags of the desired form are not equivalent.

We proceed by induction on the length t of the flag. For $t = 2$, the result follows from Theorem 3.1. Assume, then, that $t > 2$ and that for any N, any flag of length less than t is $\Gamma_0(n, N)$-equivalent to a flag of the desired form.

Let

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{t-1} \subsetneq V_t = \mathbb{Q}^n$$

be a flag of length t, in which each V_i has dimension k_i. For $1 \le i \le t$, set $\ell_i = k_i - k_{i-1}$. When we talk about a block diagonal/lower triangular/upper triangular matrix, the diagonal blocks will have sizes $\ell_1, \dots, \ell_t$. Set Z_d to be a matrix (whose size will be determined by context) whose entries are all zero, except for a d in the $(1, 1)$ position. Note that with this notation, the matrix whose rows are the basis described above is a block upper triangular matrix of the form

$$W = \left(\begin{array}{|c|c|c|c|c|}
\hline
I_{\ell_1} & \multicolumn{4}{c|}{Z_{d_1}} \\
\hline
0 & I_{\ell_2} & \multicolumn{3}{c|}{Z_{d_2}} \\
\cline{2-5}
0 & 0 & \ddots & \multicolumn{2}{c|}{\vdots} \\
\cline{3-5}
0 & 0 & 0 & I_{\ell_{t-1}} & Z_{d_{t-1}} \\
\cline{4-5}
0 & 0 & 0 & 0 & I_{\ell_t} \\
\cline{5-5}
\end{array}\right).$$

Let M be an $n \times n$ matrix with the property that for each i, the first k_i rows of M are a basis for V_i. Multiplying M on the left by a block lower triangular matrix Q will give another matrix QM whose first k_i rows are a basis for V_i. Hence, our goal is to find such a Q and a $\gamma \in \Gamma_0(n, N)$ so that $QM\gamma = W$, for some factorization $d_1 \cdots d_t = N$.

By Theorem 3.1, we can find a block diagonal $Q_1 = Q_1' \oplus I_{n-k_1}$ with $Q_1' \in \mathrm{GL}_{k_1}(\mathbb{Q})$ and a $\gamma_1 \in \Gamma_0(n, N)$ so that the first k_i rows of $Q_1 M \gamma_1$ are the first k_i basis elements listed above (for some choice of d_1). Then there is a block lower triangular R_1 of the form

$$R_1 = \begin{pmatrix} I_{k_1} & 0 \\ * & I_{n-k_1} \end{pmatrix},$$

such that $R_1 Q_1 M \gamma_1$ has the form

$$\begin{pmatrix} I_{k_1} & Z_{d_1} \\ 0 & M_1 \end{pmatrix}.$$

Now the rows of M_1 give a basis for the flag

$$0 \subsetneq V_2\gamma_1/V_1\gamma_1 \subseteq V_{t-1}\gamma_1/V_1\gamma_1 \subseteq V_t\gamma_1/V_1\gamma_1 \cong \mathbb{Q}^{n-k_1},$$

so by the induction hypothesis, we can find a block lower triangular $Q_2 \in \mathrm{GL}_{n-k_1}(\mathbb{Q})$ and a $\gamma_2 \in \Gamma_0(n - k_1, N/d_1)$ so that (for some factorization

$d_2 \cdots d_t = N/d_1$)

$$Q_2 M_1 \gamma_2 = \left(\begin{array}{cccc} I_{\ell_2} & & Z_{d_2} & \\ 0 & \ddots & & \vdots \\ 0 & 0 & I_{\ell_{t-1}} & Z_{d_{t-1}} \\ 0 & 0 & 0 & I_{\ell_t} \end{array} \right).$$

Now we see that

$$(I_{k_1} \oplus Q_2) R_1 Q_1\, M\, \gamma_1 (I_{k_1} \oplus \gamma_2) = \left(\begin{array}{ccccc} I_{\ell_1} & & & P & \\ 0 & I_{\ell_2} & & Z_{d_2} & \\ 0 & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & I_{\ell_{t-1}} & Z_{d_{t-1}} \\ 0 & 0 & 0 & 0 & I_{\ell_t} \end{array} \right),$$

where the entries of the block P are all zeros, except for the first row, which is d_1 times the first row of γ_2.

Since $\gamma_2 \in \Gamma_0(n, N/d_1)$, we find that the entries of the first row of P (except for the first entry) are all multiples of N. The first entry of the first row of P is of the form αd_1, where α is relatively prime to N/d_1.

Let $\gamma_3 \in \Gamma_0(n, N)$ be the matrix that performs column operations using the pivot 1 in the $(1,1)$ entry of I_{ℓ_1} to zero out all but the first entry of the first row of P.

Finally, let $S = \begin{pmatrix} s_{11} & s_{22} \\ s_{21} & s_{22} \end{pmatrix} \in \Gamma_0(2, N)$ be the matrix constructed in the proof of [4, Theorem 5.1] for $a = 1$ and $b = \alpha d_1$. It has the property that

$$\begin{pmatrix} 1 & \alpha d_1 \\ 0 & 1 \end{pmatrix} S = \begin{pmatrix} 1 & d_1 \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} 1 & d_1 \\ s_{21} & 1 + d_1 s_{21} \end{pmatrix}.$$

The last equality holds because $\det S = 1$.

Let $\gamma_4 \in \Gamma_0(n, N)$ be the identity matrix with the $(1,1)$, $(1, k_1+1)$, $(k_1+1, 1)$, and (k_1+1, k_1+1) entries replaced by s_{11}, s_{12}, s_{21}, and s_{22}, respectively. Finally, let R_2 be the matrix for the row operation that subtracts s_{21} times row 1 from row $k_1 + 1$.

Setting $Q = R_2(I_{k_1} \oplus Q_2) R_1 Q_1$ and $\gamma = \gamma_1 (I_{k_1} \oplus \gamma_2) \gamma_3 \gamma_4$, we see that $QM\gamma$ has the desired form.

To prove uniqueness, we note that we have proven that V_i is $\Gamma_0(n, N)$-equivalent to the k_i-dimensional space spanned by the first k_i basis elements of the desired form. Using the methods of Theorem 3.1 one can check that the space spanned by these basis elements is $\Gamma_0(n, N)$-equivalent to the representative subspace of dimension k_i corresponding to the divisor $d_1 \ldots d_i$. Thus, the $\Gamma_0(n, N)$-orbit of each V_i uniquely determines the divisor $d_1 \ldots d_i$ of N, so $\Gamma_0(n, N)$-orbit of the flag uniquely determines the ordered factorization $d_1 \ldots d_t = N$. $\square$

Corollary 3.7. *Right multiplication by elements of $S_0^{\pm}(n, N)$ preserves the $\Gamma_0(n, N)$-orbits.*

Proof. Let

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots V_{t-1} \subsetneq V_t = \mathbb{Q}^n,$$

be a flag whose orbit corresponds to the factorization $d_1 \dots d_t = N$, and let $s \in S_0^{\pm}(n, N)$. Then each subspace $V_i s$ is $\Gamma_0(n, N)$-equivalent to V_i by Theorem 3.1, and hence corresponds to the same divisor of N. Hence, the flag

$$0 = V_0 s \subsetneq V_1 s \subsetneq V_2 s \subsetneq \cdots V_{t-1} s \subsetneq V_t s = \mathbb{Q}^n,$$

corresponds to the same factorization of N, and is in the same orbit. □

Definition 3.8. We call a flag

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{t-1} \subsetneq V_t = \mathbb{Q}^n$$

with a basis of the form described in Theorem 3.5 a *representative flag.*

Definition 3.9. For a representative flag

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{t-1} \subsetneq V_t = \mathbb{Q}^n,$$

let $k_i = \dim(V_i)$ for each i, and let $d_1 d_2 \cdots d_t = N$ be the corresponding factorization of N. Write $D = (d_1, d_2, \dots, d_t)$. Define the matrix $g_D \in \mathrm{GL}_n(\mathbb{Z})$ by

$$g_D = \hat{g}_{d_{t-1}} \cdots \hat{g}_{d_2} \hat{g}_{d_1},$$

where $\hat{g}_{d_i}$ is the $n \times n$ matrix which is the identity matrix except with d_i in the $(1 + k_{i-1}, 1 + k_i)$-place.

Theorem 3.10. *With notation as in Definition 3.9, for all i, $W_i = V_i g_D^{-1}$, where W_i is the space spanned by the first k_i standard basis vectors.*

Proof. This is easy to check. □

Definition 3.11. The stabilizer of a representative flag will be called a *representative parabolic subgroup.* If the representative flag corresponds to the factorization $d_1 d_2 \cdots d_t = N$, we will set $D = (d_1, d_2, \dots, d_t)$, and denote the stabilizer of the flag by Q_D. (Note that we suppress the dimensions of the subspaces comprising the flag in this notation.)

Since every flag is $\Gamma_0(n, N)$-equivalent to a representative flag, every parabolic subgroup is $\Gamma_0(n, N)$-conjugate to a representative parabolic subgroup.

Theorem 3.12. *Let*

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{t-1} \subsetneq V_t = \mathbb{Q}^n$$

be a representative flag in which $k_i = \dim(V_i)$ for each i, and let $d_1 d_2 \dots d_t = N$ be the corresponding factorization of N. For $1 \le i \le t$, define $\ell_i = k_i - k_{i-1}$. If Q is the stabilizer in $\mathrm{GL}_n(\mathbb{Q})$ of this representative flag, then $g_D Q g_D^{-1}$ is a standard parabolic subgroup consisting of block lower triangular matrices, with blocks of size $\ell_1, \dots, \ell_t$ on the diagonal.

Proof. The group $g_D Q g_D^{-1}$ is the stabilizer of the flag

$$0 = V_0 g_D^{-1} \subsetneq V_1 g_D^{-1} \subsetneq \cdots \subsetneq V_{t-1} g_D^{-1} \subsetneq V_t g_D^{-1} = \mathbb{Q}^n,$$

and so, by Theorem 3.10, has the desired property. □

Theorem 3.13. *Let N be a squarefree positive integer prime to p, fix integers $0 = k_0 < k_1 < \cdots < k_t = n$ and a factorization $d_1 d_2 \cdots d_t = N$, and let Q_D be the representative parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$ corresponding to this factorization. For $1 \leq i \leq t$, define $\ell_i = k_i - k_{i-1}$. Let U_D be the unipotent radical of Q_D. Then there is an exact sequence*

$$0 \to U_D \cap \Gamma_0^{\pm}(n, N) \to Q_D \cap \Gamma_0^{\pm}(n, N) \xrightarrow{\psi_D} \prod_{i=1}^{t} \Gamma_0^{\pm}(\ell_i, d_i) \to 0,$$

where ψ_D is the map taking an element $s \in Q_D$ to the Cartesian product of the diagonal blocks of $g_D s g_D^{-1}$.

Proof. We first note that this is true when $t = 2$ (i.e. for a maximal representative parabolic subgroup) by Theorem 3.3. Assume that it is true for all representative parabolic subgroups fixing a flag of length less than t (for some $t > 2$). Let Q_D be a representative parabolic subgroup fixing a flag

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots V_{t-1} \subsetneq V_t = \mathbb{Q}^n$$

corresponding to the factorization $d_1 d_2 \cdots d_t = N$.

Then $\psi_{d_1}^2(Q_D)$ is the representative parabolic subgroup corresponding to the flag

$$0 = V_1 g_{d_1}^{-1}/V_1 g_{d_1}^{-1} \subsetneq V_2 g_{d_1}^{-1}/V_1 g_{d_1}^{-1} \subsetneq \cdots V_{t-1} g_{d_1}^{-1}/V_1 g_{d_1}^{-1} \subsetneq V_t g_{d_1}^{-1}/V_1 g_{d_1}^{-1} \cong \mathbb{Q}^{n-k_1},$$

of length $t-1$ corresponding to the factorization $d_2 \cdots d_t = N/d_1$. Call this subgroup Q_{D_2}, where $D_2 = (d_2, \ldots, d_t)$. We note that $\psi_Q = \psi_{d_1}^1 \times (\psi_{D_2} \circ \psi_{d_1}^2)$ (since conjugation by $\tilde{g}_{d_i}$ for $i > 1$ does not change the top left $k_1 \times k_1$ block of a matrix).

Now let

$$(\gamma_2, \ldots, \gamma_t) \in \prod_{i=2}^{t} \Gamma_0^{\pm}(\ell_i, d_i) \text{ and } \gamma_1 \in \Gamma_0^{\pm}(\ell_1, d_1).$$

Then, by our inductive hypothesis, there is a matrix $B \in \Gamma_0(n - k_1, N/d_1)$ such that $\psi_{D_2}(B) = (\gamma_2, \ldots, \gamma_t)$. Then $(\gamma_1, B) \in \psi_{d_1}(\gamma)$ for some $\gamma \in P_{d_1} \cap \Gamma_0(n, N)$. We claim that $\gamma \in P_D$. To see this, notice that the matrix $g_{D_2} = \hat{g}_t \cdots \hat{g}_2$ in the definition of ψ_{D_2} is an $(n-k_1) \times (n-k_1)$ matrix. Write $\tilde{g}_{D_2} = I_{k_1} \oplus g_{D_2}$. We have that $g_{D_2} B g_{D_2}^{-1}$ is block lower triangular with $\gamma_2, \ldots, \gamma_t$ as the diagonal blocks. Now the matrix g_D in the definition of ψ_D is equal to $\tilde{g}_{D_2} \hat{g}_{d_1}$, so we have

$$\begin{aligned} g_D \gamma g_D^{-1} &= \tilde{g}_{D_2} (\hat{g}_{d_1} \gamma \hat{g}_{d_1}^{-1}) \tilde{g}_{D_2}^{-1} \\ &= \tilde{g}_{D_2} \begin{pmatrix} \gamma_1 & 0 \\ * & B \end{pmatrix} \tilde{g}_{D_2}^{-1} \\ &= \begin{pmatrix} \gamma_1 & 0 \\ * & g_{D_2} B g_{D_2}^{-1} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \gamma_1 & 0 & \dots & 0 \\ * & \gamma_2 & \dots & 0 \\ * & * & \ddots & \\ * & * & \dots & \gamma_t \end{pmatrix} \in P_0,$$

so $\gamma \in g_D^{-1} Q_0 g_D = Q_D$. Hence, $\gamma \in Q_D \cap \Gamma_0(n, N)$, and $\psi_D(\gamma) = (\gamma_1, \dots, \gamma_t)$, so ψ_D is surjective.

One then checks easily that the kernel of ψ_D is $U_D \cap \Gamma_0^{\pm}(n, N)$. □

Definition 3.14. If T is a subsemigroup of $\mathrm{GL}_n(\mathbb{Q})$, let T_N denote the subsemigroup of $t \in T$ such that $\det(t)$ is prime to N.

Theorem 3.15. *Let N be a squarefree positive integer prime to p, fix integers $0 = k_0 < k_1 < \cdots < k_t = n$, and a factorization $d_1 d_2 \cdots d_t = N$, and let Q_D be the representative parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$ corresponding to this factorization. For $1 \leq i \leq t$, define $\ell_i = k_i - k_{i-1}$. Then there is a surjective map*

$$Q_D \cap S_0^{\pm}(n, N) \xrightarrow{\psi_D} \prod_{i=1}^{t} S_0^{\pm}(\ell_i, d_i)_N,$$

where ψ_D is the map taking an element $s \in Q_D$ to the Cartesian product of the diagonal blocks of $g_D s g_D^{-1}$.

In fact any element in the product has a preimage s such that $g_D s g_D^{-1}$ is block diagonal modulo p.

Proof. We will prove this for $t = 2$. The proof for general t follows by induction, in a similar fashion to the proof of Theorem 3.13, and will be omitted. Let $d = d_1$. Then we wish to prove that the map

$$Q_D \cap S_0^{\pm}(n, N) \xrightarrow{\psi_d^1 \times \psi_d^2} S_0^{\pm}(\ell_1, d)_N \times S_0^{\pm}(\ell_2, N/d)_N \to 0$$

is surjective. Parts (2) and (3) of Theorem 3.3 show that the image of $\psi_d^1 \times \psi_d^2$ is contained in $S_0^{\pm}(\ell_1, d)_N \times S_0^{\pm}(\ell_2, N/d)_N$. (The determinants of the two components are prime to N because the determinant of the image of $s \in S_0(n, N)$ is prime to N and is the product of the two determinants of the matrices comprising the image of s.) Hence, the map $\psi_d^1 \times \psi_d^2$ has image contained in the indicated codomain.

We now prove surjectivity. Let $A = (a_{ij}) \in S_0^{\pm}(\ell_1, d)_N$ and $B = (b_{ij}) \in S_0^{\pm}(\ell_2, N/d)_N$. We define an $\ell_2 \times \ell_1$ matrix $C = (c_{ij})$ as follows:

- For $i > 1$ and for all j, $c_{ij} = 0$.
- c_{11} is a solution to the congruence $c_{11} d \equiv a_{11} - b_{11} \pmod{N/d}$. Note that since d is a unit modulo N/d, this congruence has a unique solution modulo N/d. Using the Chinese remainder theorem, we may also choose $c_{11} \equiv 0 \pmod{p}$.
- For $i = 1$ and $j > 1$, choose $c_{ij} \equiv a_{ij}/d \pmod{N/d}$. Note that since $A \in S_0^{\pm}(\ell_1, d)$, $a_{ij}/d \in \mathbb{Z}$. In addition, we may use the Chinese remainder theorem to choose $c_{ij} \equiv 0 \pmod{p}$.

Construct the block matrix

$$X = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix},$$

and set $s = g_d^{-1} X g_d$. Then $s \in S_0^{\pm}(n,N) \cap Q_D$. The determinant of s is prime to pN since $\det(A)$ and $\det(B)$ are prime to pN. Further, s has been constructed so that $\psi_d^1 \times \psi_d^2$ maps s to (A,B). We note that since we have chosen $C \equiv 0 \pmod p$, we have that $g_d s g_d^{-1}$ is block diagonal modulo p. $\square$

4. Parabolic congruence subgroups

In this section we correct an error in [4]. In [4, Definition 7.1], $\Gamma_P(N)$ and $S_P(N)$ are defined, and in [4, Lemma 7.13] the Hecke pairs $(\Gamma_P(pN), S_P(pN))$ and $(\Gamma \cap P, S \cap P)$ are asserted to be compatible, where (Γ, S) is a certain Hecke pair. The proof of this assertion is flawed when the flag stabilized by P has a one-dimensional subquotient. In this section we rectify the situation by defining replacements for $\Gamma_P(N)$ and $S_P(N)$ which serve our purpose of proving that a Galois representation attached to the homology of a parabolic subgroup of length t has at least t irreducible components (see Theorems 8.3 and 8.5).

Definition 4.1. Let N be a positive squarefree integer prime to p. Let M be a positive divisor of N, and let

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{t-1} \subsetneq V_t = \mathbb{Q}^n$$

be a representative flag for $\Gamma_0(n,M)$, with $k_i = \dim(V_i)$ for $0 \le i \le t$, and $d_1 \dots d_t = M$ the factorization of M corresponding to the flag. Set $\vec{k} = (k_1, \dots, k_t)$, and $D = (d_1, \dots, d_t)$, and let Q be the stabilizer of the flag.

We make the following definitions.

1. $\Gamma(p)$ is the set of $A \in \mathrm{SL}_n(\mathbb{Z})$ such that $A \equiv I_n$ modulo p.
2. $S_{\vec{k}}(p)$ is the set of matrices $A \in M_n(\mathbb{Z})$ with determinant prime to p such that $A \equiv \mathrm{diag}(1,1,\dots,*,1,\dots,*,\dots) \pmod p$, with a $*$ in the k_i position for each $1 \le i \le t$. If $\vec{k}$ has length 1, we omit it and write merely $S(p)$.
3. $S_0(n,M)_Q^+$ is the set of $s \in S_0(n,M)_N \cap Q$ such that the diagonal blocks of $g_D s g_D^{-1}$ all have positive determinant (necessarily prime to pN).
4. $S_Q(p,M) = \{s \in S_0(n,M)_Q^+ : g_D s g_D^{-1} \in S_{\vec{k}}(p)\}$.
5. $\Gamma_0(n,M)_Q^+$ is the set of $x \in \Gamma_0(n,M) \cap Q$ such that the diagonal blocks of $g_D x g_D^{-1}$ all have positive determinant (necessarily equal to 1).
6. $\Gamma_Q(p,M) = \Gamma(p) \cap \Gamma_0(n,M)_Q^+$.

Remark 4.2. If $t = 1$ (so that $Q = \mathrm{GL}_n(\mathbb{Q})$), we take $g_D = I_n$, so that $S_0(n,M)_Q^+ = S_0(n,M)_N$ and $S_Q(p,M) = S_0(n,M)_N \cap S(p)$.

Remark 4.3. Note that both $S_Q(p,M)$ and $S_0(n,M)_Q^+$ have an implicit dependence on N, even though the notation does not indicate this.

Lemma 4.4. *With notation as above,*

(a) $\Gamma_Q(p,M) = \{s \in S_Q(p,M) : \det(s) = 1\}$.
(b) $\Gamma_Q(p,M)$ *is a normal subgroup of finite index in both* $\Gamma_0(n,M) \cap Q$ *and* $\Gamma_0^{\pm}(n,M) \cap Q$.

Proof. (a) Clearly the left hand side is a subset of the right. Let $s \in \{S_Q(p,M) : \det(s) = 1\}$. Then s has integer entries and determinant 1, and since $s \in S_0(n,M) \cap Q$, we see that $s \in \Gamma_0(n,M) \cap Q$. Further, the determinants of the diagonal blocks of $g_D s g_D^{-1}$ are all positive, and the product of these determinants is 1, so they are all equal to 1. Finally each diagonal block is congruent to $\mathrm{diag}(1,\ldots,1,*) \pmod p$; since each diagonal block has determinant 1, the $*$ must be 1 modulo p, so that $s \in \Gamma(p)$. Hence, $s \in \Gamma_Q(p,M)$.

(b) $\Gamma_Q(p,M)$ is the intersection of the kernel of the map taking a matrix $x \in \Gamma_0(n,M)^{\pm} \cap Q$ to the tuple of determinants of the diagonal blocks of $g_D^{-1} x g_D$, and the kernel of the map taking x to its mod p reduction. Hence, it is normal in $\Gamma_0^{\pm}(n,M) \cap Q$. Since the image of each map is finite, the intersection of the kernels has finite index in $\Gamma_0^{\pm}(n,M) \cap Q$. The statement about $\Gamma_0(n,M) \cap Q$ follows immediately since $\Gamma_Q(p,M) \subseteq \Gamma_0(n,M) \cap Q \subset \Gamma_0^{\pm}(n,M) \cap Q$. □

A key point in the definition of $S_Q(p,M)$, and the reason that it needs to be defined with the implicit dependence on N, is that this definition lends itself to induction, via the following theorem.

Theorem 4.5. *Let N be a positive squarefree integer prime to p. Let M be a positive divisor of N, and let*

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{t-1} \subsetneq V_t = \mathbb{Q}^n$$

be a representative flag for $\Gamma_0(n,M)$, with $k_i = \dim(V_i)$ for $0 \le i \le t$, and $d_1 \ldots d_t = M$ the factorization of M corresponding to the flag. Set $\vec{k} = (k_1,\ldots,k_t)$, and $D = (d_1,\ldots,d_t)$.

Then the map $\psi^1_{d_1} \times \psi^2_{d_1} : S_Q(p,M) \to S_0^{\pm}(k_1,d_1)_N \times S_0^{\pm}(n-k_1,M/d_1)_N$ has image $(S_0(k_1,d_1)_N \cap S(p)) \times S_{Q'}(p,M/d_1)$, where Q' is the parabolic subgroup fixing the flag

$$0 = V_1 g_{d_1}^{-1}/V_1 g_{d_1}^{-1} \subsetneq V_2 g_{d_1}^{-1}/V_1 g_{d_1}^{-1} \subsetneq \cdots \subsetneq V_t g_{d_1}^{-1}/V_1 g_{d_1}^{-1} \cong \mathbb{Q}^{n-k_1}.$$

Further, the map

$$\psi^1_{d_1} \times \psi^2_{d_1} : \Gamma_Q(p,M) \to \Gamma_0^{\pm}(k_1,d_1) \times \Gamma_0^{\pm}(n-k_1,M/d_1)$$

has image $(\Gamma_0(k_1,d_1)_N \cap \Gamma(p)) \times \Gamma_{Q'}(p,M/d_1)$.

Proof. As in the proof of Theorem 3.15, is easy to see that the image is contained in the given semigroup. To prove the other containment, let $A \in S_0(k_1,d_1)_N \cap S(p)$, and let $B \in S_{Q'}(p,M/d_1)$. Then $A \in S_0^{\pm}(k_1,d_1)_N$ and $B \in S_0^{\pm}(n-k_1,M/d_1)_N$. Choosing C as in the proof of Theorem 3.15 (with N there replaced by M), and noting that we have chosen $C \equiv 0 \pmod p$, we see that

$$s = g_{d_1}^{-1} \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} g_{d_1}$$

is in $S_0(n,M)^+_Q$, and that $g_D s g_D^{-1} \in S_{\vec{k}}(p)$ and is block diagonal with determinant prime to pN. Thus, $s \in S_Q(p,M)$ and the image of s under $\psi^1_{d_1} \times \psi^2_{d_1}$ is (A,B).

The assertion about $\Gamma_Q(p, M)$ follows immediately by restricting to matrices of determinant 1. $\square$

5. Compatibility of Some Hecke Pairs

We recall the definition of a Hecke pair, and of compatible Hecke pairs.

Definition 5.1. [1, page 238] We say that (Γ, S) is a Hecke pair in $\mathrm{GL}_n(\mathbb{Q})$ if Γ is a subgroup of G, S is a subsemigroup of G, and

(1) $\Gamma \subseteq S$,
(2) for each $s \in S$, the groups Γ and $s^{-1}\Gamma s$ are commensurable.

Definition 5.2. [8, Definition 1.1.2] A Hecke pair (Γ, S) is said to be *compatible* to the Hecke pair (Γ', S') if

(1) $\Gamma \subseteq \Gamma'$ and $S \subseteq S'$,
(2) $\Gamma' \cap S^{-1}S = \Gamma$,
(3) $S\Gamma' = S'$.

We will typically indicate that (Γ, S) is compatible to (Γ', S'), by saying that the Hecke pairs $(\Gamma, S) \subseteq (\Gamma', S')$ are compatible.

The significance of compatible Hecke pairs arises from [8, p. 194], where it is shown that if $(\Gamma', S') \subseteq (\Gamma, S)$ are compatible, then the natural map $\mathcal{H}(\Gamma, S) \to \mathcal{H}(\Gamma', S')$ of Hecke algebras is an injective algebra homomorphism. This allows us to consider any $\mathcal{H}(\Gamma', S')$-module as a $\mathcal{H}(\Gamma, S)$-module. If we take (Γ, S) to be $(\Gamma_0(n, N), S_0(n, N))$ or $(\Gamma_0^{\pm}(n, N), S_0^{\pm}(n, N))$, this allows us to apply the definition of an attached Galois representation to $\mathcal{H}(\Gamma', S')$-modules.

The following two lemmas are probably well known; we include proofs here for completeness.

Lemma 5.3. *If (Γ, S) is a Hecke pair, and $\Gamma' \subseteq \Gamma$ is a subgroup of finite index, and S' is any semigroup with $\Gamma' \subseteq S' \subseteq S$, then (Γ', S') is a Hecke pair.*

Proof. It is well known [14, Lemma I.3.2] that commensurability of subgroups is an equivalence relation. Writing commensurability with a $\sim$, we have for any $s \in S'$, $\Gamma' \sim \Gamma \sim s^{-1}\Gamma s \sim s^{-1}\Gamma' s$. $\square$

Lemma 5.4. *Suppose $(\Gamma_i, S_i), i = 1, 2, 3$, are Hecke pairs, with $\Gamma_i \subseteq \Gamma_{i-1}$ and $S_i \subseteq S_{i-1}$ for $i = 2, 3$.*

(a) Suppose that (Γ_3, S_3) is compatible to (Γ_1, S_1) and (Γ_2, S_2) is compatible to (Γ_1, S_1). Then (Γ_3, S_3) is compatible to (Γ_2, S_2).

(b) Suppose that (Γ_3, S_3) is compatible to (Γ_2, S_2) and (Γ_2, S_2) is compatible to (Γ_1, S_1). Then (Γ_3, S_3) is compatible to (Γ_1, S_1).

Proof. (a) :

(1) $\Gamma_3 \subseteq \Gamma_2$ and $S_3 \subseteq S_2$ is given.
(2) $\Gamma_2 \cap S_3^{-1}S_3 = \Gamma_3$ because $\Gamma_2 \subseteq \Gamma_1$.
(3) $S_3\Gamma_2 = S_2$: One inclusion is obvious. For the other, suppose $s_2 \in S_2$. Since $S_2 \subseteq S_1$ and $S_3\Gamma_1 = S_1$, we can write $s_2 = s_3\gamma_1$ for some $s_3 \in S_3$ and $\gamma_1 \in \Gamma_1$. Then $\gamma_1 = s_3^{-1}s_2 \in \Gamma_1 \cap S_2^{-1}S_2 = \Gamma_2$.

(b) :

(1) $\Gamma_3 \subseteq \Gamma_1$ and $S_3 \subseteq S_1$ is given.
(2) $\Gamma_1 \cap S_3^{-1} S_3 = \Gamma_3$ because $\Gamma_1 \cap S_2^{-1} S_2 = \Gamma_2$ and then $\Gamma_2 \cap S_3^{-1} S_3 = \Gamma_3$.
(3) $S_3 \Gamma_1 = S_1$: One inclusion is obvious. For the other, suppose $s_1 \in S_1$. Then $s_1 = s_2\gamma_1$ for some $s_2 \in S_2$ and $\gamma_1 \in \Gamma_1$. Then $s_2 = s_3\gamma_2$ for some $s_3 \in S_3$ and $\gamma_2 \in \Gamma_2$. So $s_1 = s_3\gamma_2\gamma_1$. □

Definition 5.5. If T is a subsemigroup of $\mathrm{GL}_n(\mathbb{Q})$ and Q is a parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$, define $T_Q = T \cap Q$.

Theorem 5.6. *If Γ is a subgroup of finite index in $\mathrm{GL}_n(\mathbb{Z})$ and S is a subsemigroup of $\mathrm{GL}_n(\mathbb{Q})$ and $\Gamma \subseteq S$ and Q is a parabolic subgroup, then (Γ_Q, S_Q) is a Hecke pair.*

Proof. Obviously $\Gamma_Q \subset S_Q$. The given conditions imply that for any $s \in S$, we we have that Γ and $s^{-1}\Gamma s$ are commensurable.

Let $s \in S_Q$. We must show the groups Γ_Q and $s^{-1}\Gamma_Q s$ are commensurable. Since conjugation by s is an automorphism of the ambient group $\mathrm{GL}_n(\mathbb{Q})$, and $s \in Q$, we have $s^{-1}(\Gamma \cap Q)s \cap (\Gamma \cap Q) = s^{-1}\Gamma s \cap s^{-1}Qs \cap \Gamma \cap Q = s^{-1}\Gamma s \cap \Gamma \cap Q$. We must show this has finite index in $\Gamma \cap Q$ and in $s^{-1}(\Gamma \cap Q)s = s^{-1}\Gamma s \cap Q$.

Since Γ and $s^{-1}\Gamma s$ are commensurable, Γ contains a normal subgroup Δ of finite index such that $\Delta \subset \Gamma \cap s^{-1}\Gamma s$. Clearly $\Gamma \cap Q/\Delta \cap Q \to \Gamma/\Delta$ is injective. Thus $\Gamma \cap Q$ contains the normal subgroup $\Delta \cap Q$ of finite index and $\Delta \cap Q \subset s^{-1}\Gamma s \cap \Gamma \cap Q$.

The proof that the other index is finite is similar. □

Theorem 5.7. *Let N be a squarefree positive integer prime to p, and let M be a positive divisor of N. Let Q be a representative parabolic subgroup for $\Gamma_0(n, M)$ and let*

$$(\Gamma, S) = (\Gamma_0(n, M), S_0(n, M)_N) \text{ or } (\Gamma_0^{\pm}(n, M), S_0^{\pm}(n, M)_N).$$

(a) $(\Gamma_Q(p, M), S_Q(p, M))$ is a Hecke pair.
(b) The Hecke pairs $(\Gamma_Q(p, M), S_Q(p, M)) \subseteq (\Gamma_Q, S_Q)$ are compatible.

Proof. Recall that the matrix $g = g_D \in \mathrm{GL}_n(\mathbb{Z})$ from Definition 3.9 is such that $Q_0 = gQg^{-1}$ is a standard parabolic subgroup (i.e. block lower triangular).

(a) We know that $(\Gamma_0(n, M), S_0(n, M)_N)$ is a Hecke pair. Hence, by Theorem 5.6, $(\Gamma_0(n, M) \cap Q, S_0(n, M) \cap Q)$ is a Hecke pair. Now $\Gamma_0(n, M)_Q^+$ has finite index inside $\Gamma_0(n, M) \cap Q$ (being the kernel of the map that takes a matrix x to the vector of determinants of its diagonal blocks of gxg^{-1}, all of which are ± 1). Hence, by Lemma 5.3, $(\Gamma_0(n, M)_Q^+, S_0(n, M)_Q^+)$ is a Hecke pair. Finally, $\Gamma_Q(p, M)$ has finite index in $\Gamma_0(n, M)_Q^+$ (being the kernel of reduction modulo p). Thus, by Lemma 5.3 again, $(\Gamma_Q(p, M), S_Q(p, M))$ is a Hecke pair.

(b) We have three conditions to prove compatibility. We do both cases $(\Gamma, S) = (\Gamma_0(n, M), S_0(n, M)_N)$ and $(\Gamma, S) = (\Gamma_0^{\pm}(n, M), S_0^{\pm}(n, M)_N)$ simultaneously.

(1) We have already seen that containment holds.

(2) We need to prove:

$$S_Q(p,M)^{-1}S_Q(p,M) \cap \Gamma_Q = \Gamma_Q(p,M).$$

The right hand side is obviously in the left hand side. Conversely, suppose

$$x \in S_Q(p,M)^{-1}S_Q(p,M) \cap \Gamma_Q.$$

Recall that $S_Q(p,N) = \{s \in S_0(n,N)_Q^+ : g_D s g_D^{-1} \in S_{\vec{k}}(p)\}$ and $\Gamma_Q(p,M) = \Gamma(p) \cap \Gamma_0(n,M)_Q^+$.

Obviously $x \in \Gamma$ and $x \in Q$ and the diagonal blocks of $g_D x g_D^{-1}$ have positive determinant, so $x \in \Gamma_0(n,M)_Q^+$. We must therefore show $x \in \Gamma(p)$. But $g_D x g_D^{-1} \in S_{\vec{k}}(p)$. Since the diagonal blocks are congruent to $\operatorname{diag}(1,1,\dots,*)$ modulo p and have determinant 1, we see that $* \equiv 1 \pmod p$, so $x \in \Gamma(p)$. Hence $x \in \Gamma_0(n,M)_Q^+ \cap \Gamma(p) = \Gamma_Q(p,M)$.

(3) We must prove:

$$S_Q(p,M)\Gamma_Q = S_Q.$$

The left hand side is obviously contained in the right hand side. Conversely, suppose $s \in S_Q$.

We first assume that $\det(s)$ is positive (this will always be the case if $S = S_0(n,M)_N$). We must find $\gamma \in \Gamma_Q$ such that $s\gamma \in S_Q(p,M)$. We do this by induction on the length t of the flag defining Q.

In the case where $t = 1$, we see that $Q = \mathrm{GL}_n(\mathbb{Q})$, so $\Gamma_Q(p,M) = \Gamma_0(n,M)_N \cap \Gamma(p)$, and $S_Q(p,M) = S_0(n,M)_N \cap S(p)$. Let $s \in S_Q$. By the Chinese Remainder Theorem, we may choose $\gamma \in \mathrm{SL}_n(\mathbb{Z})$ modulo p and modulo M independently. We will choose γ to be congruent to the identity modulo M, so that $s\gamma \equiv s \pmod M$. Then clearly $s\gamma \in S_0(n,M)_N$. We also choose γ congruent modulo p to $s^{-1}\operatorname{diag}(1,\dots,1,\det(s))$. Then $s\gamma \in S(p)$ and $\det(s\gamma) = \det(s) > 0$, so $s\gamma \in S_Q(p,M)$, as desired.

If $\det(s) < 0$ (which can only happen if $(\Gamma, S) = (\Gamma_0^{\pm}(n,M), S_0^{\pm}(n,M)_N)$), then we choose a $\gamma' \in \Gamma_0^{\pm}(n,M)$ with $\det(\gamma') = -1$. Then $s\gamma'$ has positive determinant, so we can (as above) find a $\gamma \in \Gamma_0(n,M)$ such that $s\gamma'\gamma \in S_Q(p,M)$. Since $\gamma'\gamma \in \Gamma_0^{\pm}(n,M)$, we are done.

We now assume that $t > 1$, and the theorem is true for all flags of length less than t, and suppose that Q stabilizes a representative flag of length t.

Let Q' be the parabolic subgroup stabilizing the representative flag

$$0 = V_1 g_{d_1}^{-1}/V_1 g_{d_1}^{-1} \subsetneq V_2 g_{d_1}^{-1}/V_1 g_{d_1}^{-1} \subsetneq \cdots V_t g_{d_1}^{-1}/V_1 g_{d_1}^{-1} \cong \mathbb{Q}^{n-k_1}$$

corresponding to the factorization $d_2 \cdots d_t = M/d_1$.

Let $s \in S_Q$. Then

$$g_{d_1} s g_{d_1}^{-1} = \begin{pmatrix} s_1 & 0 \\ * & s_2 \end{pmatrix},$$

where $s_1 \in S_0^{\pm}(k_1, d_1)_N$, and $s_2 \in S_0^{\pm}(n-k_1, M/d_1)_N$. By the inductive hypothesis, we may find $\gamma_1 \in \Gamma_0^{\pm}(k_1, d_1)$ and $\gamma_2 \in \Gamma_0^{\pm}(n-k_1, M/d_1)$ so that $s_1\gamma_1 \in S_0(k_1, d_1)_N \cap S(p)$ and $s_2\gamma_2 \in S_{Q'}(p, M/d_1)$. By the surjectivity of the map in Theorem 3.13, we may find $\gamma \in \Gamma_0^{\pm}(n, M)_N$ such that

$$g_{d_1}\gamma g_{d_1}^{-1} = \begin{pmatrix} \gamma_1 & 0 \\ * & \gamma_2 \end{pmatrix}.$$

Hence, $\gamma \in \Gamma_0^{\pm}(n, M) \cap Q$. We note that if $\det(s)$ is positive, then $\det\gamma$ must also be positive, since $\det s_1\gamma_1$ and $\det s_2\gamma_2$ must both be positive, so we see that γ is in Γ_Q (regardless of whether $\Gamma = \Gamma_0(n, M)$ or $\Gamma = \Gamma_0^{\pm}(n, M)$). Then

$$g_{d_1}s\gamma g_{d_1}^{-1} = \begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \in M_n(\mathbb{Z})$$

for some $A \in S_0(k_1, d_1)_N \cap S(p)$, $B \in S_{Q'}(p, M/d_1)$, and $Y \in M_{n-k_1,k_1}(\mathbb{Z})$. Since B has determinant prime to p, the matrix B^{-1} has entries in $\mathbb{Z}_p$, and we can choose an integer matrix B' such that $B' \equiv B^{-1} \bmod p$. Then

$$\gamma' = g_{d_1}^{-1}\begin{pmatrix} I_{k_1} & 0 \\ -B'Y & I_{n-k_1} \end{pmatrix} g_{d_1}$$

is in Γ_Q, and we see that

$$\begin{aligned} g_{d_1}s\gamma\gamma' g_{d_1}^{-1} &= \begin{pmatrix} A & 0 \\ Y & B \end{pmatrix}\begin{pmatrix} I_{k_i} & 0 \\ -B'Y & I_{n-k_1} \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ Y - BB'Y & B \end{pmatrix} \\ &\equiv \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \pmod{p}, \end{aligned}$$

so that $s\gamma\gamma' \in S_Q(p, M)$. □

We now prove a lemma to simplify the proofs of Theorem 5.9 and 5.10, and a second lemma needed for the proof of Theorem 5.9.

Lemma 5.8. *Let* $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, M)_N)$. *For each* $s \in S$, *the double coset* $\Gamma s\Gamma$ *contains a matrix* $D = \mathrm{diag}(c_1, \ldots, c_n)$ *with* $c_1|c_2|\cdots|c_n$.

The same conclusion holds if $(\Gamma, S) = (\Gamma_0^{\pm}(n, N), S_0^{\pm}(n, M)_N)$.

Proof. We follow [16, p. 66]. Let $(\Gamma', S') = (\mathrm{SL}_n(\mathbb{Z}), \Delta_N)$, where

$$\Delta_N = \{x \in M_n(\mathbb{Z}) : \det(x) > 0, \gcd(\det x, pN) = 1\}.$$

Let a bar denote the reduction map modulo N from Δ_N to $\mathrm{GL}_n(\mathbb{Z}/N)$. Let G be the subgroup of $\mathrm{GL}_n(\mathbb{Z}/N)$ consisting of matrices whose first row has the form $(*, 0, \ldots, 0)$.

Let

$$\Phi = \{\alpha \in \Delta_N : \overline{\Gamma\alpha} = \overline{\alpha\Gamma}\}.$$

Let $\alpha \in S$. Then $\overline{\alpha} \in G$. For any $\gamma \in \Gamma$, $\overline{\gamma\alpha} \in G$. Further, $\overline{\alpha}^{-1} \in G$, so $\overline{\alpha}^{-1}\overline{\gamma\alpha} \in G$. Now $\det(\overline{\alpha}^{-1}\overline{\gamma\alpha}) = 1$, so it lifts to a $\gamma' \in \Gamma'$ such that

$\overline{\alpha}^{-1}\overline{\gamma\alpha} = \overline{\gamma'}$. Since $\overline{\gamma'} \in G$, we see that $\gamma' \in \Gamma$ and $\overline{\gamma\alpha} = \overline{\alpha\gamma'}$. This shows that $\overline{\Gamma\alpha} \subseteq \overline{\alpha\Gamma}$. The reverse inclusion is proven similarly. Thus, $S \subseteq \Phi$.

Now let $s \in S$ and suppose that we have a coset decomposition

$$\Gamma s \Gamma = \coprod_i \Gamma s_i$$

with each $s_i \in S$. By Shimura, Lemma 3.29(5), we see that

$$\Gamma' s \Gamma' = \coprod_i \Gamma' s_i.$$

By Smith Normal Form, we know that $\Gamma' s \Gamma'$ contains a diagonal matrix D as in the statement of the lemma. Hence, $D = \gamma' s_i$ for some i and some $\gamma' \in \Gamma'$. Since both $\overline{D}, \overline{s_i} \in G$, we see that $\overline{\gamma}' \in G$, so that $\gamma' \in \Gamma$. Then $D \in \Gamma s_i \subseteq \Gamma s \Gamma$, as desired.

The final sentence holds since, if $(\Gamma, S) = (\Gamma_0^{\pm}(n, M), S_0^{\pm}(n, M)_N)$, each double coset $\Gamma s \Gamma$ contains an element $s' \in S_0(n, M)_N$. Then

$$\Gamma s \Gamma = \Gamma s' \Gamma \supset \Gamma_0(n, M) s' \Gamma_0(n, M),$$

and this last double coset contains a matrix of the desired form. □

Theorem 5.9. *Let P be a maximal parabolic subgroup, and let*

$$(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N)) \text{ or } (\Gamma_0^{\pm}(n, N), S_0^{\pm}(n, N)).$$

Then the Hecke pairs

$$(\Gamma_P, S_P) \subseteq (\Gamma, S)$$

are compatible.

Proof. We first prove the theorem for $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$.

Let $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$. There are three conditions to prove for compatibility.

(1) We must show that $\Gamma_P \subset \Gamma$ and $S_P \subset S$. This is obvious.

(2) We must show that $\Gamma \cap S_P^{-1} S_P = \Gamma_P$. Since $S_P, S_P^{-1} \subset P$, the left side is clearly contained in the right side. The other containment follows from the fact that $\Gamma_P \subseteq \Gamma \cap S_P$, and $I \in S_P^{-1}$.

(3) We must show that $S_P \Gamma = S$. We note that $S_P \Gamma \subseteq S$ is obvious. Hence, let $s \in S$. Then by Lemma 5.8, the double coset $\Gamma s \Gamma$ contains a diagonal matrix $\gamma_i s \gamma_j$ that is a product of matrices of the form $s(\ell, k) = \mathrm{diag}(1, \ldots, 1, \ell, \ldots, \ell)$ where ℓ is a prime, $\ell \nmid pN$, and there are k ℓ's, where k can vary.

In [4, Theorem 8.6], we show that we can write

$$\Gamma s(\ell, k) \Gamma = \coprod_i t_i \Gamma,$$

with the $t_i \in S_P$. Then the top of page 52 in Shimura [16, Section 3.1] shows that the single coset representatives of $\Gamma s \Gamma$ can be chosen to be products of coset representatives in S_P of the single cosets in $\Gamma s(\ell, k)\Gamma$ for the various $s(\ell, k)$, and so can all be chosen to be in S_P. (Take into account the fact that Shimura uses right cosets and we use left cosets.) Since s is contained in one of these single cosets, we have that $s = s'\gamma$ for some $s' \in S_P$ and some $\gamma \in \Gamma$. Hence, the theorem is true for $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$.

If $(\Gamma, S) = (\Gamma_0^{\pm}(n,N), S_0^{\pm}(n,N))$, conditions (1) and (2) are true with the same proof as above. For condition (3), we wish to prove that $S_P\Gamma = S$. The containment $S_P\Gamma \subseteq S$ is obvious. For the other containment, let $s \in S$, and choose an element $\gamma_1 \in \Gamma$ such that $\det(s\gamma_1) > 0$. Then $s\gamma_1 \in S_0(n,N)$, and by the first part of the theorem, it has the form $s'\gamma_2$ with $s' \in S_0(n,N) \cap P \subset S_P$, and $\gamma_2 \in \Gamma_0(n,N) \subset \Gamma$. Then $s = s'(\gamma_2\gamma_1^{-1})$ with $\gamma_2\gamma_1^{-1} \in \Gamma$, as desired.

Note that for both possible choices of (Γ, S), the element $s' \in S_P$ that we find has positive determinant. □

Theorem 5.10. *Let Q be a representative parabolic subgroup of length t stabilizing the flag*

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_t = \mathbb{Q}^n$$

and corresponding to the factorization $d_1 \cdots d_t = N$. Let P be the stabilizer of the flag $0 \subsetneq V_1 \subsetneq \mathbb{Q}^n$. Set $(\Gamma, S) = (\Gamma_0(n,N), S_0(n,N))$ or $(\Gamma, S) = (\Gamma_0^{\pm}(n,N), S_0^{\pm}(n,N))$. Then

(a) The Hecke pair (Γ_Q, S_Q) is compatible with (Γ, S).

(b) The Hecke pair (Γ_Q, S_Q) is compatible with (Γ_P, S_P).

Proof. Let $d = d_1$, and let Q' be the parabolic subgroup stabilizing the flag

$$0 = V_1 g_d^{-1}/V_1 g_d^{-1} \subseteq V_2 g_d^{-1}/V_1 g_d^{-1} \subseteq \cdots \subseteq V_t g_d^{-1}/V_1 g_d^{-1} \cong \mathbb{Q}^{n-k_1}.$$

Note that Q' corresponds to the factorization $d_2 \cdots d_t = N/d$ and has length $t-1$.

(a) We prove this by induction on the length t of $\mathbb{Q}$. The case when $t = 1$ is trivial (since $\Gamma_Q = \Gamma$ and $S_Q = S$), and the case when $t = 2$ is Theorem 5.9.

Let $t > 2$, and assume that the theorem is true for all representative parabolic subgroups of length less than t (in particular, the theorem is true for Q'). We prove conditions (1), (2), and (3) of compatibility.

(1) Clearly $\Gamma_Q \subseteq \Gamma$ and $S_Q \subseteq S$.

(2) We need to prove that $\Gamma \cap S_Q S_Q^{-1} = \Gamma_Q$. That the right side is contained in the left is clear.

To prove the other containment, let $s \in \Gamma \cap S_Q S_Q^{-1}$. Then s is contained in both Γ and Q, so $s \in \Gamma_Q$.

(3) We wish to prove that $S_Q\Gamma = S$. Clearly, $S_Q\Gamma \subseteq S$. Now let $s \in S$. By Theorem 5.9, since (Γ_P, S_P) and (Γ, S) are compatible, we have that $S_P\Gamma = S$, so that there is some $\gamma_1 \in \Gamma$ with $s = s_1\gamma_1$ and $s_1 \in S_P$. If $\det(s_1)$ is negative, we may choose an element $\gamma' \in \Gamma_P \subset \Gamma$ of determinant -1, and replace s_1 by $s_1\gamma' \in S_P$ and γ_1 by $(\gamma')^{-1}\gamma_1 \in \Gamma$, so we may as well assume that $\det(s_1)$ is positive.

Then

$$g_{d_1} s_1 g_{d_1}^{-1} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

with $A \in S_0^{\pm}(k_1, d_1)$ and $C \in S_0^{\pm}(n-k_1, N/d_1)$. Since $\det(s_1) > 0$, we see that $\det(A)$ and $\det(C)$ have the same sign.

Note that $\gamma_2 = \mathrm{diag}(-1, 1, \cdots, 1, -1, \cdots,)$ (with -1 in the 1 and $k_1 + 1$ positions) commutes with g_{d_1} and is in Γ. If $\det(A)$ and

$\det(C)$ are negative, replacing s_1 by $s_1\gamma_2$ (and replacing γ_1 with $\gamma_2^{-1}\gamma_1$) will make $\det(A)$ and $\det(C)$ positive, so that $A \in S_0(k_1, d_1)$ and $C \in S_0(n-k_1, N/d_1)$.

By the inductive hypothesis, there is some $\gamma_3 \in \Gamma_0(n-k_1, N/d_1)$ such that $C\gamma_3 \in S_0(n-k_1, N/d_1) \cap Q'$. Let

$$\gamma_4 = g_{d_1}^{-1} \begin{pmatrix} I & 0 \\ 0 & \gamma_3 \end{pmatrix} g_{d_1} \in \Gamma.$$

Set $s_2 = s_1\gamma_4$. Then $s_2 \in S_Q$, since $C\gamma_3 \in Q'$. Then $s = s_2(\gamma_4^{-1}\gamma_1)$, and $\gamma_4^{-1}\gamma_1 \in \Gamma$, as desired. ∎

(b) follows from Lemma 5.4.

For use in the next section, we prove:

Lemma 5.11. *Let Q be a parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$, with U the unipotent radical of Q. Suppose $(\Gamma, S) \subseteq (\Gamma_0^{\pm}(n,N), S_0^{\pm}(n,N))$ are compatible Hecke pairs. Then $S_U = \Gamma_U$.*

Proof. First note that $\Gamma = \{x \in S : \det x = \pm 1\}$. This is because if $x \in \Gamma$ then it is in S and has determinant ± 1. Conversely, if $x \in S$ has determinant ± 1 then since the entries of x are integers, $x \in \mathrm{GL}_n(\mathbb{Z})$. But $x \in S \subset S_0^{\pm}(n,N)$, so $x \in \Gamma_0^{\pm}(n,N)$. But compatibility implies that $\Gamma_0^{\pm}(n,N) \cap S^{-1}S = \Gamma$, and clearly $\Gamma_0^{\pm}(n,N) \cap S \subset \Gamma_0^{\pm}(n,N) \cap S^{-1}S$. Thus, $x \in \Gamma$.

Since the elements of U all have determinant 1 it follows that $S \cap U$ is contained in $\Gamma \cap U$. Since $\Gamma \subset S$, the other inclusion is trivial. ∎

6. Admissibility of certain group actions

In this section, we review the definition of an admissible module, and prove that certain homology groups are admissible modules. Because we use the Hecke pair $(\Gamma_Q(p,N), S_Q(p,N))$ in this paper, we must reprove some of the Lemmas in [4] for this Hecke pair. In particular, [4, Theorem 7.10] needs to be revised, and we also give a more detailed proof of [4, Theorem 11.3].

We recall the definition of an admissible module.

Definition 6.1. A (p,N)-*admissible* S-module M is an $\mathbb{F}[S]$-module of the form $M' \otimes \mathbb{F}_\epsilon$, where M' is an $\mathbb{F}[S]$-module on which $S \subset \mathrm{GL}_n(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z}_p)$ acts via its reduction modulo p, and $\epsilon : S \to \mathbb{F}^\times$ is a character which factors through the reduction of S modulo N. Here $\mathbb{F}_\epsilon$ is the vector space $\mathbb{F}$, with S acting as multiplication via ϵ. Moreover, if $S \subseteq S_0^{\pm}(n,N)$ then we require ϵ to be defined by $\epsilon(s) = \eta(s_{11})$, for some homomorphism $\eta : (\mathbb{Z}/N)^\times \to \mathbb{F}^\times$ (where s_{11} denotes the $(1,1)$-entry of s). If p and N are understood from the context, we will just use the term *admissible*.

The following theorem proves the equivalent of [4, Theorem 7.10] for the Hecke pair $(\Gamma_Q(p,N), S_Q(p,N))$.

Theorem 6.2. *Let Q be a representative parabolic subgroup, and let $P = P_d$ be a representative maximal parabolic subgroup containing Q. Let U be the unipotent radical of P. Let (Γ, S) be either $(\Gamma_0(n,N), S_0(n,N))$ or*

$(\Gamma_Q(p,N), S_Q(p,N))$. *Let* $z \in H_*(\Gamma \cap U, M)$. *Then the action of individual elements of* $S \cap P$ *on* $H_*(\Gamma \cap U, M)$ *described in* [4, Def. 7.2] *yields a semigroup action of* $S \cap P$ *under which* $S \cap U$ *acts trivially.*

Proof. By the Lemmas preceding [4, Thm. 7.10], we need only show that $S \cap P$ lies in a subgroup T of $\mathrm{GL}_n(\mathbb{Q})$ such that every element of $T \cap U(\mathbb{Q})$ is congruent modulo p to an element of $\Gamma \cap U$.

For $(\Gamma, S) = (\Gamma_0(n,N), S_0(n,N))$, we take T to be the group of all rational matrices in P with determinant prime to pN, coefficients having denominator prime to pN, and top row congruent to $(*, 0, \cdots, 0)$ modulo N. Then for $t \in T \cap U$, we have that (in block form)

$$g_d t g_d^{-1} = \begin{pmatrix} I & 0 \\ A & I \end{pmatrix},$$

where A is a rational matrix with coefficients having denominators prime to pN. Each entry of A is then congruent to some integer modulo p, so A is congruent to an integer matrix modulo p.

Now any $g \in \Gamma \cap U$, has the form

$$g = g_d^{-1} \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} g_d,$$

where B is an integer matrix such that the top row of B is congruent to 0 modulo N/d. Since there is no restriction on B modulo p, we can choose it to have the same mod p reduction as A above. Then t will be congruent to g modulo p.

For $(\Gamma, S) = (\Gamma_Q(p,N), S_Q(p,N))$, we first remark that $\Gamma_P = \Gamma$ and $S_P = S$. Take T to be the group of all matrices $t \in Q(\mathbb{Q})$ with determinant prime to pN and denominators prime to pN such that $g_D t g_D^{-1}$ is diagonal modulo p, and the top row of t is congruent to $(*, 0, \ldots, 0)$ modulo N. Clearly, $S \subseteq T$.

We note that any $g \in \Gamma \cap U$ is congruent to 1 modulo p. Hence, we need to show that every $t \in T \cap U$ is congruent to 1 modulo p. Let $t \in T \cap U$, and let y be the mod p reduction of $g_D t g_D^{-1}$, which is a diagonal matrix. Since $t \in U$, $g_D t g_D^{-1}$ is lower triangular with all diagonal entries equal to 1, so all of its eigenvalues are 1. Hence the eigenvalues of its mod p reduction are all 1, so y is the identity mod p. Hence, t is congruent to the identity modulo p, as desired. □

Remark 6.3. In either of the two cases considered, we apply [4, Corollary 7.9] to our situation, and we find that there is a semigroup action of $T \cap P$ (and hence of $S \cap P$) on $H_k(\Gamma_U, M)$. Moreover, the action of $t \in T \cap P$ is given by a certain Hecke operator as described below in the proof of Theorem 6.4. Note that since T is a group, and the identity element clearly acts trivially, we have an actual group action of $T \cap P$ on $H_k(\Gamma_U, M)$.

Now, [4, Theorem 11.3] implies Theorem 6.4. Because the proof in [4] is very brief, we give additional details here.

Theorem 6.4. *Let* Q*,* P*, and* U *be as in the previous theorem, and let* $(\Gamma, S) = (\Gamma_Q(p,N), S_Q(p,N))$. *Note that* $S_P = S$. *Let* ξ *be a character*

$S \to \mathbb{F}^\times$ such that $\xi|_{\Gamma_U} = 1$. Then $H_k(\Gamma_U, \mathbb{F}_\xi)$ is an admissible S-module, with the action of S described in [4, Definition 7.2].

Moreover, the S-action on $H_k(\Gamma_U, \mathbb{F}_\xi)$ is diagonalizable, so that $H_k(\Gamma_U, \mathbb{F}_\xi)$ is isomorphic as an S-module to a direct sum of one-dimensional modules $\mathbb{F}_\chi$ where χ runs over some sequence of characters of S that are trivial on S_U.

Proof. First of all, since $\xi|_{\Gamma_U} = 1$, there is a canonical isomorphism $\phi : H_k(\Gamma_U, \mathbb{F}) \to H_k(\Gamma_U, \mathbb{F}_\xi)$ as $\mathbb{F}$-vector spaces. Let $z \in H_k(\Gamma_U, \mathbb{F})$ and $s \in S$. Directly from [4, Definition 7.2], we see that

$$\phi(z)s = \xi(s)zs.$$

So without loss of generality we may assume $\xi = 1$.

Let $s \in S$. By [4, Lemma 7.4], for any coefficient module M, the S-action on $H_k(\Gamma_U, M)$ is given by $(1/e_s)[\Gamma_U s \Gamma_U]$, where $e_s = [\Gamma_U : \Gamma_U \cap s\Gamma_U s^{-1}]$ and $[\Gamma_U s \Gamma_U]$ denotes the Hecke action of this double coset on the homology.

In [4, Lemma 11.2], we show that for an element $u \in \Gamma_U = H_1(\Gamma_U, \mathbb{Z})$, the action of the Hecke operator $[\Gamma_U s \Gamma_U]$ on u is given by

$$u[\Gamma_U s \Gamma_U] = s^{-1}(u^{e_s})s.$$

Consider Γ_U as a free abelian group A written additively. Then right conjugation by s on Γ_U induces a map $A \to A$ which (in terms of a $\mathbb{Z}$-basis of A) is given by a matrix $M(s) \in \mathrm{GL}_n(\mathbb{Q})$, all of whose denominators must divide e_s (because $M(s)$ maps $e_s A$ to A. Also, all the denominators of the entries of $M(s)$ are prime to p. To see this, note that $M(s)$ is the same as the matrix that describes the action of s (in terms of the same basis) on $U(\mathbb{Q})$. Now s and $u \in U$ satisfy

$$g_d s g_d^{-1} = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix},$$

$$g_d s^{-1} g_d^{-1} = \begin{pmatrix} A^{-1} & 0 \\ C' & B^{-1} \end{pmatrix},$$

$$g_d u g_d^{-1} = \begin{pmatrix} I & 0 \\ Z & I \end{pmatrix}.$$

So

$$g_d s^{-1} u s g_d^{-1} = \begin{pmatrix} I & 0 \\ (C' + B^{-1}Z)A & I \end{pmatrix}.$$

Since the determinant of s is prime to pN, the denominators in the entries of s^{-1} are prime to p, as are the entries of $s^{-1}us$. Further, the entries of $M(s)$ are given by polynomials in the entries of s and s^{-1} with integer coefficients.

It follows from the preceding sentence that if $s \equiv s' \pmod p$, then $M(s) \equiv M(s') \pmod p$.

Let $\overline{M(s)}$ denote the mod p reduction of $M(s)$. Then the conjugation action of s on A is given by the matrix $e_s\overline{M(s)}$ and the action defined by [4, Definition 7.2] of s on $H_1(\Gamma_U, \mathbb{F}) = H_1(\Gamma_U, \mathbb{Z}) \otimes \mathbb{F} = A \otimes \mathbb{F}$, which is $1/e_s$ times the previous action, is given by $\overline{M(s)}$. Therefore this action is admissible.

Extending this action via the wedge product to $H_k(\Gamma_U, \mathbb{F}) = \wedge^k H_1(\Gamma_U, \mathbb{F})$, we find that s acts via the Pontrjagin product as $\wedge^k \overline{M(s)}$, and thus yields an admissible action of S on $H_k(\Gamma_U, \mathbb{F})$. We need to show that this extension of the action via the Pontrjagin product matches the action defined by [4, Definition 7.2]. If the Hecke operators were induced by some homomorphism of the group Γ_U, this would follow from naturality of the Pontrjagin product, but unfortunately they are not so induced.

Let $H = \Gamma_U \cap s\Gamma_U s^{-1}$, and recall that $e_s = [\Gamma_U : H]$. Then the action of the Hecke operator $[\Gamma_U s \Gamma_U]$ is given on $H_k(\Gamma_U, \mathbb{Z}) \cong \Gamma_U$ by the composition

$$H_k(\Gamma_U, \mathbb{Z}) \xrightarrow{\text{transfer}} H_k(H, \mathbb{Z}) \xrightarrow{\alpha} H_k(s^{-1}Hs, \mathbb{Z}) \xrightarrow{\iota} H_k(\Gamma_U, \mathbb{Z}),$$

where the first map is the transfer, the second is induced by conjugation by s on H, and the third is induced by inclusion of $s^{-1}Hs$ into Γ_U. The last two are induced by group homomorphisms, but the transfer map is not induced by a group homomorphism. Hence, naturality does not apply to it. There is one case, however, in which the transfer is induced by a group homomorphism; namely when $e_s = 1$, so that the transfer is just the identity map. In this case the action of s on $H_k(\Gamma, \mathbb{F}) = H_k(\Gamma, \mathbb{Z}) \otimes \mathbb{F}$, which equals $(1/e_s)[\Gamma_U s \Gamma_U]$, will match the action via $\wedge^k M(s)$, and will depend only on the mod p reduction of s.

In the proof of the previous theorem, we defined a group T containing $S_Q(p, N)$, and from Remark 6.3 we know that the action of $T \cap P$ on $H_k(\Gamma, \mathbb{F})$ via the Hecke operators is a group action. We will now show that $T \cap P$ is generated by elements t that have $e_t = 1$, so that the action via the Hecke operators matches the action via the wedge product. Since the action of elements of $T \cap P$ is a group action, this will show that these two actions are the same for every element of $T \cap P$, so that every element of $T \cap P$ acts admissibly.

Lemma 6.5. *Let T be the group of all matrices $t \in Q(\mathbb{Q})$ with determinant prime to pN and denominators prime to pN such that $g_D t g_D^{-1}$ is diagonal modulo p, and the top row of t is congruent to $(*, 0, \ldots, 0)$ modulo N. Then $T \cap P$ is generated as a group by elements t that have $e_t = 1$.*

Proof. Recall that $\Gamma = \Gamma_Q(p, N)$. Let $f : \mathrm{GL}_n(\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{Q})$ be given by $f(t) = g_D t g_D^{-1}$. Then f is an isomorphism. Define $K = f(\Gamma_U) = f(\Gamma \cap U) = f(\Gamma) \cap f(U) = f(\Gamma) \cap U_0$. Set $T_0 = f(T \cap P) = f(T) \cap P_0$.

For $t \in T$, note that $e_t = 1$ if $t\Gamma_U t^{-1} \subseteq \Gamma_U$, which happens if and only if $f(t) f(\Gamma_U) f(t)^{-1} \subseteq f(\Gamma_U)$.

Then T_0 consists of all matrices $m \in P(\mathbb{Q})$ with determinant prime to pN and denominators prime to pN such that m is diagonal mod p and the top row of $g_D^{-1} m g_D \equiv (*, 0, \ldots, 0) \pmod{N}$.

On the other hand, K consists of matrices of the form

$$k = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix},$$

such that

(1) the entries of B are in $\mathbb{Z}$,

(2) $B \equiv 0 \pmod p$,
(3) the top row of B is congruent to 0 modulo N/d,

(so that $f^{-1}(k) \in \Gamma_Q(p,N) \subseteq \Gamma_0(n,N)$).

Now choose any element $m \in T_0$, and write it in block form as

$$m = \begin{pmatrix} r & 0 \\ u & s \end{pmatrix}.$$

Then

$$mkm^{-1} = \begin{pmatrix} I & 0 \\ (u+sB)r^{-1} - ur^{-1} & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ B' & I \end{pmatrix},$$

where $B' = sBr^{-1}$. Note that since $B \equiv 0 \pmod p$ we see that $B' \equiv 0 \pmod p$.

Let μ be the least common multiple of the denominators of coefficients of r^{-1} and s. Then μ is prime to pN, and for any integer $e \geq 2$, $\mu^e B'$ has integer entries. Let $b = \varphi(pN)$ so that $\mu^b \equiv \mu^{-b} \equiv 1 \pmod{pN}$, and let $\Delta = \mathrm{diag}(\mu^{-b} I_k, \mu^b I_{n-k})$. Then $\Delta \in T_0$ since Δ is congruent to the identity matrix modulo pN.

Set

$$u = \Delta mkm^{-1}\Delta^{-1}$$

Again, because Δ is congruent to the identity matrix modulo pN, we see that $u \in K = f(\Gamma_U)$. Hence, setting $z_m = \Delta m$, we have

$$z_m k z_m^{-1} = u \in f(\Gamma_U).$$

Since $k \in f(\Gamma_U)$ was arbitrary, we see that $z_m f(\Gamma_U) z_m^{-1} \subseteq f(\Gamma_U)$. Hence $e_{z_m} = 1$.

Similarly, for any $k \in K$, $\Delta k \Delta^{-1} \in f(\Gamma_U)$, so $\Delta f(\Gamma_U)\Delta^{-1} \subseteq f(\Gamma_U)$, and $e_\Delta = 1$. Since m is arbitrary, and is equal to $\Delta^{-1} z_m$, we see that T_0 is generated as a group by elements t with $e_t = 1$, as desired. ☐

It remains to prove the claim about diagonalizability. Let overline denote reduction modulo p. We have shown that the S-action on $H_k(\Gamma_U, \mathbb{F}_\xi)$ factors through the reduction of S modulo pN and if we tensor this action with $\mathbb{F}_{\xi^{-1}}$, it factors through $\overline{S}$. Since Γ_U acts trivially, and by Lemma 5.11 $\Gamma_U = S_U$, it factors through $\overline{S}/\overline{S}_U$ (note that $\overline{S}_U$ is actually trivial). This latter group is finite and is isomorphic to a group of invertible diagonal matrices mod p of order prime to p, so this action on the $\mathbb{F}$-vector space $H_k(\Gamma_U, \mathbb{F}_\xi)$ is diagonalizable. Since $\overline{S}/\overline{S}_U$ is abelian, $H_k(\Gamma_U, \mathbb{F}_\xi)$ is isomorphic as an S-module to a direct sum of one-dimensional modules $\mathbb{F}_\chi$ where χ runs over some sequence of characters of S that are trivial on S_U. ☐

7. Hecke operators and the Künneth formula

Because we have changed our Hecke pair to $(\Gamma_Q(p,N), S_Q(p,N))$, we need to reprove [4, Theorem 10.5] in this new context.

Definition 7.1. Let $(\Gamma, S) = (\Gamma_0(n,N), S_0(n,N))$, let ℓ be a prime, and for $0 \leq r \leq n$, let $s = s(\ell, r, n)$ be the diagonal matrix with the first r diagonal entries equal to 1, and the rest equal to ℓ. We recall the definition of two different sets of left coset representatives of $\Gamma s \Gamma$.

(1) $\mathcal{T}_n(\ell, r)$ is the set of lower triangular coset representatives described in [4, Theorem 8.1].
(2) $\mathcal{T}^n(\ell, r)$ is the set of upper triangular coset representatives described in [4, Theorem 8.1]
(3) For $0 < k < n$, we define $\mathcal{T}(s, k)$ to be the collection of right coset representatives of $\Gamma s \Gamma$ described in [4, Theorem 8.4].
(4) $T_n(\ell, m)$ is the Hecke operator corresponding to the double coset $\Gamma s(\ell, m, n)\Gamma$.

We note that if M is an S-module, and $(\Gamma', S') \subseteq (\Gamma, S)$ is a compatible Hecke pair, then $T_n(\ell, m)$ acts naturally on $H_*(\Gamma', M)$

In [4], we proved the following.

Theorem 7.2. [4, Theorem 8.6] *Let $d|N$ and let P_d be a representative maximal parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$ that is the stabilizer of a subspace of dimension k corresponding to the divisor d. For every $t \in \mathcal{T}(s, k)$, there is an element $\gamma_t \in \Gamma_0(n, N)$ such that $t\gamma_t \in P_d$.*

We also recall the following definition ([4, Definition 8.7]).

Definition 7.3. Let N and k be positive integers. Set $C_{k,N}$ equal to the set of left cosets of $\Gamma_0(k, N)$ inside $S_0(k, N)$. Denote by $F_{k,N}$ the free $\mathbb{Z}$-module on the elements of $C_{k,N}$. For a collection $\mathcal{S}$ of matrices in $S_0(k, N)$ we will write $\overline{\mathcal{S}}$ for the element

$$\sum_{s \in \mathcal{S}} s\Gamma_0(k, N) \in F_{k,N}.$$

In [4], we proved the following theorem ([4, Theorem 8.8]; note that we have corrected some minor typographical errors). We give a new (less computational) proof of this theorem.

Theorem 7.4. *Let $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$, and let P be a representative maximal parabolic subgroup of type $(k, n-k)$ corresponding to the divisor d of N. Let ℓ be a prime number not dividing N, let $1 \le r \le n$ and let $s = s(\ell, r, n)$. Let $\mathcal{T}(s, k)$ be the set of left coset representatives of $\Gamma s \Gamma$ described in* [4, Theorem 8.4]*. Then in the tensor product $F_{k,d} \otimes_{\mathbb{Z}} F_{n-k,N/d}$, we have*

$$\sum_{t \in \mathcal{T}(s,k)} \psi_d^1(t\gamma_t)\Gamma_0(k, d) \otimes \psi_d^2(t\gamma_t)\Gamma_0(n-k, N/d)$$
$$= \sum_{m=\max(0, r-(n-k))}^{\min(r,k)} \ell^{(k-m)(r-m)} \overline{\mathcal{T}^k(\ell, m)} \otimes \overline{\mathcal{T}_{n-k}(\ell, r-m)},$$

where $T^k(\ell, m)$ is a collection of upper triangular left $\Gamma_0(k, d)$-coset representatives of $s(\ell, m, k)$, and $\mathcal{T}_k(\ell, m)$ is a collection of lower triangular left $\Gamma_0(k, N/d)$-coset representatives of $s(\ell, m, k)$.

Proof. We note that the coset representatives $\mathcal{T}(s, k)$ have the property that they are also a complete system of right coset representatives for the double coset $\mathrm{GL}_n(\mathbb{Z})s\,\mathrm{GL}_n(\mathbb{Z})$.

Now, we see that as t runs through $\mathcal{T}(s,k)$ the elements $g_d t\gamma_t g_d^{-1}$ also run through a complete set of right coset representatives of $\mathrm{GL}_n(\mathbb{Z})s\,\mathrm{GL}_n(\mathbb{Z})$. Hence, there is a permutation $t \mapsto t'$ of the elements of $\mathcal{T}(s,k)$ such that $g_d t\gamma_t g_d^{-1} = t'\delta_t$ with $\delta_t \in \mathrm{GL}_n(\mathbb{Z})$. Since both $g_d t\gamma_t g_d^{-1}$ and t' are in P_0, we see that δ_t is as well. Further, we can write

$$t' = \begin{pmatrix} t_1 & 0 \\ * & t_2 \end{pmatrix}$$

with $t_1 \in \Gamma_0(k,d)$ and $t_2 \in \Gamma_0(n-k, N/d)$. Hence, δ_t has the form

$$\delta_t = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

with $A \in \Gamma_0(k,d)$ and $C \in \Gamma_0(n-k, N/d)$.

Hence, $\psi_d^1(t\gamma_t)$ is equal to $t_1 A \in t_1\Gamma_0(k,d)$ and similarly $\psi_d^2(t\gamma_t)$ is equal to $t_2 C \in t_2\Gamma_0(n-k, N/d)$. As t runs through $\mathcal{T}(s,k)$, so does t'. A given pair (t_1, t_2), where t_1 has m diagonal elements equal to ℓ and t_2 has $r-m$ diagonal elements equal to ℓ, will occur for exactly $\ell^{(k-m)(r-m)}$ different representatives t'. The formula follows. □

Now we recall from Theorem 4.5 that $\psi_d^1(S_Q(p,N)) = S_0(k_1,d)_N \cap S(p)$ and $\psi_d^2(S_Q(p,N)) = S_{Q_1}(p, M/d)$.

The Hecke pairs

$$(\Gamma_0(k_1,d_1) \cap \Gamma_{k_1}(p), S_0(k_1,d)_N \cap S(p)) \subset (\Gamma_0(k_1,d_1), S_0(k_1,d_1))$$

and

$$(\Gamma_{Q_1}(p, N/d_1), S_{Q_1}(p, N/d_1)) \subseteq (\Gamma_0(n-k_1, N/d_1), S_0(n-k_1, N/d_1))$$

are compatible. The first assertion is the case $t=1$ in Theorem 5.7, and the second follows from Lemma 5.4, Theorem 5.7 and Theorem 5.10.

Therefore we may adjust the coset representatives in $T^k(\ell,m)$ to a set $\mathcal{T}_{d_1}(\ell,m) \subset S_0(k_1,d)_N \cap S(p)$ and the coset representatives in $T_{n-k}(\ell,m)$ to a set $\mathcal{T}_{N/d_1}(\ell,m) \subset S_{Q_1}(p, N/d_1)$.

Since the Hecke pairs $(\Gamma_0(k,d)\cap\Gamma(p), S_0(k,d)\cap S(p)) \subset (\Gamma_0(k,d), S_0(k,d))$ and $(\Gamma_{Q_1}(p,N/d), S_{Q_1}(p,N/d)) \subseteq (\Gamma_0(n-k,N/d), S_0(n-k,N/d))$ are compatible, we see that for $s = s(\ell,r,n)$ and an element $t \in \mathcal{T}(s,k)$ we can find $\delta_1 \in \Gamma_0(k,d)$ and $\delta_2 \in \Gamma_0(n-k, N/d)$ with $\psi_d^1(t\gamma_t)\delta_1 \in S_0(k,d) \cap S(p))$ and $\psi_d^2(t\gamma_t)\delta_2 \in S_{Q_1}(p, N/d)$. By the surjectivity of the map in Theorem 3.3(4), we can find a $\delta_t \in \Gamma_0^{\pm}(n,N)$ with $\psi_d^i(\delta_t) = \delta_i$. Note that since both δ_1 and δ_2 have positive determinant, so will δ_t. We note that we then have $\psi_d^1(t\gamma_t\delta_t) \in S_0(k,d) \cap S(p))$ and $\psi_d^2(t\gamma_t\delta_t) \in S_{Q_1}(p, N/d)$, and $\gamma_t\delta_t \in \Gamma_0(n,N)$.

Setting F_1 to be the free $\mathbb{Z}$-module on the left cosets of $\Gamma_{L_1} = \psi_d^1(\Gamma_Q(p,N)) = \Gamma_0(\ell_1,d_1) \cap \Gamma_{\ell_1}(p)$, and F_2 to be the free $\mathbb{Z}$ module on the left cosets of $\Gamma_{L_2} = \psi_D^2(\Gamma_Q(p,N)) = \Gamma_{Q_1}(p, N/d_1)$ in $S_{Q_1}(p, N/d_1)$, we get the following corollary.

Corollary 7.5. *Let $(\Gamma, S) = (\Gamma_0(n,N), S_0(n,N))$, and let P be a representative maximal parabolic subgroup of type $(k, n-k)$ corresponding to the divisor d of N. Let ℓ be a prime number not dividing N, let $1 \le r \le n$ and* *let $s = s(\ell, r, n)$. Let $\mathcal{T}(s,k)$ be the set of left coset representatives of $\Gamma s \Gamma$ described in* [4, Theorem 8.4]. *Then in $F_1 \otimes_{\mathbb{Z}} F_2$, we have*

$$\sum_{t \in \mathcal{T}(s,k)} \psi_d^1(t\gamma_t)\Gamma_{L_1} \otimes \psi_d^2(t\gamma_t)\Gamma_{L_2}$$

$$= \sum_{m=\max(0,r-(n-k))}^{\min(r,k)} \ell^{(k-m)(r-m)} \overline{\mathcal{T}_{d_1}(\ell, m)} \otimes \overline{\mathcal{T}_{N/d_1}(\ell, r-m)},$$

The Kúnneth theorem allows us to write

$$H_k(\Gamma_L, M_1 \otimes M_2) \cong \bigoplus_{i+j=k} H_i(\Gamma_{L_1}, M_1) \otimes H_j(\Gamma_{L_2}, M_2).$$

As in [4, Section 10], we obtain the following theorem (compare [4, Theorem 10.5]).

Theorem 7.6. *Let Q be a representative parabolic subgroup stabilizing the flag*

$$\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_t = \mathbb{Q}^n$$

associated to the factorization $d_1 \cdots d_t = N$. Let P be the representative maximal parabolic subgroup stabilizing

$$\{0\} = V_0 \subseteq V_1 \subseteq \mathbb{Q}^n.$$

Note that P corresponds to the factorization $d_1(N/d_1) = N$.

Write $\Gamma_{L_1} = \psi_{d_1}^1(\Gamma_Q(p,N))$ and $\Gamma_{L_2} = \psi_{d_1}^2(\Gamma_Q(p,N))$, and $\Gamma_L = \psi_P(\Gamma_Q(p,N))$.

Let M_1 be an admissible $S_0(k_1, N)$-module, and let M_2 be an admissible $S_0(n-k_1, N)$-module.

As in [4, Section 10], *since*

$$(\Gamma_Q(p,N), S_Q(p,N)) \subset (\Gamma_0(n,N) \cap P, S_0(n,N) \cap P) \subseteq (\Gamma_0(n,N), S_0(n,N)),$$

are compatible, there is a natural action of $T_n(\ell, r) \in \mathcal{H}(\Gamma_0(n,N), S_0(n,N))$ on

$$H_t(\Gamma_L, M_1 \otimes M_2).$$

It is given on the Künneth component

$$H_r(\Gamma_{L_1}, M_1) \otimes H_{t-r}(\Gamma_{L_2}, M_2)$$

by

$$(f_1 \otimes f_2)|T_n(\ell, r) = \sum_{m=\max(0,r-(n-k))}^{\min(r,k)} \ell^{(k-m)(r-m)} f_1|T_{k_1}(\ell, m) \otimes f_2|T_{n-k_1}(\ell, r-m).$$

Corollary 7.7. *If $f \in H_t(\Gamma_L, M_1 \otimes M_2)$ is an eigenvector with system of eigenvalues Φ, then $f = f_1 \otimes f_2$ for some $f_1 \in H_r(\Gamma_{L_1}, M_1)$ and $f_2 \in H_{t-r}(\Gamma_{L_2}, M_2)$ eigenvectors of the Hecke algebras $\mathcal{H}(\Gamma_0(k_1, d_1), S_0(k_1, d_1))$ and $\mathcal{H}(\Gamma_0(n-k_1, N/d_1), S_0(n-k_1, N/d_1))$, respectively.*

In addition, if f_1 and f_2 have attached Galois representations ρ_1 and ρ_2, respectively, then f has an attached Galois representation $\rho_1 \oplus \omega^{k_1}\rho_2$.

Proof. The proof of this theorem and corollary mimics the proofs of Theorems 10.1, and 10.2 in [4], substituting Theorem 7.4 in place of [4, Theorem 8.8]. □

8. Reducibility of Galois Representations Attached to Parabolic Homology

In this section, our goal is to prove that for any parabolic subgroup Q of $\mathrm{GL}_n(\mathbb{Q})$ stabilizing a flag of length t, any homology eigenclass in the homology of $\Gamma_0^\pm(n,N) \cap Q$ is attached to a Galois representation that is a sum of at least t irreducible Galois representations. We assume throughout this section that N is a squarefree positive integer prime to p.

Lemma 8.1. *Let (Γ, S) be any Hecke pair in $\mathrm{GL}_n(\mathbb{Q})$ such that the determinants and denominators of matrices in S are relatively prime to pN, and let P a maximal parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$. Let M be an admissible S-module. Let U be the unipotent radical of P, $S_P = S \cap P$, $\Gamma_P = \Gamma \cap P$ $S_U = S \cap U$, $\Gamma_U = \Gamma \cap U$, $\Gamma_L = \Gamma_P/\Gamma_U$. Suppose that $S_U = \Gamma_U$. Then*

(1) for every $s \in S_U$, $[\Gamma_U s \Gamma_U]$ acts trivially on $H_*(\Gamma_U, M)$;

(2) the actions of the individual $s \in S_P$ on $H_*(\Gamma_U, M)$ given by [4, Definition 7.2] *compile into a semi-group action under which S_U acts trivially;*

(3) the Hecke algebra $\mathcal{H}(\Gamma_P, S_P)$ acts equivariantly on the Lyndon-Hochschild-Serre spectral sequence

$$E^2_{ij} = H_i(\Gamma_L, H_j(\Gamma_U, M)) \implies H_{i+j}(\Gamma_P, M)$$

and a given packet of Hecke eigenvalues occurs in $H_k(\Gamma_P, M)$ if and only if it appears in

$$\bigoplus_{i+j=k} E^\infty_{ij}.$$

Proof. Statement (1) is obvious, since for $s \in S_U = \Gamma_U$, the Hecke operator $[\Gamma_U s \Gamma_U]$ is the identity. Statements (2) and (3) follow, using (1), exactly as in the proofs of [4, Corollary 7.9] and [4, Theorem 7.11]. □

Theorem 8.2. *Let Q be a representative parabolic subgroup, and let $(\Gamma, S) = (\Gamma_0(n,N), S_0(n,N))$ or $(\Gamma_0^\pm(n,N), S_0^\pm(n,N))$. Assume that $(\Delta, T) \subset (\Gamma_Q, S_Q) \subset (\Gamma, S)$ are compatible Hecke pairs, with Δ normal of finite index in Γ_Q. Then any system of Hecke eigenvalues occurring in $H_k(\Gamma_Q, M)$ also appears in $H_j(\Delta, M)$ for some $j \le k$.*

Proof. The Hochschild Serre spectral sequence for the exact sequence $0 \to \Delta \to \Gamma_Q \to \Gamma_Q/\Delta \to 0$ computes $H_i(\Gamma_Q, M)$ in two different ways (as in [10, VII.7.6] and [3, Theorem 4.6]) by using two spectral sequences to compute the total homology of the double complex

$$F_\bullet \otimes_{\Gamma_Q/\Delta} (C_\bullet \otimes_\Delta M),$$

where $F_\bullet$ is the standard resolution of $\mathbb{Z}$ over the finite group Γ_Q/Δ, and $C_\bullet$ is the standard resolution of $\mathbb{Z}$ over $\mathrm{GL}(n,\mathbb{Q})$.

Let $\mathcal{H}$ denote the Hecke algebra $\mathcal{H}(\Gamma, S)$. By compatibility, any $\mathcal{H}(\Delta, T)$-module is a $\mathcal{H}(\Gamma, S)$ module.

Then $\mathcal{H}$ acts on the double complex by its natural action on $C_\bullet \otimes_\Delta M$ (with the trivial action on $F_\bullet$). This action commutes with the differentials of the double complex, and hence the spectral sequence is Hecke equivariant. Therefore, any system of Hecke eigenvalues appearing in the abutment

$H_k(\Gamma_Q, M)$ of the first spectral sequence must occur in the E^1 page of the other, i.e., in $F_i \otimes_{\Gamma_Q/\Delta} H_j(\Delta, M)$ for some $i+j=k$. This uses the fact that F_i has finite rank over $\mathbb{Z}$ for each i, and the finite-dimensionality of each $H_j(\Delta, M)$. Since the Hecke algebra acts trivially on F_i, the desired system of eigenvalues must appear in $H_j(\Delta, M)$ for some $j \leq k$. $\square$

Our application of this theorem will set $(\Delta, T) = (\Gamma_Q(p, N), S_Q(p, N))$. We note that $\Gamma_Q(p, N)$ is normal of finite index in Γ_Q, by Lemma 4.4.

To facilitate the inductive proof of the next theorem we will consider the full group GL_n to be a representative parabolic subgroup Q stabilizing the trivial flag $0 = V_0 \subsetneq V_1 = \mathbb{Q}^n$ of length 1. This is consistent with the terminology of Borel and Serre in [9]. In this case we note that $(\Gamma_Q(p, N), S_Q(p, N))$ equals $(\Gamma_0(n, N) \cap \Gamma(p), S_0(n, N) \cap S_n(p))$.

Theorem 8.3. *Let Q be any representative parabolic subgroup stabilizing a flag of length t. Assume that Φ is a packet of $\mathcal{H}(\Gamma_Q(p, N), S_Q(p, N))$-eigenvalues that appears in*

$$H_*(\Gamma_Q(p, N), M).$$

Then Φ has an attached Galois representation that is the direct sum of at least t irreducible Galois representations. In particular, for $t > 1$, the Galois representation attached to Φ is reducible.

Remark 8.4. Note that this theorem is an improvement on [4, Theorem 11.5], which only applied to maximal representative parabolic subgroups.

Proof. The proof proceeds by an inductive argument on the length t of the flag stabilized by Q, using many of the same techniques as the proof of [4, Theorem 11.5]. The base case of $t = 1$ is a consequence of [15], so let $t > 1$.

Let V_1 be the smallest subspace in the flag stabilized by Q and denote its dimension by k. Let P be the stabilizer of the flag $0 \subsetneq V_1 \subsetneq \mathbb{Q}^n$. Then P is a maximal parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$. Denote by U the unipotent radical of P. For any congruence subgroup Γ of $\mathrm{GL}_n(\mathbb{Z})$, note that $\Gamma_U = \Gamma \cap U$ is a free abelian group of finite rank.

Since Q is a representative parabolic subgroup, P is a representative maximal parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$.

By Theorem 5.10, we have compatibility of the Hecke pairs

$$(\Gamma_0(n, N) \cap Q, S_0(n, N) \cap Q) \subset (\Gamma_0(n, N) \cap P, S_0(n, N) \cap P) \subset (\Gamma_0(n, N), S_0(n, N)).$$

By Theorem 5.7 we have compatibility of the Hecke pairs

$$(\Gamma_Q(p, N), S_Q(p, N) \subset (\Gamma_0(n, N) \cap Q, S_0(n, N) \cap Q).$$

If the dimension of V_i in the flag is k_i, let $\ell_i = k_i - k_{i-1}$. Since Γ_U is a normal subgroup of Γ_P, Γ_U is a normal subgroup of Γ_Q. From Theorem 3.13 in the case $t = 2$, it follows that we have the exact sequence

$$0 \to U \cap \Gamma_0^{\pm}(n, N) \to P \cap \Gamma_0^{\pm}(n, N) \xrightarrow{\psi^P} \Gamma_0^{\pm}(\ell_1, d_1) \times \Gamma_0^{\pm}(n - \ell_1, N/d_1) \to 0.$$

From the same theorem we have

$$0 \to R_u(Q) \cap \Gamma_0^{\pm}(n,N) \to Q \cap \Gamma_0^{\pm}(n,N) \xrightarrow{\psi^Q} \prod_{i=1}^{t} \Gamma_0^{\pm}(\ell_i, d_i) \to 0,$$

where $R_u(Q)$ is the unipotent radical of Q.

Since ψ^P is given by conjugation with $\hat{g}_{d_1}$ and ψ^Q is given by conjugation with $g_D = \hat{g}_{d_{t-1}} \cdots \hat{g}_{d_1}$, it follows that we have the exact sequence

$$0 \to U \cap \Gamma_0^{\pm}(n,N) \to Q \cap \Gamma_0^{\pm}(n,N) \xrightarrow{\psi^P} \Gamma_0^{\pm}(\ell_1, d_1) \times (Q_1 \cap \Gamma_0^{\pm}(n-\ell_1, N/d_1)) \to 0,$$

where Q_1 is a representative parabolic subgroup of $\mathrm{GL}_{n-\ell_1}$ corresponding to the factorization $N/d_1 = d_2 \cdots d_t$.

In a similar way, from Theorem 3.15 we obtain a surjective map

$$Q \cap S_0^{\pm}(n,N) \xrightarrow{\psi^P} S_0^{\pm}(\ell_1, d_1)_N \times (Q_1 \cap S_0^{\pm}(n-\ell_i, N/d_1)_N.$$

Clearly, $x, y \in Q \cap S_0^{\pm}(n,N)$ have the same image under ψ^P if and only if $xy^{-1} \in U$.

Now let $\Gamma = \Gamma_Q(p,N)$ and $S = S_Q(p,N)$. Set $\Gamma_{L_1} = \Gamma_0(\ell_i, d_1)_N \cap \Gamma(p)$ and $\Gamma_{L_2} = \Gamma_{Q_1}(p, N/d_1)$. Then

$$\Gamma/\Gamma_U \cong \Gamma_{L_1} \times \Gamma_{L_2}.$$

There is a Lyndon-Hochschild-Serre spectral sequence for the exact sequence

$$1 \to \Gamma_U \to \Gamma \to \Gamma_{L_1} \times \Gamma_{L_2} \to 1,$$

namely

$$E_{ij}^2 = H_i(\Gamma_{L_1} \times \Gamma_{L_2}, H_j(\Gamma_U, M)) \implies H_{i+j}(\Gamma, M).$$

Lemma 5.11 says that $S \cap U = \Gamma \cap U$. Then by Lemma 8.1, the semigroup action of $S = S_P$ on $H_*(\Gamma_U, M)$, when restricted to S_U, is trivial. So S acts on $H_*(\Gamma_U, M)$, with S_U acting trivially.

Lemma 8.1(3) says that $\mathcal{H}(\Gamma, S)$ acts equivariantly on the spectral sequence, and a given packet of eigenvalues occurs in $H_k(\Gamma, M)$ if and only if it appears in the infinity page of the spectral sequence.

Because each term of the E^2-page of the spectral sequence is finite dimensional, we see that since Φ appears in the infinity page of the spectral sequence, it must appear in some term of the E^2-page.

So, we can now assume that Φ appears as a system of $\mathcal{H}(\Gamma, S)$-eigenvalues in some

$$H_i(\Gamma_{L_1} \times \Gamma_{L_2}, H_j(\Gamma_U, M)).$$

Let overline denote reduction modulo p. By the definition of $S = S_Q(p,N)$, we see that $\bar{S}/\overline{S_U}$ is isomorphic to a group of invertible diagonal matrices modulo p, and thus that $\overline{S}/\overline{S_U}$ is a finite abelian group, of order prime to p.

Since M is admissible, $M = M' \otimes \mathbb{F}_\epsilon$ as in Definition 6.1. Since any element of Γ_U is congruent to the identity modulo p, it acts trivially on M'. Since $S_U = \Gamma_U$ the S-action on M' factors through $\bar{S}/\overline{S_U}$ and therefore is diagonalizable. Therefore M' is isomorphic as an S-module to a direct sum of one-dimensional modules $\mathbb{F}_{\xi'}$ where ξ' runs over some sequence of characters of S. Moreover, each ξ' is trivial when restricted to Γ_U.

We now claim that ϵ is trivial on Γ_U. To see this, note that by the definition of admissibility, $\epsilon(s) = \eta(s_{11})$ for some character $\eta : (\mathbb{Z}/N)^\times \to \mathbb{F}^\times$. If $s \in \Gamma_U$, then $s \in U$ and the characteristic polynomial of s is $(x-1)^n$. On the other hand, since $s \in S_0(n, N)$, its characteristic polynomial mod N vanishes on s_{11}. Since N is squarefree, we deduce that $s_{11} \equiv 1$ modulo N, and therefore $\epsilon(s) = \eta(s_{11}) = 1$.

Let $\xi = \xi' \otimes \epsilon$. Then M is isomorphic as an S-module to a direct sum of one-dimensional modules $\mathbb{F}_\xi$ and each ξ is trivial when restricted to Γ_U.

By Theorem 6.4, for each ξ, $H_j(\Gamma_U, \mathbb{F}_\xi)$ is a (p, N)-admissible S-module and the S-action on $H_j(\Gamma_U, \mathbb{F}_\xi)$ is itself diagonalizable, so that $H_j(\Gamma_U, \mathbb{F}_\xi)$ is isomorphic as an S-module to a direct sum of one-dimensional modules $\mathbb{F}_\chi$ where χ runs over some sequence of characters of S.

Since $S \cap U$ acts trivially on $H_j(\Gamma_U, \mathbb{F}_\xi)$, it makes sense to view the latter as a module for $\Gamma_{L_1} \times \Gamma_{L_2}$. By linear algebra, we deduce that Φ appears as a system of eigenvalues in

$$H_i(\Gamma_{L_1} \times \Gamma_{L_2}, \mathbb{F}_\chi)$$

for some χ. We may factor $\chi = \chi_1 \cdot \chi_2$, as in the proof of [4, Thm 11.5]. Applying Corollary 7.7, we have that Φ appears in

$$\bigoplus_{r+t=i} H_r(\Gamma_{L_1}, \mathbb{F}_{\chi_1}) \otimes H_t(\Gamma_{L_2}, \mathbb{F}_{\chi_2})$$

and that Φ arises from some eigenvector $f_1 \otimes f_2$ with

$$f_i \in H_i(\Gamma_{L_i}, \mathbb{F}_{\chi_i}),$$

and each f_i an eigenvector of the appropriate Hecke algebra.

By Scholze [15], we know that f_1 has an attached Galois representation $\rho_1 : G_\mathbb{Q} \to \mathrm{GL}(k, \bar{\mathbb{F}}_p)$, and by the inductive hypothesis, f_2 has an attached Galois representation $\rho_2 : G_\mathbb{Q} \to \mathrm{GL}(n-k, \bar{\mathbb{F}}_p)$ that is a sum of at least $t-1$ irreducible constituents. As in [4, Theorem 10.2], we see that $f_1 \otimes f_2$ has $\rho_1 \oplus \omega^k \rho_2$ attached. Hence, $\rho_1 \oplus \omega^k \rho_2$ is reducible with at least t irreducible constituents. □

Theorem 8.5. *Let Q be a $\mathbb{Q}$-parabolic subgroup of $\mathrm{GL}(n, \mathbb{Q})$ of length t, and let $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$ or $(\Gamma_0^\pm(n, N), S_0^\pm(n, N))$. Let M be an irreducible (p, N)-admissible $\mathbb{F}[S]$-module, and let Φ be a system of $\mathcal{H}(\Gamma, S)$-eigenvalues occurring in $H_*(\Gamma_Q, M)$. Then there is a Galois representation ρ attached to Φ, and ρ has at least t irreducible constituents.*

Proof. Conjugating by an element of Γ, we may assume that Q is a representative parabolic subgroup of length t. By Theorem 5.7, $(\Gamma_Q(p, N), S_Q(p, N)) \subset (\Gamma_Q, S_Q)$ are compatible Hecke pairs, and by Lemma 4.4(b), $\Gamma_Q(p, N) \lhd \Gamma_Q$ with finite index, so by Theorem 8.2, Φ appears in

$$H_*(\Gamma_Q(p, N), M).$$

Finally, by Theorem 8.3, Φ has an attached Galois representation that is reducible with at least t irreducible constituents. □

9. The Tits building and its spectral sequence

Let X be a finite dimensional vector space over a field K with $n = \dim(X) \geq 2$. We review the construction of the Tits building T of X, and the Steinberg module $\mathrm{St}(X)$.

The Tits building is the $(\dim(X) - 2)$-dimensional simplicial complex T whose vertices are nonzero proper subspaces of X and whose i-simplices consist of sequences of $i+1$ nonzero proper subspaces $V_1, V_2, \ldots, V_{i+1}$ satisfying

$$0 \neq V_1 \subsetneq \cdots \subsetneq V_{i+1} \neq X.$$

We denote by T_i the free $\mathbb{Z}$-module generated by the i-simplices.

An i-simplex is a face of a j-simplex if and only if the subspaces comprising it are a subset of those comprising the j-simplex. We choose an orientation on each simplex by ordering the vertices of each simplex by containment.

There is a natural action of $\mathrm{GL}(X)$ on T inherited from its action on X. Note that the stabilizer of an i-simplex under this action is the parabolic subgroup of type $(n_1, \ldots, n_{i+2})$, with $n_1 + \cdots + n_{i+2} = n$, stabilizing the flag

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{i+1} \subsetneq V_{i+2} = X$$

corresponding to the i-simplex in question, and we set $n_j = \dim(V_j) - \dim(V_{j-1})$. Our chosen ordering of the vertices in a simplex is clearly preserved by the action of $\mathrm{GL}(X)$.

By the Solomon-Tits theorem [17], the reduced homology of the Tits building is trivial in all dimensions except dimension $n-2$. We define the Steinberg module

$$\mathrm{St}(X) = \widetilde{H}_{n-2}(T, \mathbb{Z}).$$

Note that the Steinberg module inherits an action of $\mathrm{GL}(X)$ from the action of $\mathrm{GL}(X)$ on T. The additive group of the Steinberg module is free abelian.

From now on, assume that $n \geq 3$. The homology of the Tits building in dimension 0 is $H_0(T, \mathbb{Z}) = \mathbb{Z}$. For our purposes, it will be better to use reduced homology; rather than using the homology of the complex

$$0 \to T_{n-2} \to T_{n-3} \to \cdots \to T_0 \to 0,$$

we use the homology of the augmented complex

$$0 \to T_{n-2} \to T_{n-3} \to \cdots \to T_0 \to \mathbb{Z} \to 0,$$

where $\mathbb{Z}$ is acted on trivially by $\mathrm{GL}(X)$. We will call this augmented complex C, and note that

$$C_i = \begin{cases} T_{i-1} & \text{for } i > 0, \\ \mathbb{Z} & \text{for } i = 0. \end{cases}$$

We then have that

$$H_i(C) = \begin{cases} \mathrm{St}(X) & \text{for } i = n-1, \\ 0 & \text{for } i \neq n-1. \end{cases}$$

For a subgroup $\Gamma \subset \mathrm{GL}(X)$ and $i > 0$, we can choose a set of representatives of Γ-orbits of simplices in $C_i = T_{i-1}$. Let $\mathcal{P}_i$ be the set of stabilizers in $\mathrm{GL}(X)$ of these flags; so $\mathcal{P}_i$ is a set of parabolic subgroups of $\mathrm{GL}(X)$. Note that $\mathcal{P}_1$ consists of maximal parabolic subgroups of $\mathrm{GL}_n(\mathbb{Q})$; in general, for

$i > 0$, $\mathcal{P}_i$ consists of stabilizers of flags of length $i+1$. For $i = 0$, we may take $\mathcal{P}_0 = \{\mathrm{GL}(X)\}$, since C_0 is generated by a single element stabilized by $\mathrm{GL}(X)$. With this notation, we find that as a Γ-module,

$$C_i = \bigoplus_{P \in \mathcal{P}_i} \mathrm{Ind}_{\Gamma \cap P}^{\Gamma} \mathbb{Z}.$$

From now on set $X = \mathbb{Q}^n$, and let (Γ, S) be a Hecke pair contained in $\mathrm{GL}(X) = \mathrm{GL}_n(\mathbb{Q})$ with $\Gamma \subseteq \mathrm{GL}_n(\mathbb{Z})$. Let W be a $\mathbb{Z}[S]$-module. We get a complex $C \otimes_{\mathbb{Z}} W$ on which S acts with the diagonal action. We also choose a projective resolution F of $\mathbb{Z}$ as a $\mathrm{GL}_n(\mathbb{Q})$-module.

Now the double complex $(C \otimes_{\mathbb{Z}} W) \otimes_\Gamma F$ has two spectral sequences associated to it, as in [10, VII.5, p. 169]. The first of these is obtained by taking the homology in the C direction first, so we get

$$E^1_{ij} = H_j(C \otimes_{\mathbb{Z}} W) \otimes_\Gamma F_i \cong \begin{cases} (\mathrm{St}(X) \otimes_{\mathbb{Z}} W) \otimes_\Gamma F_i & \text{if } j = n-1, \\ 0 & \text{if } j \neq n-1. \end{cases}$$

Here, we have used the universal coefficient theorem [18, Theorem 3.6.2], which states that

$$H_j(C \otimes_{\mathbb{Z}} W) \cong H_j(C) \otimes_{\mathbb{Z}} W \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{j-1}(C), W).$$

Since each $H_{j-1}(C)$ is either 0 or free abelian, the second term vanishes, and we get $H_j(C \otimes_{\mathbb{Z}} W) \cong H_j(C) \otimes W$.

At this point, the spectral sequence has collapsed, with nonzero terms only when $j = n-1$. To pass to page 2 of the spectral sequence, we take the homology in the F direction and find that the only nonzero terms on this page are

$$E^2_{i(n-1)} = H_i(\Gamma, \mathrm{St}(X) \otimes_{\mathbb{Z}} W).$$

Taking the abutment of the spectral sequence, we find that the total homology of the original double complex in dimension k is 0 for $k < n-1$, and is given by

$$H_{k-(n-1)}(\Gamma, \mathrm{St}(X) \otimes_{\mathbb{Z}} W),$$

for $k \geq n-1$.

For the second spectral sequence, we first take the homology in the F direction, and so obtain

$$\begin{aligned}
\mathcal{E}^1_{ij} &= H_i(\Gamma, C_j \otimes_{\mathbb{Z}} W) \\
&= H_i\left(\Gamma, \bigoplus_{P\in\mathcal{P}_j} \operatorname{Ind}^{\Gamma}_{\Gamma\cap P} \mathbb{Z} \otimes_{\mathbb{Z}} W\right) \\
&= \bigoplus_{P\in\mathcal{P}_j} H_i\left(\Gamma, \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma\cap P]} \mathbb{Z} \otimes_{\mathbb{Z}} W\right) \\
&= \bigoplus_{P\in\mathcal{P}_j} H_i\left(\Gamma, \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma\cap P]} W\right) \\
&= \bigoplus_{P\in\mathcal{P}_j} H_i\left(\Gamma, \operatorname{Ind}^{\Gamma}_{\Gamma\cap P} W\right) \\
&= \bigoplus_{P\in\mathcal{P}_j} H_i(\Gamma\cap P, W),
\end{aligned}$$

where we use Shapiro's lemma for the last step.

Since this spectral sequence also converges to the total homology of the double complex, we see that

$$\mathcal{E}^1_{ij} = \bigoplus_{P\in\mathcal{P}_j} H_i(\Gamma\cap P, W) \implies \begin{cases} 0 & \text{if } i+j < n-1, \\ H_{i+j-(n-1)}(\Gamma, \operatorname{St}(X)\otimes_{\mathbb{Z}} W) & \text{if } i+j \geq n-1. \end{cases}$$

We note that the action of S commutes with both differentials in the original double complex, so the action of the Hecke operators commutes with the differentials in each of the spectral sequences. In fact, if the S-orbits and Γ-orbits of flags in $\mathbb{Q}^n$ are equal, we see that the Hecke operators act on the individual summands of each $\mathcal{E}^1_{ij}$. Assume this equality of orbits (which holds in particular for $(\Gamma, S) = (\Gamma_0^{\pm}(n,N), S_0^{\pm}(n,N))$ and for $(\Gamma, S) = (\Gamma_0(n,N), S_0(n,N))$ by Theorem 3.1). Assume now that W is a (p,N)-admissible $\mathbb{F}[S]$-module. Then Theorem 8.5 implies that the Hecke eigenvectors in $\mathcal{E}^1_{ij}$ are attached to Galois representations that have at least $j+1$ irreducible constituents. In addition, any system of Hecke eigenvalues in $\mathcal{E}^1_{ij}$ that survives to the infinity page of the spectral sequence appears in the abutment, and hence for some $k = i+j \geq n-1$ in $H_{k-(n-1)}(\Gamma, \operatorname{St}(X)\otimes_{\mathbb{Z}} W)$.

We summarize all this in a theorem for the arithmetic groups we are especially interested in.

Theorem 9.1. *Let $(\Gamma, S) = (\Gamma_0^{\pm}(n,N), S_0^{\pm}(n,N))$ or $(\Gamma_0(n,N), S_0(n,N))$. Then there is a spectral sequence*

$$\mathcal{E}^1_{ij} = \bigoplus_{P\in\mathcal{P}_j} H_i(\Gamma\cap P, W) \implies \begin{cases} 0 & \text{if } i+j < n-1, \\ H_{i+j-(n-1)}(\Gamma, \operatorname{St}(X)\otimes_{\mathbb{Z}} W) & \text{if } i+j \geq n-1. \end{cases}$$

Let W be a (p,N)-admissible $\mathbb{F}[S]$-module. Then the Hecke eigenvectors in $\mathcal{E}^1_{ij}$ are attached to Galois representations that have at least $j+1$ irreducible constituents. Any system of Hecke eigenvalues that survives to the infinity *page of the spectral sequence appears in* $H_{k-(n-1)}(\Gamma, \mathrm{St}(X) \otimes_{\mathbb{Z}} W)$ *for some* $k = i + j \geq n - 1$.

Definition 9.2. We call the spectral sequence in Theorem 9.1 the *Tits spectral sequence.*

10. First consequences of the Tits spectral sequence

Definition 10.1. Say that a Galois representation *fits* a Hecke module Y if it is attached to a Hecke eigenvector in Y.

Theorem 10.2. *Let N be a positive squarefree integer prime to p, and let $n > 1$. Let $0 \leq k < n - 1$, let $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$ or $(\Gamma, S) = (\Gamma_0^{\pm}(n, N), S_0^{\pm}(n, N)$, and let M be an admissible $\mathbb{F}[S]$-module. Any Galois representation fitting $H_k(\Gamma, M)$ must be reducible.*

Proof. Let $0 \leq k < n - 1$. The $\mathcal{E}^1_{k0}$ term of the Tits spectral sequence with coefficient module M is equal to $H_k(\Gamma, M)$. Because the spectral sequence converges to the total homology of the double complex, which is 0 in degree less than $n - 1$, we see that all Hecke eigenvectors in $\mathcal{E}^1_{k0} = H_k(\Gamma, M)$ must fail to persist to the infinity page. However, they can only be killed by the differentials if there are Hecke eigenvectors in some $\mathcal{E}^*_{ij}$ with $j > 0$ with the same package of eigenvalues. Any Galois representation fitting such an $\mathcal{E}^*_{ij}$ must fit $\mathcal{E}^1_{ij}$, and since $j > 0$, it is reducible, by Theorem 8.5. Hence, any Galois representations fitting $H_k(\Gamma, M)$ must be reducible. □

Next, we use the Tits spectral sequence to prove a type of duality theorem for systems of eigenvalues appearing in homology and corresponding to irreducible Galois representations.

Theorem 10.3. *Let N be a positive squarefree integer prime to p, and let $n > 1$. Assume that p is not a torsion prime for $\Gamma_0^{\pm}(n, N)$. Let $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$ or $(\Gamma, S) = (\Gamma_0^{\pm}(n, N), S_0^{\pm}(n, N)$, and let M be an admissible $\mathbb{F}[S]$-module. If $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{F})$ is an irreducible Galois representation fitting $H_k(\Gamma, M)$ for some $k \in \mathbb{Z}$ and coefficient module M, then ρ also fits $H_{\ell-k}(\Gamma, M)$, where $\ell = (n + 2)(n - 1)/2$.*

Proof. Suppose that ρ is attached to $z \in H_k(\Gamma, M)$. We note that $H_k(\Gamma, M) = \mathcal{E}^1_{k0}$ in the Tits spectral sequence. Since ρ is irreducible, it cannot be killed off by elements of any $\mathcal{E}^1_{ij}$ with $j \geq 1$, since any eigenclass in $\mathcal{E}^1_{ij}$ is attached to a reducible Galois representation with at least $j + 1$ irreducible constituents. Hence z survives to the infinity page, and contributes to the abutment; hence, there is an eigenclass in $H_{k-(n-1)}(\Gamma, \mathrm{St}(\mathbb{Q}^n) \otimes M)$ with ρ attached. Then by Borel-Serre duality([9, Theorem 11.4.2] and [7, Theorem 2]) and [3, Lemma 2.4] (which says that if a Galois representation fits $H^i(\Gamma, M)$ it also fits $H_i(\Gamma, M)$), we find that ρ is attached to an eigenclass in $H_{\nu-k+(n-1)}(\Gamma, M)$, where $\nu = n(n - 1)/2$. Since $\nu + (n - 1) = \ell$, the theorem follows. □

Remark 10.4. This theorem can also be derived using Lefschetz duality and the homology of the Borel-Serre boundary of the locally symmetric space for Γ, given Theorem 8.5.

11. The Lyndon-Hochschild-Serre spectral sequence

We recall the main result of [2]. We will then use this in the next section.

Definition 11.1. Let P be a parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$ with unipotent radical U. Let Γ be a subgroup of $\mathrm{GL}_n(\mathbb{Z})$, and let $G = \Gamma \cap P$. A (U, p)-*admissible* G-module M is a G-module of the form $V \otimes \mathbb{F}_\epsilon$ where V is an irreducible module for $\mathbb{F}\mathbf{GL}_n(\mathbb{Z}/p)$ on which G acts via its reduction modulo p, and $\epsilon : G \to \mathbb{F}^\times$ is a character that is trivial on $G \cap U$.

Theorem 11.2. *Let Γ be a subgroup of $\mathrm{GL}_n(\mathbb{Z})$ determined by congruence conditions modulo an integer N such that p does not divide N. Let $\mathbf{P} = \mathbf{LU}$ be a maximal $\mathbb{Q}$-parabolic subgroup of $\mathbf{GL}_n$, where $\mathbf{U}$ is its unipotent radical and $\mathbf{L}$ is a Levi-factor. Let $G = \mathbf{P} \cap \Gamma$, $H = \mathbf{U} \cap \Gamma$, $Q = G/H$. Let M be a (U, p)-admissible G-module.*

(a) For any m, the natural map $H_m(G, M^H) \to H_m(G, M)$ is injective.

(b) If M' is any submodule of M, consider the LHS spectral sequence $E(M')$ with coefficients in M' for the exact sequence

$$1 \to H \to G \to Q \to 1.$$

Let d be the rank of the free abelian group H, and assume that $p > d + 1$. Suppose there is a nonzero $z \in E^2_{jd}(M^H) = H_j(Q, H_d(H, M^H))$ for some j. Then z survives to a nonzero element of $E^\infty_{jd}(M^H)$.

Let P be a representative maximal parabolic subgroup, and let $\Gamma = \Gamma_0(n, N)$ or $\Gamma = \Gamma_0^\pm(n, N)$. Given a character $e : \mathbb{Z}/N \to \mathbb{F}^\times$, we obtain a nebentype character $\epsilon : \Gamma \to \mathbb{F}^\times$ by defining $\epsilon(\gamma) = e(\gamma_{11})$. Using Theorem 3.3(1), and the fact that for $\gamma \in \Gamma \cap U$, both $\psi^1_d(\gamma)$ and $\psi^2_d(\gamma)$ are the identity, we see that ϵ is trivial on $\Gamma \cap U$, so that the (p, N)-admissible modules that we have been dealing with are (U, p)-admissible.

Corollary 11.3. *Let P be a representative maximal parabolic subgroup, set $G = \Gamma \cap P$, and define Q and H as in Theorem 11.2. Let $(\Gamma, S) = (\Gamma_0^\pm(n, N), S_0^\pm(n, N))$ and let M be an (N, p)-admissible S-module that is also (U, p)-admissible. Let $z \in H_j(Q, H_d(H, M))$ be an eigenclass of the Hecke algebra for the Hecke pair (Γ, S). Then there is an eigenclass $z' \in H_{j+d}(G, M)$ with the same system of Hecke eigenvalues as z.*

Proof. As in [4, Theorem 9.1], $H_d(H, M) \cong H_d(H, M^H)$ as S_P-modules. Hence, there is a Hecke-equivariant isomorphism

$$H_j(Q, H_d(H, M)) \cong H_j(Q, H_d(H, M^H)).$$

We identify z with its image under this map.

As in [4, Section 7], the Lyndon-Hochschild-Serre spectral sequence $E(M^H)$ is Hecke-equivariant. Since z persists to the infinity page of this spectral sequence by Theorem 11.2(b), z contributes to the abutment; hence, there is some $z' \in H_{j+d}(G, M^H)$ with the same system of Hecke eigenvalues as z.

The natural map $H_{j+d}(G, M^H) \to H_{j+d}(G, M)$ is Hecke equivariant, and by Theorem 11.2(a), it is injective. Identifying z' with its image, this yields the desired element of $H_{j+d}(G, M)$. $\square$

12. Sums of Galois representations

We now prove the main theorem of this paper. The term "predicted" was defined in Section 2. Recall that a Galois representation ρ is called "odd" if and only if p is even or p is odd and the eigenvalues of $\rho(c)$ are equal to $\pm(1,-1,1,-1,\dots)$, where c is complex conjugation.

Theorem 12.1. *Let n_1, n_2 be positive integers, $n = n_1 + n_2$, $p > \max(n_1 + n_2 + 1, n_1 n_2 + 1)$, and for $i = 1, 2$, let*

$$\rho_i : G_{\mathbb{Q}} \to \mathrm{GL}_{n_i}(\mathbb{F}_p)$$

be odd irreducible Galois representations such that:

1. *ρ_1 has predicted level N_1, predicted nebentype ϵ_1, and predicted weight $M_1 = F(a_1 + n_2, \dots, a_{n_1} + n_2)$, and ρ_2 has predicted level N_2, predicted nebentype ϵ_2, and predicted weight $M_2 = F(a_{n_1+1}, \dots, a_n)$;*
2. *the n-tuple $(a_1, \dots, a_n)$ is chosen so that $0 \le a_{n_1} - a_{n_1+1} \le p - 1$ and $N_1 N_2$ is squarefree;*
3. *ρ_1 is attached to a Hecke eigenclass in*
$$H_{s_1}(\Gamma_0^{\pm}(n_1, N_1), (M_1)_{\epsilon_1})$$
and $\rho_2 \otimes \omega^{-n_1}$ is attached to a Hecke eigenclass in
$$H_{s_2}(\Gamma_0^{\pm}(n_2, N_2), (M_2 \otimes \det{}^{-n_1})_{\epsilon_2}).$$

Set $k = s_1 + s_2 + n_1 n_2$ and $\nu = \frac{n(n-1)}{2}$. Let $M = F(a_1, \dots, a_n)$ and $\epsilon = \epsilon_1 \epsilon_2$.

Then $\rho_1 \oplus \rho_2$ is odd, and is attached to a Hecke eigenclass in at least one of

$$H_k(\Gamma_0^{\pm}(n, N_1 N_2), M_\epsilon)$$

or

$$H_{\nu-k+n-2}(\Gamma_0^{\pm}(n, N_1 N_2), M_\epsilon).$$

Moreover, $N_1 N_2$, ϵ, and M are the predicted level, nebentype, and a predicted weight for $\rho_1 \oplus \rho_2$.

Proof. We begin by demonstrating that under the given assumptions, $\rho_1 \oplus \rho_2$ is odd.

Note that by hypothesis p is odd. By the main result of [11], neither ρ_1 or ρ_2 can be even, so we may assume that both are odd. Also, if either n_1 or n_2 is even, then $\rho_1 \oplus \rho_2$ is automatically odd. So we may assume that both n_1 and n_2 are odd.

Let m be odd and let W be an irreducible $\mathrm{GL}_m(\mathbb{F}_p)$-module with central character η. Then $-I \in \Gamma_0^{\pm}(m, M)$, so it acts trivially on $H_*(\Gamma_0^{\pm}(m, M), W)$, but $-I$ also centralizes $\Gamma_0^{\pm}(m, M)$ so it acts on the homology only through its action on W via the central character η. Hence, if $H_*(\Gamma_0^{\pm}(m, M), W) \neq 0$, we must have $\eta(-I) = 1$.

Now, let c denote complex conjugation, and suppose that σ is attached to some eigenclass in $H_*(\Gamma_0^{\pm}(m, M), W)$. Then

$$\det \sigma(c) = (-1)^{m(m-1)/2} \eta(-I) = (-1)^{m(m-1)/2}$$

because of the attachment (see [5, Lemma 3.3]). By [11] we know σ is odd. If $m \equiv 1 \pmod 4$, then we have that $\det \sigma(c) = 1$, and we see that the alternating eigenvalues of $\sigma(c)$ must be $1, -1, \cdots, 1$. On the other hand, if $m \equiv 3 \pmod 4$, then $\det \sigma(c) = -1$, and again, the alternating eigenvalues of $\sigma(c)$ must be $1, -1, \cdots, 1$.

Apply this to ρ_1 and $\omega^{-n_1}\rho_2$. The eigenvalues of $\rho_1(c)$ are $1, -1, \ldots, 1$ and the eigenvalues of $\omega^{-n_1}(c)\rho_2(c)$ are also $1, -1, \ldots, 1$. Since $\omega(c) = -1$ and n_1 is odd, the eigenvalues of $\rho_2(c)$ are $-1, 1, \ldots, -1$. Therefore $\rho_1 \oplus \rho_2$ is odd.

We now proceed to the proof that $\rho_1 \oplus \rho_2$ is attached to a homology eigenclass.

Let $\Gamma^{\pm} = \Gamma_0^{\pm}(n, N_1N_2)$ and let $P = P_{N_1}^{n_1}$ be the representative maximal parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$ with unipotent radical U and Levi quotient L corresponding to the divisor N_1 of N_1N_2. Then $\Gamma_L^{\pm}$ is isomorphic to the direct product of the two components $\Gamma_{L_1}^{\pm} \cong \Gamma_0^{\pm}(n_1, N_1)$ and $\Gamma_{L_2}^{\pm} \cong \Gamma_0^{\pm}(n_2, N_2)$. Let $M_2' = M_2 \otimes \det^{-n_1}$. Since ρ_1 is attached to an eigenclass in

$$H_{s_1}(\Gamma_{L^1}^{\pm}, (M_1)_{\epsilon_1}),$$

and $\rho_2 \otimes \omega^{-n_1}$ is attached to an eigenclass in

$$H_{s_2}(\Gamma_{L^2}^{\pm}, (M_2')_{\epsilon_2}),$$

then by [4, Corollary 10.2], we see that $\rho_1 \oplus \rho_2$ is attached to an eigenclass in

$$H_{s_1+s_2}(\Gamma_L^{\pm}, (M_1 \otimes M_2')_{\epsilon}).$$

By [4, Theorem 9.1], we see that $M_\epsilon \cong H_{n_1n_2}(\Gamma_U^{\pm}, (M_1 \otimes M_2')_{\epsilon})$. Taking $G = \Gamma_P^{\pm}$, $H = \Gamma_U^{\pm}$, and thus $Q = \Gamma_L^{\pm}$, it follows that $\rho_1 \oplus \rho_2$ is attached to an eigenclass in

$$H_{s_1+s_2}(Q, H_{n_1n_2}(H, M_\epsilon)).$$

This homology group is the term $E^2_{s_2+s_2,n_1n_2}$ of the Lyndon-Hochschild-Serre spectral sequence for the exact sequence $1 \to H \to G \to Q \to 1$. Since n_1n_2 is the rank of the free abelian group H, and $p > 1 + n_1n_2$, Theorem 11.2 and Corollary 11.3 apply, and we see that $\rho_1 \oplus \rho_2$ is attached to an eigenclass z in

$$H_{s_1+s_2+n_1n_2}(\Gamma_P^{\pm}, M_\epsilon).$$

This homology group is a summand of the term $\mathcal{E}^1_{1,s_1+s_2+n_1n_2}$ of the first page of the Tits spectral sequence.

Note that z (or any nonzero descendants in higher pages of the spectral sequence) cannot be in the image of any of the higher differentials, since the higher differentials mapping to z would have source in column j of the spectral sequence for $j \geq 2$, and any eigenclass w in column j would be attached to a Galois representation with at least $j + 1 \geq 3$ irreducible constituents (by Theorem 8.5). Therefore the package of eigenvalues of w could not match with the eigenvalues of z. (This uses the Chebotarev density theorem and Scholze's theorem that w has an attached Galois representation.)

Hence, either z is in the kernel of the differential d_1 and survives to the infinity page of the spectral sequence, or else $d_1(z)$ is nonzero and hence $d_1(z)$ is an eigenvector in

$$\mathcal{E}_{0,s_1+s_2+n_1n_2} = H_{s_1+s_2+n_1n_2}(\Gamma^{\pm}, M_\epsilon)$$

that has $\rho_1 \oplus \rho_2$ attached.

If z survives to the infinity page, then we find that $\rho_1 \oplus \rho_2$ is attached to an eigenvector in the abutment, and hence to an eigenvector in

$$H_{s_1+s_2+n_1n_2+1-(n-1)}(\Gamma^{\pm}, \mathrm{St}(X) \otimes_{\mathbb{Z}} M_\epsilon).$$

Note that this can only happen when $s_1 + s_2 + n_1n_2 + 1 - (n-1) \geq 0$, or equivalently when $s_1 + s_2 + n_1n_2 > n - 2$. By Borel-Serre duality (which requires $p > n_1 + n_2 + 1$) and [3, Lemma 2.4] which permits us to switch homology and cohomology in the same degree, we find that $\rho_1 \oplus \rho_2$ is attached to an eigenvector in

$$H_{\nu-s_1-s_2-n_1n_2+n-2}(\Gamma^{\pm}, M_\epsilon).$$

Hence, we see that $\rho_1 \oplus \rho_2$ is attached to an eigenvector in either

$$H_{s_1+s_2+n_1n_2}(\Gamma^{\pm}, M_\epsilon)$$

or

$$H_{\nu-s_1-s_2-n_1n_2+n-2}(\Gamma^{\pm}, M_\epsilon).$$

The level and nebentype of $\rho_1 \oplus \rho_2$ are N_1N_2 and ϵ, and one easily checks that M is a predicted weight for $\rho_1 \oplus \rho_2$. □

We now apply this theorem in several examples. The first two of these examples have $n_1 = n_2$, and are already proven in [4, 5]. The remaining examples are new, taking advantage of the fact that our new spectral sequence deals easily with the case where $n_1 \neq n_2$.

In each of these examples, we will specify values for n_1, n_2, s_1, s_2, and assume that ρ_1 is attached to an eigenclass in $H_1(\Gamma_0^{\pm}(n_1, N_1), (M_1)_{\epsilon_1})$, and $\omega^{-n_1}\rho_2$ is attached to an eigenclass in $H_1(\Gamma_0^{\pm}(n_2, N_2), (M_2 \otimes \det^{-n_1})_{\epsilon_2})$ where N_i, ϵ_i, and M_i are predicted for ρ_i.

Example 12.2. Let $n_1 = n_2 = 2$ so that $n = 4$ and $\nu = 6$. Choose $p > 5$. Since ρ_1 and ρ_2 are irreducible, they can only fit homology with degree 1, so, we take $s_1 = s_2 = 1$. Then $s_1+s_2+n_1n_2 = 6$ and $\nu-s_1-s_2-n_1n_2+n-2 = 2$. Hence, we have that $\rho_1 \oplus \rho_2$ appears in $H_k(\Gamma_0^{\pm}(4, N_1N_2), (M_1 \otimes M_2)_{\epsilon_1\epsilon_2})$ with $k = 2$ or $k = 6$. This matches the result of [4, Theorem 12.1].

Example 12.3. Let $n_1 = n_2 = 3$ so that $n = 6$ and $\nu = 15$. Choose $p > 10$. Since ρ_1 and ρ_2 are irreducible, they fit homology only in degree 2 or 3 [1, Theorems 4.1.4 and 4.1.5], so we have that s_1 and s_2 are either 2 or 3. There are then three possibilities.

(1) If $s_1 = s_2 = 3$, then $k_1 = s_1 + s_2 + n_1n_2 = 15$ and $k_2 = \nu - s_1 - s_2 - n_1n_2 + n - 2 = 4$.
(2) If one of s_1, s_2 equals 3 and the other equals 2, then $k_1 = s_1 + s_2 + n_1n_2 = 14$ and $k_2 = \nu - s_1 - s_2 - n_1n_2 + n - 2 = 5$.
(3) If $s_1 = s_2 = 2$, then $k_1 = s_1 + s_2 + n_1n_2 = 13$ and $k_2 = \nu - s_1 - s_2 - n_1n_2 + n - 2 = 6$.

In each case, $\rho_1 \oplus \rho_2$ is attached to an eigenclass in

$$H_k(\Gamma_0^{\pm}(6, N_1N_2), (M_1 \otimes M_2)_{\epsilon_1\epsilon_2})$$

for $k = k_1$ or $k = k_2$. This is consistent with, but stronger than, the results of [5, Corollary 8.3], when we take into account the fact that ρ_1 is attached to homology in degree 3 if and only if it is attached to homology in degree 2 (by Theorem 10.3 above).

Example 12.4. Let $n_1 = 3$ and $n_2 = 2$, so that $n = 5$ and $\nu = 10$. Choose $p > 7$. Then, as above, $s_1 = 2$ or 3, and $s_2 = 1$. There are thus two cases:

(1) $s_1 = 3$, $s_2 = 1$. Then $k_1 = s_1 + s_2 + n_1 n_2 = 10$ and $k_2 = \nu - s_1 - s_2 - n_1 n_2 + n - 2 = 3$.
(2) $s_1 = 2$, $s_2 = 1$. Then $k_1 = s_1 + s_2 + n_1 n_2 = 9$ and $k_2 = \nu - s_1 - s_2 - n_1 n_2 + n - 2 = 4$.

In both cases, we have that $\rho_1 \oplus \rho_2$ is attached to an eigenclass in

$$H_k(\Gamma_0^{\pm}(5, N_1 N_2), (M_1 \otimes M_2)_{\epsilon_1 \epsilon_2})$$

for $k = k_1$ or $k = k_2$. This result is new, since the techniques of [4, 5] required $n_1 = n_2$.

Example 12.5. Let $n_1 = 5$ and $n_2 = 3$, so that $n = 8$ and $\nu = 28$. Choose $p > 16$. It is likely that an irreducible Galois representation can fit only cuspidal degrees of homology (this is plausible but not known for $n \geq 4$). Then ρ_1 may fit homology of degree 6, 7, or 8 (the cuspidal degrees for $n_1 = 5$) and ρ_2 may fit homology of degree 2 or 3. We thus get 6 possible cases, which we summarize in the following table:

s_1	s_2	k_1	k_2
6	2	23	11
6	3	24	10
7	2	24	10
7	3	25	9
8	2	25	9
8	3	26	8

For each row of the table, we find that if ρ_1 appears in degree s_1, and ρ_2 appears in degree s_2, then $\rho_1 \oplus \rho_2$ appears in

$$H_k(\Gamma_0^{\pm}(8, N_1 N_2), (M_1 \otimes M_2)_{\epsilon_1 \epsilon_2})$$

for $k = k_1$ or $k = k_2$. As in the previous example, this result is new.

If $n_1 = n_2 = 2$, we proved in [4] a version of Theorem 12.1 for $\Gamma_0(n, N)$ (as opposed to $\Gamma_0^{\pm}(n, N)$). The proof uses a theorem of Khare and Wintenberger (i.e. Serre's conjecture) plus some elementary properties of classical modular forms. The problem with proving a version of Theorem 12.1 for $\Gamma_0(n, N)$ when one or both of n_1, n_2 is even and greater than 2 is that we do not know that a package of Hecke eigenvalues for $\Gamma_0(n, N)$ also appears in $\Gamma_0^{\pm}(n, N)$. Although a similar statement is known for cuspidal automorphic forms, we don't know how to derive it for mod p homology. But if both n_1 and n_2 are odd, we can prove a version of Theorem 12.1 for $\Gamma_0(n, N)$, instead of $\Gamma_0^{\pm}(n, N)$. (If one of n_1 and n_2 is odd and the other equals 2 then there is a similar theorem, which we leave to the reader to formulate.)

Theorem 12.6. *Let n_1, n_2 be positive odd integers, $n = n_1 + n_2$, $p > \max(n_1 + n_2 + 1, n_1 n_2 + 1)$, and for $i = 1, 2$, let*

$$\rho_i : G_{\mathbb{Q}} \to \mathrm{GL}_{n_i}(\mathbb{F}_p)$$

be odd irreducible Galois representations such that $\rho_1 \oplus \rho_2$ is odd. Further, assume that

(1) ρ_1 has level N_1, nebentype ϵ_1, and one of its predicted weights is $M_1 = F(a_1 + n_2, \dots, a_{n_1} + n_2)$, and ρ_2 has level N_2, nebentype ϵ_2, and one of its predicted weights is $M_2 = F(a_{n_1+1}, \dots, a_n)$;

(2) that the n-tuple $(a_1, \dots, a_n)$ is chosen so that $0 \le a_{n_1} - a_{n_1+1} \le p-1$ and that $N_1 N_2$ is squarefree;

(3) ρ_1 is attached to a Hecke eigenclass in

$$H_{s_1}(\Gamma_0(n_1, N_1), (M_1)_{\epsilon_1})$$

and ρ_2 is attached to a Hecke eigenclass in

$$H_{s_2}(\Gamma_0(n_2, N_2), (M_2)_{\epsilon_2}).$$

Set $k = s_1 + s_2 + n_1 n_2$ and $\nu = \frac{n(n-1)}{2}$. Let $M = F(a_1, \dots, a_n)$ and $\epsilon = \epsilon_1 \epsilon_2$. Then $\rho_1 + \rho_2$ is attached to a Hecke eigenclass in at least one of

$$H_k(\Gamma_0(n, N_1 N_2), M_\epsilon) \quad \text{or} \quad H_{\nu-k+n-2}(\Gamma_0(n, N_1 N_2), M_\epsilon).$$

Moreover, $N_1 N_2$, ϵ, and M are the level, nebentype, and a predicted weight for $\rho_1 \oplus \rho_2$.

Proof. By [3, Lemma 2.4] and [6, Theorem 3.6], if a Galois representation τ fits $H_k(\Gamma_0(n, N), M_\epsilon)$ then $\tau \otimes \omega$ fits $H_k(\Gamma_0(n, N), M_\epsilon \otimes \det)$.

Hence, since n_1 is odd, we may assume (by twisting both ρ_1 and ρ_2 by ω if needed – this uses the fact that the main conjecture of [6] is preserved under twisting by ω, which is [6, Theorem 3.6]) that $-I$ acts trivially on M_1. By the same argument as at the beginning of the proof of Theorem 12.1, we see that the eigenvalues of $\rho_1(c)$ are $1, -1, \dots, 1$. Since $\rho_1 \oplus \rho_2$ is odd the eigenvalues of $\rho_2(c)$ must be $-1, 1, \dots, -1$. Since n_1 is odd, the eigenvalues of $(\omega^{-n_1}\rho_2)(c)$ are $1, -1, \dots, 1$. Therefore $-I$ also acts trivially on $M_2 \otimes \det^{-n_1}$.

Then by [5, Theorem 3.5], we see that ρ_1 is attached to an eigenclass in

$$H_{s_1}(\Gamma_0^{\pm}(n_1, N_1), (M_1)_{\epsilon_1}),$$

and $\omega^{-n_1}\rho_2$ is attached to an eigenclass in

$$H_{s_2}(\Gamma_0^{\pm}(n_2, N_2), (M_2 \otimes \det{}^{-n_1})_{\epsilon_2}).$$

So by Theorem 12.1, $\rho_1 \oplus \rho_2$ is attached to an eigenclass in at least one of

$$H_k(\Gamma_0^{\pm}(n, N_1 N_2), M_\epsilon) \quad \text{or} \quad H_{\nu-k+n-2}(\Gamma_0^{\pm}(n, N_1 N_2), M_\epsilon).$$

Finally, by use of the corestriction from Γ_0 to $\Gamma_0^{\pm}$ as in the proof of [4, Theorem 10.4], we obtain that $\rho_1 \oplus \rho_2$ is attached to an eigenclass in at least one of

$$H_k(\Gamma_0(n, N_1 N_2), M_\epsilon) \quad \text{or} \quad H_{\nu-k+n-2}(\Gamma_0(n, N_1 N_2), M_\epsilon). \qquad \square$$

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