2 - Permutations and Combinations

Permutations. In how many ways can I draw a five-card hand from a deck of fifty-two cards? Assume, for the moment, that we are very interested in the *order* in which the five cards are drawn. Thus the sequence $9\clubsuit, A\heartsuit, 9\diamondsuit, 3\clubsuit, 5\spadesuit$ would be counted as a *different* result than if we drew $3\clubsuit, 9\diamondsuit, 9\clubsuit, 5\spadesuit, A\heartsuit$, even though the same five cards are drawn in both cases. Such ordered *sequences* of distinct elements chosen from a given set are sometimes called *permutations*. They are also called *ordered selections without replacement* (since the chosen cards are not put back into the deck before the next card is drawn).

Of course we're more interested in discovering a formula than in this particular five-card problem, but before making generalizations, it is often wise to investigate a smaller version of the problem, where we can actually list the items to be counted. Suppose I have a deck with just four cards, labelled 1, 2, 3 and 4. Suppose I intend to draw three cards, in order, from the *set* $\{1, 2, 3, 4\}$. In how many ways can I do it? In order to keep track, and to make sure I don't list any possibility more than once, I will list the results *as if they were words* in alphabetical order:

1	2	3
1	2	4
1	3	2
1	3	4
1	4	1
1	4	3
2	1	3
2	1	4
2	3	1
2	3	4
2	4	1
2	4	3
3	1	2
3	1	4
3	2	1
3	2	4
3	4	1
3	4	2
4	1	2
4	1	3
4	2	1
4	2	3
4	3	1
4	3	2

Thus there are 24 ways to select three items in order from a set of four, without replacement. This table is simple, with obvious patterns, but *please take the time to stop and study it at length.*

Problems which are far more complicated than this one can often be solved by visualizing the resulting list arranged in a patterned way – even if you can't actually write down the list. Practice visualizing small lists like this, which you can actually write down, and you'll soon learn to visualize astronomically long lists *arranged in sensible patterns.*

What are the patterns in the list above? Note that, each starting pair (like 3 1), appears twice in the table (as in 3 1 2 and 3 1 4) – because there are two elements remaining in the original set (2 and 4 in this case), to be be drawn for the third position.

Similarly, for each element which can appear in the first position (e.g. 3), there are three possibilities for the second element (as in 3 1, 3 2, and 3 4), each of which then has two completions by the argument in the previous paragraph. Thus, altogether, there are four times three times two possible ordered selections of three things (without replacement) from our set $\{1, 2, 3, 4\}$. By a similar argument, you can now solve the problem of selecting an ordered 5-card hand out of a 52-card deck, but it seems redundant – since we can now write down a much more general formula.

Suppose we wish to choose from a set of n things, an ordered list of m distinct elements. Clearly, there are n choices for the first element of our list, and $n-1$ choices for the second. Thus there are $n(n-1)$ pairs to put in the first two positions. Then there are $n-2$ choices for the third position. By *induction*, though we won't do it formally, we may therefore prove that the number of ways to make our ordered selection is $n(n-1)(n-2)\cdots(n-m+1)$. The reader should explain why the smallest factor is $(n-m+1)$ instead of $(n-m)$. This expression gives the number of *m-element sequences* chosen from a set of n things, which we denote by

$$P(n, m) = n(n-1)(n-2)\cdots(n-m+1).$$

Thus $P(4, 3) = 24$ and $P(52, 5) = 311875200$.

In the case where $n = m$, these are simply called *permutations* of the set. The number of permutations of a set of n things is

$$P(n, n) = n(n-1)(n-2)\cdots 2 \cdot 1 = n!$$

We use the *factorial* notation $n!$ to denote the product of all positive integers up to and including n.

Exercise 2.1. Show that $P(n, m) = \frac{n!}{(n-m)!}$ for $n > m$.

Since factorials are important in combinatorics, we examine them more closely. Note if we try to use the formula from Exercise 2.1 to compute the number of permutations of n things, we get $P(n, n) = \frac{n!}{0!}$, where the answer should be simply $n!$. Does this mean we're extending the formula for $P(n, m)$ where it isn't applicable, or can we simply define $0! = 1$, and make the formula work even when $n = m$? There are reasons why the latter solution is not only practical, but logical. First, since $n! = n \cdot (n-1)!$, then $1! = 1 \cdot 0!$ (requiring that $0! = 1$). A somewhat more compelling reason is provided by the following exercise:

Exercise 2.2. Assume $0! = 1$, then prove, by induction, that

$$\int_0^\infty x^n e^{-x}\, dx \;=\; n!$$

for $n \geq 0$. [Hint: Use integration by parts.]

Note that the improper integral from Exercise 2 converges for every non-negative, real n, even if n is not an integer. Thus we may calculate factorials of positive real numbers which aren't integers – an interesting and useful fact. For example, the n-volume of an n-dimensional sphere of radius r can be shown to be $\left(\frac{\pi^{(n/2)}}{(n/2)!}\right) r^n$. Note that the 1-volume of a 1-sphere (an interval) is its length, the 2-volume of a 2-sphere (circle) is its area, etc. Check this formula out for the cases $n = 1, 2, 3$. [Note: It can be shown by any of several methods, that $\int\limits_0^\infty \sqrt{x} e^{-x}\, dx = \frac{1}{2}\sqrt{\pi}$.]

Exercise 2.3. Find the formula for the (four-dimensional) volume of a four-dimensional sphere of radius r.

Combinations. Now suppose, as we draw five cards from our fifty-two card deck, that we no longer care in what order they are drawn. In that case, the sequence $9\clubsuit, A\heartsuit, 9\diamondsuit, 3\clubsuit, 5\spadesuit$ would be effectively the same as if we drew $3\clubsuit, 9\diamondsuit, 9\clubsuit, 5\spadesuit, A\heartsuit$, since we may rearrange them in our hand however we wish. When order doesn't matter, we call this an *unordered selection without replacement.* We see that, in fact, we are talking about *sets* and *subsets.* Thus our hand $\{9\clubsuit, A\heartsuit, 9\diamondsuit, 3\clubsuit, 5\spadesuit\}$ is a subset of the 52-card deck. How many such five-element subsets are there?

We may answer this question by referring to the permutation problem we've already solved. We know there are $P(52, 5) = \frac{52!}{47!} = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311875200$ five card sequences which can be drawn from a 52-card deck. Each five card *subset* will be repeated many times in that list of sequences – as many times as it can be rearranged. In how many ways can a five-card sequence be rearranged? That's just the number of permutations of a set of five things, namely 5!. Thus if we divide the number of 5-card sequences by 5!, to account for repetitions, we should get precisely the number of 5-element subsets of our 52-card deck. More generally:

Theorem 2.1 *If n and m are non-negative integers, with $n \geq m$, the number of m-element subsets of an n-element set is given by*

$$C(n, m) = \frac{P(n, m)}{m!} = \frac{n!}{(n-m)!\, m!}.$$

Proof. Since there are $P(n, m)$ m-element sequences, and each m-element subset will appear as a sequence $m!$ times, the claim is established.

Note, subsets are sometimes called *combinations*, thus the notation $C(n, m)$. We also often use the notation $\binom{n}{m}$ instead of $C(n, m)$. We shall use both notations, so that students will get used to them (as they are both in common usage).

Exercise 2.4. How many five-card hands (unordered) can be drawn from a deck of 52 cards?

Exercise 2.5. How many n-digit sequences of zeros and ones (such as 0010110010...1101) have exactly k 1's? Explain.

Exercise 2.6. If n is a non-negative integer, prove that $C(n, 0) = C(n, n) = 1$.

Exercise 2.7. If n and m are non-negative integers,with $n \geq m$, prove that $\binom{n}{m} = \binom{n}{n-m}$.

Exercise 2.8. If n and m are positive integers, with $n > m$, prove that $C(n, m) = C(n-1, m) + C(n-1, m-1)$. (This is called Pascal's identity.)

Exercise 2.9. If n is a non-negative integer, prove that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$. (We'll prove this another way later.)

Exercise 2.10. If n and k are positive integers, and $k \leq [\frac{n}{2}]$, prove that $C(n, k) > C(n, k-1)$. Note that $[x]$ denotes the integer part of x.

Exercise 2.11. If n and k are positive integers, and $0 \leq k \leq n$, prove that $\binom{n}{[\frac{n}{2}]} \geq \binom{n}{k}$.