Lecture Notes on
The History of Mathematics

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Numbers and Counting

General

- Counting is arguably the most fundamental/primitive mathematical action.
  - Modern science of combinatorics. Counting arrangements.
    - ⋆ E.g., how many ways can 67 be written as the unordered sum of positive integers?
  - More generally, assessing the quantity of objects in a collection.
    - ⋆ In competition for most basic with assessing the magnitude of a physical quantity.
      - · Difference is discreteness versus continuity.
      - · Parallel to time versus space.
      - · Which do we see first in children?
    - ⋆ Story of comparing hairs on two animals
      - · 1-to-1 correspondence
      - · cardinality
    - ⋆ 1-to-1 correspondence with
      - · physical tokens (pebbles, finger-positions)
      - · sounds
      - · written symbols (numerals)
- Natural Numbers = $\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$ (abstract quantity, not the symbol)
- Prehistoric
  - Tallies
  - Off-topic by definition
  - Claims are:
    - ⋆ Speculative
    - ⋆ Tentative
    - ⋆ Unconvincing
  - Calinger: "far more fundamental than language"
    - ⋆ Not terribly advanced animals have ability to count into the high single digits.
      - · Anecdote about crows being able to count to five but not six.
      - · The canonical (but not only) work is *The Number Sense* by Stanislas Dehaene.
- For the historic era, we now address the question, what numeral systems did various ancient civilizations develop and when? In a bow to history over heritage, we look at:
  - Mesopotamia
  - Egypt
  - Indus Valley
  - Yellow River
  - Greek
  - Mesoamerican
  - Roman
  - Independence? Still a matter of dispute.

Mesopotamia

- Babylonian?
- Land between the rivers: Tigris and Euphrates
- Present-day Iraq
- Oldest extant writings: ca. 3000 BC (cuneiform=wedge)
- Oldest purely mathematical texts date to Old Babylonian or Hammurabic period (ca. 1900 BC)
- Claims that system developed by 2350 BC
- Sexagesimal (base 60)
  - Remnants in our measurement of time and angle (DMS)
  - Why?
    - ⋆ Avoid messy fractions.
    - ⋆ 6 times 60 is close to 365???
- Number ideograms (abstract symbols)
  - Not just tallies
- Place-value notation
  - Originally no place-holder.
  - Relied on context.
  - Or left a gap.
  - Seleucid period (300 BC): medial placeholder: omicron = o
  - No terminal placeholder
- Wedge = 1 = $\curlyvee$
- Boomerang = crescent = 10 = $\prec$

Egyptian

- Nile river
- Old Kingdom started in about 2700 BC
- Oldest mathematical scrolls and tablets date to about 2000 BC
- Three number systems
  - Decimal
  - Hieroglyphic system
    - ⋆ Symbols for $10^n, n = 0 \ldots 6$
    - ⋆ Stroke, loaf, coil, lotus flower, bent thumb, fish, man with raised hands
    - ⋆ Additive
    - ⋆ Typically right to left
  - Hieratic and Demotic systems
    - ⋆ Ciphered = symbols for $b^n k$, base = b, $k < b$
      - · Intrinsically additive
      - · Advantage: compact
      - · Disadvantage: Lots of symbols

Indus Valley

- Modern day Pakistan
- Aryans settled in about 1500 BC (supplanting previous civilization in place for a millennium).
- Sanskrit
- Brahmi numerals for 1 through 9 date to about 300 BC
  - Evolved into ours through time.
    - ⋆ Adopted by Arab/Islamic Mathematicians
    - ⋆ Introduced to west in Iberia and in studying Arab Mathematicians
    - ⋆ Won us over by 1500 AD
  - But they used in cipher system
- Place-value
  - Sometime in the next 1000 years.

Yellow River

- Xia dynasty started about 2100 BC
- Like India, ancient Chinese math seems to have little intermixing, unlike other Eastern hemisphere civilizations.
  - Almost totally independent until 400 BC.
  - Principally indigenous until 17th century.
- Written math dates to 1300 BC. (Late Shang dynasty)
- Two number systems
  - Multiplicative Grouping (Shang and later)
    - ⋆ Symbols for 1-9 (1-3 are easy) and powers of 10
    - ⋆ Communicate same as place-value more explicitly but less efficiently.
- Counting-rod numerals (Han, 100 BC)
  - Looks like ...
  - Different for odd or even place

Greek

- Sparta settled in 1200 BC
- Two number systems
  - Attic
    - ⋆ 500 BC
    - ⋆ Acrophonic (used first letters of corresponding words)
    - ⋆ Additive like Hieroglyphic and Roman
  - Ionian
    - ⋆ Traced back to 450 BC, won out about 300 BC
    - ⋆ Cipher
    - ⋆ Used Greek alphabet (plus 3 old letters) in cipher system
    - ⋆ How does this extended system in order match up with ours?
    - ⋆ Primes

Mesoamerican

- Olmec = 1200 BC
- Maya = 300 BC
- Numeral system
  - Unknown date of origin
  - Vigesimal = base 20
    - ⋆ But $18 \cdot 20$, $18 \cdot 20^2$, etc.
  - Reminiscent of ? Chinese counting-rods? Mesopotamian?
  - Stacked vertically
  - Had a zero

Roman

- 753 BC
- Threw out the Etruscans in 509 BC
- Roman numerals
  - I, V, and X borrowed from Etruscans
  - L, C, D, and M come after 100 BC
  - Additive
  - Calinger: "Subtractive principle was used only sparingly in ancient and medieval times"

Counting Basic Astronomical Events

- What are the three most basic in order of most obvious?
  - Day
  - Lunar Month
  - Year
- Why don't we think much about calendars and motions of moon, earth, and sun?
  - Given to us
  - Poor visibility of night sky
  - Non-agricultural
  - Other things to think about!

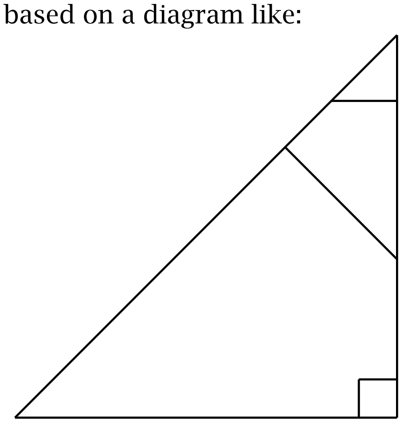
Other Numbers

Positive Rationals

- Why are they natural objects to consider?
  - Sharing.
  - The problem of measurement.
- Around 1000 BC, the Egyptians used unit fractions (and $2/3$).
  - Unit fractions were written with an ellipse over the denominator.
  - Writing proper fractions as sum of distinct unit fractions is an interesting exercise.
    - ⋆ Nonuniqueness: $1/n = 1/(n+1) + 1/(n(n+1))$
    - ⋆ Simplicity
      - · $2/n$ table in the A'hmosè Papyrus for odd n from 3 to 101.
        1. Frequently uses the identity
        $$\frac{2}{n} = \frac{1}{(n+1)/2} + \frac{1}{n(n+1)/2}.$$
        2. When possible, it uses
        $$\frac{2}{3m} = \frac{1}{2m} + \frac{1}{6m}.$$
        3. Multiplication based on doubling made this table especially handy.
- In ancient Babylon, the sexagesimal place-value notation extended to fractions.
  - Boyer: "the best that any civilization afforded until the time of the Renaissance."
- Ancient Greeks were fine with geometric concept of a ratio but didn't really work with common (as opposed to unit) fractions notationally.
- By 100 AD, the Chinese were handling fractions much as we do today.
- The use of decimal fractions really got entrenched in the West around 1600. (This late date would help explain the old British monetary system, etc.)

Irrationals

- That 2 and other integers weren't the square of any positive rational number was discovered by the Greeks around 400 BC.
  - Arguably precipitated by their lousy number system. (Compare to epicycles.)
  - Evidence seems to indicate that our standard algebraic proof (known to Aristotle) wasn't used.
  - Possibly a geometric proof based on a diagram like:



    and a calculation like $p^2 = 2q^2 \Rightarrow p/q = (2q-p)/(p-q)$.
- This indicated that there were irrational lengths, but it didn't explain what all lengths *do* look like.

Zero

- Recall first used as a medial place-holder (but not alone) by Babylonians in 300 BC.
- Mayans also had it about 300 BC.
- Ptolemy used a genuine zero in 130 AD in a hybrid Greek/Babylonian system.
- Brahmagupta in 628 AD used first decimal zero, and this got transmitted to Arabian mathematicians and subsequently to the West. (Brahmagupta himself was influenced by the Babylonians' medial zero and Ptolemy's sexagesimal zero.)

Negative Numbers

- Hard for us to imagine doing without them, but there is a sense in which it is natural do so. We have been trained to see their usefulness. (Flatland?)
- Needed for solving linear equations.
- Useful in commerce and were thus used before completely accepted by mathematicians.
- Used in first century BC by Chinese. (Before they had zero!) Red and black rods for positive and negative, respectively, (reverse of us) but a rod of length zero easily gets lost.
- Not used by medieval Arabic scholars.
- Not fully accepted in Europe until much later (17th century). (Consequently there were cubic and quartic formula*s*.)
- How could -1 be to 1 as 1 is to -1?
- Often called "false", "absurd", "fictitious", or "impossible". (Compare to "extraneous".)

Complex Numbers

- Leibniz called them half-existent, half-nonexistent and likened them to the Holy Ghost.
- Complex numbers were accepted *before* real numbers were well-understood. (Common theme. E.g., Euler was doing an awful lot of advanced stuff with integration before Riemann's integral was defined.)
- Stemmed not from a desire to say every number has a square root (or every quadratic equation has a solution) but to make sense of formulas for *real* solutions of cubic equations that contained square roots of negative numbers. See author's example and quote on next to last paragraph of p. 206.
- See author's smart-alec remark on p. 207 about Wallis on negative and imaginary numbers and tracts of land re-reclaimed by the sea.
- Euler gave us his formula $e^{i\theta} = \cos\theta + i\sin\theta$ in 1727 (or 1748) and, later, the use of i. It was a Norwegian surveyor who gave us the geometrical interpretation in 1797. (Or was it? Kline gives us Wallis as the originator 100 years earlier. Wallis apparently just dealt with purely imaginary numbers.) And Gauss's name is associated with it.
- Hypercomplex numbers discovered by Hamilton in 1843. Higher-dimensional analogue of complex numbers. Hamilton carved the formula $i^2 = j^2 = k^2 = ijk = -1$ on bridge in Dublin (now effaced) he was crossing when he came up with it. Used in control theory, etc., but largely supplanted by vectors.

Real Numbers

- Real analysis studies more pathological functions than complex analysis, so more secure footing was necessary.
- Notion of (positive) real number arose early as a ratio of lengths.
- Existence of transcendental (non-algebraic) real numbers:
  - $1/10 + 1/10^2 + 1/10^6 + 1/10^{24} + \cdots$ (Liouville, 1851)
  - e (Hermite, 1873)
  - π (Lindemann, 1882)
- Dedekind's geometric construction *published* in 1878.
- Dedekind's method had roots in work done 1500 years earlier by Eudoxus: $D_1 : S_1 = D_2 : S_2$ iff for every $m, n \in \mathbb{N}$

$$mD_1 \begin{bmatrix} < \\ = \\ > \end{bmatrix} mD_2 \Leftrightarrow nS_1 \begin{bmatrix} < \\ = \\ > \end{bmatrix} nS_2.$$

- Cantor's analytic construction (about 1872)
- What's wrong with axiomatic approach?
- What's wrong with infinite decimals?

Infinite Ordinals and Cardinals

- For finite ordinals and cardinals, the distinction is linguistic and conceptual, not mathematical.
- Transfinite ordinal numbers were introduced by Cantor in 1883. Order types of well-ordered sets. Noncommutative arithmetic. (More to come on the reason for their discovery.)
- Transfinite cardinal numbers were introduced by Cantor in 1895. Some algebraic rules.

Hyperreal Numbers

- More or less invented by Abraham Robinson in 1960s to allow working with differentials in a rigorous matter. "Nonstandard analysis." Not widely used. (Maybe more to come.)

Computation

General

- I mean mainly addition, subtraction, multiplication, and division. (Not algebraic problems.)
- Abacus
- Slide Rule
- Mechanical devices post slide rule belong in History of Computer Science class.

Egyptians

- Example of Multiplication through doubling and Peasant Multiplication
- Example of Division with and without fractions.

Babylonians

- Boyer: "It is clear that the fundamental arithmetic operations were handled by the Babylonians in a manner not unlike that which would be employed today, and with comparable facility."
- Square Root Procedure

Romans

- Illustrate multiplying Roman numerals abacus-like. Really nothing to memorize. No scratch paper necessary.

Arabians

- Do multiplication example. (Different orientations in different books.)

Europe

- Eves: "Many of the computing patterns used today in elementary arithmetic, such as those for performing long multiplications and divisions, were developed as late as the fifteenth century."
  - Media: Pencil-and-paper arithmetic requires paper?
    - Eves: "[O]ur common machine-made pulp paper is little more than a hundred years old."
    - Eves: "The older rag paper was made by hand, was consequently expensive and scarce, and even at that, was not introduced into Europe until the twelfth century."
    - Papyrus = Matted reeds dried and smoothed.
    - Parchment = skin of sheep
    - Vellum = skin of calves
  - When were Arabic numerals adopted?
  - When were decimal fractions adopted?

Number Theory

General

- It is at this point that we encounter a change: people working on mathematics of no obvious value to common people doing their daily work.
- Number theory is used in cryptography and cosmology. Strong students with knowledge of number theory are in demand at the National Security Agency. But these uses came late and as an afterthought.
- What is/was the justification?
  - Stobaeus, *Extracts*: “A youth who had begun to read geometry with Euclid, when he had learnt the first proposition, inquired, ‘What do I get by learning these things?’ So Euclid called a slave and said ‘Give him threepence, since he must make a gain out of what he learns.”’
  - Henri Poincare, *The Value of Science*: “The search for truth should be the goal of our activities; it is the sole end worthy of them. Doubtless we should first bend our efforts to assuage human suffering, but why? Not to suffer is a negative ideal more surely attained by the annihilation of the world. If we wish more and more to free man from material cares, it is that he may be able to employ the liberty obtained in the study and contemplation of truth.”
  - Analogy to music
  - The ability to think straight and to appreciate that ability.
  - Ascertaining things true in every possible universe.
- Number theory is the study of the properties and relationships of the natural numbers.
- “Arithmetic” versus “Logistic” versus “Number Theory”
- Plimpton 322? Babylonians were interested in number theory (Diophantine equations) more than 1500 years before Christ.
- Least coherent, most understandable, and hardest to justify of all branches of mathematics?

Important Persons and their Publications

- Pythagoras: 6th century BC
  - Everything we have is second-hand.
- Euclid: *Elements, Books 7, 8, 9* (300 BC)
  - Not just geometry.
  - Language very unfamiliar (and not just because it’s Greek).
- Nichomachus: *Introductio arithmeticae* (100)
  - Contains more of "Pythagorean lore" than does *Elements.*
  - Errors, few innovations, standard text for 1500 years.
- Diophantus: *Arithmetica* (250)
- Fermat: 17th century
- Euler: *Anleitung zur Algebra* (1770)
- Legendre: *Théorie des nombres* (1798)
- Gauss: *Disquisitiones arithmeticae* (1801)
- Dirichlet: 19th century
- Riemann: 19th century
- Wiles: 20th century

Definitions

- Background: "Natural numbers" means "numbers".
- $m|n$, m *divides* n, m is a *factor* of n, m is a *divisor* of n, n is a *multiple* of m: There is a number q such that $qm = n$.
- m is *prime* if it has exactly two divisors. (Thus, 1 is not prime.)
- m is *composite* if it has more than 2 divisors. (Thus, 1 is not composite.)
- m and n are *relatively prime* if 1 is their greatest common divisor.

Figurate Numbers

- The Pythagoreans were also impressed by numbers that naturally correspond to simple geometric figures.
- Triangular numbers
- Square numbers
- Etc. (Cooke is wrong about the regularity of these?)

Perfect Numbers

- Ascribed to Pythagoras (6th century BC).
- m is *perfect* if the sum of its divisors other than itself equals itself.
- m and n are *amicable* if the sum of m's divisors other than m equals n, and the sum of n's divisors other than n equals m.
- Given mystical significance by the Pythagoreans.
- Examples 6 and 28 are (individually) perfect. 284 and 220 are amicable.
- Progress on Perfect Numbers
    - Proposition 36 in Book IX of *Elements* is that if $2^n - 1$ is prime then $(2^n - 1)2^{n-1}$ is perfect.
    - Proof is by listing.
    - Euler showed that *all* even perfect numbers are given by Euclid's formula.
        - ⋆ V. Lebesgue's Proof
    - A prime number of the form $2^n - 1$ is called a *Mersenne Prime*.
        - ⋆ Why must n be prime for $2^n - 1$ to be prime?
        - ⋆ What's the smallest prime n such that $2^n - 1$ is *not* prime? (11)
    - There are 43 know Mersenne primes, and therefore 43 known even perfect numbers.
    - The largest known even perfect number is $(2^{30,402,457} - 1)2^{30,402,457-1}$.
    - It is not know if there are any odd perfect numbers.
        - ⋆ In 2003, former BYU student (and future BYU professor) Paul Jenkins proved that every odd perfect number has a prime factor bigger than 10^7.
        - ⋆ In 2003, former BYU student (and future BYU professor) Pace Nielsen proved that if an odd perfect number N has k distinct prime factors, then $N < 2^{(4^k)}$.
- Progress on Amicable Numbers
    - Fermat discovered the second pair of (imperfect) amicable numbers (17,296 and 18,416) in 1636.
    - René Descartes discovered the third pair.
    - Euler lengthened the list to sixty pairs.
    - As of June 5, 2006, there were 11,222,079 known pairs.
    - As of October 5, 2005, the largest know pairs had 24,073 digits each.

Pythagorean Triples

- Is more or less Proposition 29 in Book X of *Elements* (300 BC).
- A *Pythagorean triple* is a triple of numbers (a, b, c) such that $a^2 + b^2 = c^2$.
- A Pythagorean triple is *primitive* If a, b, and c have no nontrivial common factors.
- The Theorem: Every primitive Pythagorean triple is of the form $(2mn, m^2 - n^2, m^2 + n^2)$ (or $(m^2 - n^2, 2mn, m^2 + n^2)$) for some numbers $m > n$. Conversely, If $m > n$ are numbers, the formula $(2mn, m^2 - n^2, m^2 + n^2)$ always produces a (not-necessarily primitive) Pythagorean triple.
- The Proof

The Euclidean Algorithm

- Is more or less Propositions 1 and 2 in Book VII of *Elements* (300 BC).
- The Theorem: The greatest common factor of two numbers can be found by repeating the following process:
  - Divide the smaller number into the larger number.
  - If the remainder is zero, the smaller number is the GCD.
  - If the remainder isn't zero, replace the larger number by the smaller number and the smaller number by the remainder, and go back to the first step.
- It presumes the division algorithm.
- Example using Cooke's notation.

The Fundamental Theorem of Arithmetic

- Is more or less Proposition 14 in Book IX of *Elements* (300 BC).
- An equation $m = p_1^{q_1} \cdots p_n^{q_n}$ is a *prime factorization* of m if each p_i is prime.
- The Theorem: Every number greater than 1 has a prime factorization that is unique up to exchanging the order of the p_i.

The Infinitude of Primes

- Is more or less Proposition 20 in Book IX of *Elements* (300 BC).
- The Theorem: There is no largest prime.
- The Proof: If $p_1, p_2, \ldots p_n$ were all the primes, what could $p_1 p_2 \cdots p_n + 1$ be? Prime? Composite? (What about the Fundamental Theorem of Arithmetic?)

Formulas for Primes

- No polynomial formula can produce primes for every natural number input, but formulas can be produced that give primes for the first n inputs.

The Sieve of Eratosthenes

- Method for finding all the prime numbers less than a given number n. Given in 3rd century BC. First published in Nichomachus' *Introductio* (100).
- Example for $n = 100$

Diophantine Equations

- These are polynomial equations for which postive integer solutions are sought.
- Diophantus's *Arithmetica* (250) consisted of solutions to one or two hundred Diophantine equations, but neither complete solution sets nor systematic solution techniques were presented.
- In 1900, David Hilbert gave a list of 23 unsolved math problems he thought were especially important. These are referred to as "Hilbert's Problems".
- Hilbert's 10th Problem was to come up with a general algorithm for determining whether a Diophantine equation has a solution.
- In 1970 Yuri Matiyasevich proved that there is no such algorithm.

Fermat's Last Theorem

- The Theorem: If $n > 2$ there are no numbers x, y, and z such that $x^n + y^n = z^n$.
- Fermat wrote in the margin of his copy of Diophantus' *Arithmetica*: "It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain."
- He almost certainly didn't have such a proof.
- He did have a proof for $n = 4$.
- Euler (1753) proved $n = 3$.
- Dirichlet and Legendre (1825) handled $n = 5$.
- Kummer (1857) handled all $n < 100$.
- The general case was proved by Andrew Wiles in 1995, for which he used some very sophisticated theoretical tools.

The Twin Prime Conjecture

- Conjectured by Euclid?
- If p and $p + 2$ are both primes, they form a pair of *twin primes*.
- The Conjecture: There are infinitely many pairs of twin primes of the form $\{p, p + 2\}$.
- Unsolved.

Goldbach's Conjecture

- Conjectured in a 1742 letter from Prussian mathematician Christian Goldbach to Euler.
- The Conjecture: Every even number bigger than 2 is the sum of two primes.
- Unsolved.
- It is one-half of Hilbert's 8th Problem.
- So far it has been verified for every even number less than 3×10^{17}.

The Prime Number Theorem

- The Theorem: If we define A_n to be the number of primes less than n, then $A_n \sim n/(\ln n)$ as $n \to \infty$. That is, $\lim_{n\to\infty}(A_n \ln n)/n = 1$.
- Conjectured by Legendre (1796).
- Proved independently by two mathematicians in 1896.

Modular Arithmetic

- Formalized and systematized by Gauss in *Disquisitiones arithmeticae* (1801).
- m and n are congruent modulo p ($m \equiv n \pmod{p}$) if p is a divisor of the difference of m and n.
- Clock arithmetic.
- How is it similar to or different from ordinary arithmetic?
- How can it be used to solve Diophantine equations like $x^2 - 5y^2 = 2$?
- Chinese Remainder Theorem
  - The Theorem: If $p_1, p_2, \ldots, p_k$ are relatively prime and $a_1, a_2, \ldots, a_k$ are given numbers. There is a number x such that $x \equiv a_i \pmod{p_i}$ for each i (and there is a practical way to compute such an x).
  - Practicality
  - Example
- Quadratic Reciprocity
  - m is a *quadratic residue mod p* if there is a number x such that $m \equiv x^2 \pmod{p}$.
  - The Legendre symbol:

$$\left(\frac{p}{q}\right) = \begin{cases} 1 & \text{if } p \text{ is a quadratic residue modulo } q \\ -1 & \text{otherwise} \end{cases}$$

  - The Theorem: If p and q are odd primes then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}.$$

  - Conjectured by Euler.
  - Proved by Gauss. Gauss called mathematics the queen of the sciences, arithmetic the queen of mathematics, and the law of quadratic reciprocity the gem of arithmetic.

Primes in Arithmetic Progressions

- An *arithmetic progression* is a sequence of the form $a, a+b, a+2b, a+3b, a+4b, \ldots$.
- if a and b have a nontrivial common factor, there's no way this arithmetic progression could contain any primes.
- The Theorem: If a and b are relatively prime, then the corresponding arithmetic progression contains infinitely many primes.
- Proved by Dirichlet in 1855.

Arithmetic Progressions of Primes

- In 2004, Ben Green and Terence Tao proved that there are arbitrarily long (finite) arithmetic progressions consisting completely of primes.

The Riemann Hypothesis

- For $s = a + bi$ with $a > 1$, the Riemann zeta function is given by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

- This function can be extended in a natural way to be defined for all s except $s = 1$.
- The extended function is equal to zero at all negative even integers. These are called the *trivial zeros*.
- The Hypothesis: If $\zeta(s) = 0$, and s is not a trivial zero, then $a = 1/2$.
- Elementary number-theoretic characterization due to Jeffrey Lagarias: Let $H_n = 1 + 1/2 + \cdots + 1/n$. Then RH is equivalent to: For every number n the sum of the divisors of n is less than or equal to $H_n + e^{H_n} \ln H_n$.
- It is the other half of Hilbert's 8th Problem.
- Solver will get million dollar prize from Clay Mathematics Institute.

Primality Testing

- The obvious brute force way to check whether a number n is prime takes $\sqrt{n}$ divisions. Thus, if n has k digits there will be about $10^{k/2}$ divisions.
- In 2002, 3 Indian mathematicians discovered the Agrawal-Kayal-Saxena primality test or AKS primality test that requires no more than $Ck^{12+\varepsilon}$ operations to check for primality.

Ancient (Non-Greek) Geometry

General

- Geometry literally means "Earth measurement".
- Societies considered here are Egyptian, Babylonian, Indian, and Chinese.

Miscellaneous

- A lot of apparent interest in unit conversion problems.
- Double-difference method of surveying common in China, and India.
  - How does it work?
  - Why wouldn't they use something simpler?
- Egyptians had right formula for a frustum of a pyramid, unlike other ancient civilizations. (It is *not* average base area times height.)
- Babylonian geometry very algebraic. Early sliding ladder problem!: "A ladder of length 30 stands against a wall; the question is, how far will the lower end move out from the wall if the upper end slips down a distance of 6 units?"
- Was geometry purely a tool for these ancient civilizations? What about the strange algebra problems connected with geometry in Mesopotamia?

Trigonometry

- Ptolemy had a table of correspondences between arc and chord lengths in 150 AD. (How is this trigonometry?)
- Indian culture appears to have made the most extensive early inroads. (Cooke: "[T]rigonometry began to assume its modern form among the Hindus some 1500 years ago.")
- Astronomy was a major motivation.
- Indian mathematicians had a power series formulation for the inverse tangent in 1550 (before Newton and Leibniz).

Pythagorean Theorem

- Currently, most historians appear to lean toward several independent discoveries in different societies.
- Cooke: "On balance, one would guess that the Egyptians *did* know the Pythagorean theorem. However, there is no evidence that they used it to construct right angles .... There are much simpler ways of doing that (even involving the stretching of ropes), which the Egyptians must have known. Given that the evidence for this conjecture is so meager, why is it so often reported as fact? Simply because it has been repeated frequently since it was originally made. We know precisely the source of the conjecture, but that knowledge does not seem to reach the many people who report it as fact."
- Cooke: "In contrast to the case of Egypt, there is clear proof that the Mesopotamians knew the Pythagorean theorem in full generality at least 1000 years before Pythagoras. ... as far as present knowledge goes, the earliest examples of the use of the 'Pythagorean' principle ... occur in the cuneiform tablets."
- In China, described in Zhou Bi Suan Jing, which traditionally dates to 800 BC but the present text is currently dated to 100 BC.
- In India in the *Sulva Sutras* (800-500 BC).

Calculating π

- Avoid looking down our noses at "inaccurate" ancient values.
  - 1 Kings 7:23: "And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about."
    - ⋆ A prominent Mormon mathematician: "Readers familiar with mathematics may be amused by [this verse] ... In other words, the Bible teaches that the mathematical constant pi (the ratio between the circumference of a circle and its diameter) is exactly 3.00, not 3.14159... as students learn today."
    - ⋆ Roger Cooke: "[T]he claim often made that the "biblical" value of π is 3, based on [this verse] is pure pedantry. It assumes more precision than is necessary in the context. The author may have been giving measurements only to the nearest 10 cubits, not an unreasonable thing to do in a literary description."
- One-, two-, and three-dimensional π.
- But plausibility argument for ancients equating 1-D and 2-D version.
- In Mesopotamia, we see an early value of 3 for the 2-D π but $25/8$ for 1-D. Why?
- In China, a mixture of good and bad values.
  - A value of 3 implied in 1st century AD documents.
  - Other first century evidence shows use of 3.15147(?).
  - Around 500 AD, some were using the very good approximation $355/113$. (Not discovered in Europe until the late 16th century.)
- Boyer: "We should bear in mind that accuracy in the value of π is more a matter of computational stamina than of theoretical insight."
- The figure of $\sqrt{10}$ was used so often in India that it is sometimes called the "Hindu value".
- Derivation using Pythagorean Theorem.

Euclidean Geometry

Precursors

- Pythagoras
  - Considered right angles to be morally exemplary.
  - Transformation of areas
    - ⋆ How to dissect a polygon and reassemble the pieces to form a square.
      - · By Pythagorean Theorem, it's sufficient to solve the case where the polygon is a triangle.
      - · We can first turn the triangle into a rectangle and then turn the rectangle into a square.
- The 3 Classical Problems
  - Squaring the Circle
    - ⋆ Proved impossible (using only Euclidean means) by Lindemann in 1882 when he proved π is transcendental.
  - Doubling a Cube
    - ⋆ Proved impossible (using only Euclidean means) by Pierre Wantzel in 1837.
  - Trisecting an Angle
    - ⋆ Proved impossible (using only Euclidean means) by Pierre Wantzel in 1837.
  - All can be done if the rules are loosened.
  - Typical complaint about the loosened rules is that they involve creating intermediate objects more complicated than the final problem to be solved.
- Zeno's Paradoxes
  - E.g., motion is impossible.
  - Should we be impressed?
  - Effect on Pythagoreans?
- Sign over the entrance to Plato's Academy: "Let no ungeometrical person enter."
- Eudoxus
  - Theory of proportion
  - Method of Exhaustion: Indirectly lets us ascertain volume by approximation.

Euclid

- *Elements*
  - Miscellaneous
    - ⋆ Oldest extant manuscript: 4th century AD?
    - ⋆ With few exceptions, the individual results are not original to Euclid, but the organization probably is.
    - ⋆ Intended as a textbook, not a research monograph.
    - ⋆ Not intended to contain all known results.
    - ⋆ Elements in the sense of fundamental, not in the sense of simple and uncomplicated.
    - ⋆ *Synthetic* rather than *analytic*.
  - Superlatives
    - ⋆ Eves: "No work, except the Bible, has been more widely used, edited, or studied, and probably no work has exercised a greater influence on scientific thinking."
    - ⋆ Over 1000 editions
    - ⋆ First printed in 1482

    - ⋆ Calinger: "Forty-year-old Abraham Lincoln mastered its first six books to make him a more exact reasoner."
    - ⋆ What does this say about mathematics?
        - · Centrality
        - · Universality
- ◦ Definitions, Axioms, and Postulates
    - ⋆ The definitions are not mathematical and of no rigorous value. They can be thought of as giving intuition.
        - · Examples
    - ⋆ The five axioms are intended to be general principles, not specific to geometry.
        1. Things that are equal to the same thing are equal to each other.
        2. If equals be added to equals, the wholes are equal.
        3. If equals be subtracted from equals, the remainders are equal.
        4. Things that coincide with one another are equal to one another.
        5. The whole is greater than the part.
    - ⋆ The five postulates are specific to geometry and are about things that can be done.
        1. It is possible to draw a straight line from any point to any other point.
        2. It is possible to produce a finite straight line indefinitely in that straight line.
        3. It is possible to describe a circle with any point as center and with a radius equal to any finite straight line drawn from the center.
        4. All right angles are equal to one another.
        5. If a straight line intersects two straight lines so as to make the interior angles on one side of it together less than two right angles, these straight lines will intersect, if indefinitely produced, on the side on which are the angles which are together less than two right angles.
- ◦ Implied tools: straightedge and (collapsing) compass.
    - ⋆ How Proposition 2 gives us a non-collapsing compass.
- ◦ Eves: "Important as are the contents of *Elements*, perhaps still more important is the formal manner in which those contents are presented. ...In spite of a considerable abandonment of the Euclidean form during the 17th and 18th centuries, the postulational method has today penetrated into almost every field of mathematics."
- ◦ 13 books
    - ⋆ 7 on plane geometry
        - · Book 2 on "geometric algebra" has been largely ignored because we have symbolic algebra and trigonometry.
        - · Book 5 is on the Eudoxan theory of proportion.
        - · Arthur Cayley: "There is hardly anything in mathematics more beautiful than the wondrous fifth book."
        - · Work involving similar figures comes after Book 5 (starting in Book 6).
        - · Euclid's proof of the Pythagorean Theorem and its converse (which comes before Book 5 at the end of Book 1)
    - ⋆ 3 on arithmetic
    - ⋆ 3 on solid geometry
- ◦ Defects in logic
    - ⋆ States a general theorem but only proves a special case or uses specially positioned data.
    - ⋆ Boyer: "[B]y modern standards of rigor the Euclidean assumptions are woefully inadequate, and in his proofs Euclid often makes use of tacit postulates."

    - ⋆ Postulate 4 tacitly assumes congruence. Russell: That without further conditions a triangle can be moved without any alteration in its structure is "a tissue of nonsense".
    - ⋆ Do "intersecting" arcs have a point in common?
    - ⋆ Russell: "The value of ... [Euclid's] work as a masterpiece of logic has been ... exaggerated."
    - ⋆ David Hilbert's *Grundlagen der Geometrie* cleared things up. Euclidean geometry rests on *fifteen* postulates and six primitive terms.
- Others

Successors

- Archimedes
  - Computed the area and volume of a sphere rigorously: unique in ancient world.
    - ⋆ Proof for the area of a sphere
  - Volume of circumscribed cylinder = volume of sphere + volume of a cone inscribed in the cylinder.
  - How were the results stated?
- Appolonius: Conics

Post-Euclidean Geometry

General

- Focus on results after Euclid closely connected to *Elements*

Hellenic Geometry

- Why was there a post-Euclidean change?
    - Roman dominance and Roman's distaste for the impractical.
- Heron
    - Apparently lived in 1st century AD.
    - His book *Metrica* was concerned with numerical calculations related to geometry, something unemphasized by his notable predecessors.
    - Unconcerned with Euclidean niceties related to working with natural units. (E.g., he had no qualms about multiplying 4 lengths together.)
    - Heron's formula for the area of a triangle in terms of its side lengths.
        - ⋆ Possibly known earlier by Archimedes.
        - ⋆ His derivation of this formula.
- Ptolemy (∼ 150 AD)
    - Ptolemy's Theorem
        - ⋆ Possibly known earlier by Hipparchus.
        - ⋆ His derivation
        - ⋆ Its trigonometric significance.
    - The *Almagest*
    - A symbol of bad astronomy?
- Pappus (∼ 340 AD)
    - Theorem: "The ratio of rotated bodies is the composite of the ratio of the areas rotated and the ratio of straight lines drawn similarly from their centers of gravity to the axes of rotation."

Non-Euclidean Geometry

- Attempts to prove the parallel postulate.
    - Why were they undertaken?
    - What was their common pitfall?
    - Saccheri (∼ 1700) and his quadrilaterals. (Hypotheses of the acute, obtuse, right angle.)
    - Legendre and his "proof" of the parallel postulate.
- Questioning the parallel postulate.
    - Philosophical issues
        - ⋆ Strict adherence to only axioms and postulates
        - ⋆ Logical necessity versus empirical justification
        - ⋆ How to give a proof of the (relative) consistency of non-Euclidean geometry.
            - · Models
            - · Reinterpretation of terms
    - Discoveries of the properties of Non-Euclidean Geometry
        - ⋆ Apparently discovered independently by Lobachevskii (,1829), Janos Bolyai (1823,1832), and Gauss (1824,).

- Farkas Bolyai (friend of Gauss and father of Janos) to Janos Bolyai upon hearing of Janos's plans: "You must not attempt this approach to parallels. I know this way to the very end. I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallels alone ... I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. I accomplished monstrous, enormous labours; my creations are far better than those of others and yet I have not achieved complete satisfaction .... I turned back when I saw that no man can reach the bottom of the night. I turned back unconsoled, pitying myself and all mankind. Learn from my example: I wanted to know about parallels, I remain ignorant, this has taken all the flowers of my life and all my time from me."
- Gauss to Farkas upon receiving Janos's work: "If I commenced by saying that I am unable to praise this work, you would certainly be surprised for a moment. But I cannot say otherwise. To praise it, would be to praise myself. Indeed the whole contents of the work, the path taken by your son, the results to which he is led, coincide almost entirely with my meditations, which have occupied my mind partly for the last thirty or thirty-five years. So I remained quite stupefied. So far as my own work is concerned, of which up till now I have put little on paper, my intention was not to let it be published during my lifetime. Indeed the majority of people have not clear ideas upon the questions of which we are speaking, and I have found very few people who could regard with any special interest what I communicated to them on the subject. To be able to take such an interest it is first of all necessary to have devoted careful thought to the real nature of what is wanted and upon this matter almost all are most uncertain. On the other hand it was my idea to write down all this later so that at least it should not perish with me. It is therefore a pleasant surprise for me that I am spared this trouble, and I am very glad that it is just the son of my old friend, who takes the precedence of me in such a remarkable manner."
- Gauss to Taurinus: "The assumption that the sum of the three angles [of a triangle] is smaller than 180° leads to a geometry which is quite different from our (Euclidean) geometry, but which is in itself completely consistent. I have satisfactorily constructed this geometry for myself so that I can solve every problem, except for the determination of one constant, which cannot be ascertained a priori. The larger one chooses this constant, the closer one approximates Euclidean geometry. ... If non-Euclidean geometry were the true geometry, and if this constant were comparable to distances which we can measure on earth or in the heavens, then it could be determined a posteriori. Hence I have sometimes in jest expressed the wish that Euclidean geometry is not true. For then we would have an absolute a priori unit of measurement."

- Riemann's *Habilitationsschrift* (1854) on the differential geometry of surfaces provided a general theory in which the study of non-Euclidean geometry could be embedded.
- Beltrami (1868) used Riemann's theory to develop models of non-Euclidean geometry.
  - These provided an equiconsistency proof of Euclidean and non-Euclidean geometry.
  - "Poincaré" disk
    - The "plane" is the interior of the unit circle
    - "Lines" are diameters and circular arcs perpendicular to the unit circle where they meet.
    - "Points" are points.
    - "Distance" between points corresponding to complex numbers z and w is

$$\tanh^{-1}\left|\frac{z-w}{1-\overline{z}w}\right|.$$

  - Other models

- Hilbert (1901) proved that there is no 2-dimensional surface in 3-dimensional Euclidean space representing hyperbolic space with lines being geodesics and natural lengths and angles.

Modern Geometries

General

- Why was there renewed interest in geometry in the 1600s?
  - Why was there a Renaissance?
  - Mathematical tools like algebra now available.

Analytic or "Algebraic" Geometry

- What is it?
  - "Analytic" is not correctly descriptive.
  - Ancient Greeks did geometric algebra.
  - Ancient mapmakers used coordinates (latitude and longitude).
  - In the 14th century, Nicole Oresme had been graphing dependent versus independent variables.
  - Eves: Real essence is the transference of a geometric investigation to an algebraic investigation.
  - By this definition, Descartes and Fermat have claim as the originators.
- How has it been appraised?
  - Cooke: "Everyone who has studied analytic geometry in school must have been struck at the beginning by how much clearer and easier it was to use than the synthetic geometry of Euclid."
  - John Stuart Mill: "[T]he greatest single step ever made in the progress of the exact sciences."
  - Paul Valéry: "[T]he most brilliant victory ever achieved by a man whose genius was applied to reducing the need for genius."
  - Descartes on Euclidean geometry: "[I]t can exercise the understanding only on condition of greatly fatiguing the imagination."
  - Newton on Descartes' algebra: If written out in words, it "would prove to be so tedious and entangled as to provoke nausea."
  - Cooke's response: "Precisely! That's why it's better to use algebraic symbolism and avoid the tedium, confusion, and nausea."
- Descartes
  - *La Géométrie* (1637) was one of three appendices to his philosophical/scientific treatise *Discourse on Method* (the source of "I think, therefore I am.")
  - Introduced the convention of using late letters for variables and early letters for constants. (Previously, had been vowels and consonants, respectively.)
  - Removed dimensional difficulties: "[U]nity can always be understood, even when there are too few dimensions; thus, if it be required to extract the cube root of $a^2b^2 - b$, we must consider the quantity a^2b^2 divided once by unity and the quantity b multiplied twice by unity."
  - Did *not* introduce the modern rectangular coordinate system. Oblique coordinates are taken for granted.
  - No formulas for distance, slope, angle between lines.
  - No negative abscissas.
  - Finding tangent lines to plane curves without calculus. Example of a parabola.
  - Kline: Showed the power of algebraic methods.
  - Focused on *locus problems*
    - ⋆ Initially posed by Pappus. Generalized later by others. (How?)

        - Pappus: Given four lines in a plane, find the locus of a point that moves so that the product of the distances from two fixed lines (along specified directions) is proportional to the square of the distance from the third line [three-line locus problem], or proportional to the product of the distances from the other lines [four-line locus problem].
        - Pappus stated without proof that the locus was a conic section.
        - Descartes showed this algebraically.
            - Descartes: "[S]ince so many lines are confusing, I may simplify matters by considering one of the given lines and one of those to be drawn ... as the principal lines, to which I shall try to refer all the others."
        - Newton later showed it geometrically.
    - Descartes "I have omitted nothing inadvertently but I have foreseen that certain persons who boast that they know everything would not miss the opportunity of saying that I have written nothing that they did not already know, were I to make myself sufficiently intelligible for them to understand me."
- Fermat
    - Involved in a priority feud with Descartes. As commonly happens, the controversy stems from the lapse of time between discovery and publication. Eventually Fermat (as opposed to Descartes) showed some generosity.
    - Investigated planar curves more systematically than Descartes.
    - No negative abscissas.
    - Equations often came before geometry for these curves.
    - Clearly conveyed the idea of equations for curves.
- Jakob Bernoulli is credited with inventing polar coordinates.
- Jan De Witt wrote the first textbook *Elementa curvarum*. Invented the word "directrix". Had focus-directrix definitions of conic sections. Reduced all second-degree equations to canonical form through translation and rotation of axes. Had a test for type of conic sections. Met a grisly end for political reasons, not for authorial ones.

Algebraic Geometry

- What is it?
    - The study of solution sets of systems of polynomial equations and their properties and natural classifications.
    - The simplest case is the study of algebraic curves $p(x, y) = 0$, where p is a polynomial in 2 variables.
    - If p has degree 2, these are conic sections.
- Newton
    - Classified all cubic curves into 72 categories. (He missed 6 categories.)
    - Phenomena not present in nondegenerate lower-order curves include inflection points and multiple points.
- Maclaurin
    - Discovered that a curve of degree m and a curve of degree n typically intersect in mn points. This counts points according to multiplicity and includes imaginary points and other complications. (Bézout's Theorem)
- Cramer
    - His Paradox
        - An nth degree curve is (allegedly) determined by $n(n + 3)/2$ of its points. Why?
        - Bézout's Theorem says two nth degree curves intersect in n^2 points.

        - If the $n(n+3)/2$ points determining an nth degree curve are chosen to be among the n^2 points that two nth degree curves share, there seems to be a lack of determinacy.
        - Resolution: The equations aren't independent.

Descriptive Geometry

- What is it?
    - The study of the representation of 3-dimensional objects using 2-dimensional figures.
    - think of mechanical drawings, drafting, etc.
- Essentially invented by Monge in the late 1700s. (Much geometry done in revolutionary France. Monge's motivation: To "pull the French nation out of its hitherto dependence on foreign industry".)
- Uses parallel projection.
- Monge went on to do nontrivial mathematics with such representations.

Projective Geometry

- What is it?
    - The study of geometric properties of and classes of geometric objects that are unchanged by projections, or changes of perspective.
    - Original motivation was a desire to help painters: A painting should be a section of the projection of the lines of light from the thing being painted to the eye.
    - Projection preserves neither congruence, nor similarity, nor angle, nor length, nor area.
    - Some quantitative things are preserved.
        - If A, B, C, and D are in order on a line, then the cross ratio $(BA/BC)/(DA/DC)$ is preserved.
    - What do original and section have in common?
    - What do two sections of original on same projection have in common?
    - What do two sections of original on different projections have in common?
- France rules, as in the beginning of analytic geometry.
- Desargues
    - One of first to think in terms of projection and section.
    - Introduced *points at infinity* where parallel lines meet.
    - These lines formed a *line at infinity*, corresponding to the horizon.
    - All parallel planes meet on one line.
    - Asymptotes of hyperbolas are tangent to them at infinity.
    - Lines are circles of infinite radius.
    - Desargues' Theorem
        - A collection of lines is *concurrent* if they have a point in common.
        - A collection of points is *collinear* if they all lie on the same line.
        - The Result: The lines passing through corresponding vertices of two triangles are concurrent if and only if the points of intersection of extended pairs of corresponding sides are collinear.
- Pascal
    - Theorem (~ 1639): The opposite sides of a hexagon inscribed in a conic intersect in 3 collinear points.
    - Proof: It suffices to show that it is true of a circle, since other conic sections project to a circle, and the relations are preserved under projection.
- No substantial progress for a century and a half.

- Brianchon ($\sim$ 1810)
  - A student of Monge.
  - His theorem: If 6 tangents to conic form a circumscribing hexagon, the 3 lines connecting opposite vertices are concurrent. Dual of Pascal's Theorem.
- Poncelet
  - A student of Monge.
  - First to view projective geometry as a new branch of mathematics.
  - Captured in Russia in Napoleonic wars. Did work in prison.
  - Principle of Duality
    - ⋆ Proposed and used by (but not really proved by) Desargues.
    - ⋆ Content: The words "line" and "point" can be interchanged in the plane. The words "plane" and "point" can be interchanged in 3-dimensional space. The language needs to be smoothed out. E.g., interchange "lie on" and "pass through".
  - Principle of Continuity
    - ⋆ More ambiguous and controversial than the Principle of Duality.
    - ⋆ Poncelet: "Let us consider an arbitrary figure in a general position and indeterminate in some way, taken from all those that one can consider without breaking the laws, the conditions, the relationships which exist between the diverse parts of the system. Let us suppose, having been given this, that one finds one or more relations or properties, be they metric or descriptive, belong to the figure by drawing on ordinary explicit reasoning, that is to say by the development of an argument that in certain cases is the only one which one regards as rigorous. Is it not evident that if, keeping the same given things, one can vary the primitive figure by insensible degrees by imposing on certain parts of the figure a continuous but otherwise arbitrary movement, is it not evident that the properties and relations found for the first system, remain applicable to successive states of the system, provided always that one has regard for certain particular modifications that may intervene, as when certain quantities vanish or change their sense or sign, etc., modifications which it will always be easy to recognize *a priori* and by infallible rules? [...] Now this principle, regarded as an axiom by the wisest mathematicians, one can call *the principle* or *law of continuity* for mathematical relationships involving abstract and depicted magnitudes."
  - Poncelet's Porism: If two conics are such that an n-gon can be simultaneously inscribed in the one and circumscribed in the other, then an n-gon can be thus inscribed/circumscribed starting at any point on the outer conic.
- Move to German dominance.
- Möbius
  - Möbius Transformations: $z \mapsto (az + b)/(cz + d)$, where $a, b, c, d \in \mathbb{C}$ are constants satisfying $ad \neq bc$. They map lines and circles to lines and circles.
- Plücker
  - Introduced algebraic approaches to projective geometry.
  - Introduced the homogeneous coordinates widely used today.
    - ⋆ All points of the form (x, y, t) correspond to points with regular coordinates $(x/t, y/t)$.
    - ⋆ Lack of uniqueness no more problematic than for polar coordinates.
    - ⋆ Curves represented by polynomial equations in regular coordinates are represented by homogeneous polynomial equations in projective coordinates.
    - ⋆ Line at infinity: $t = 0$
    - ⋆ Proved the Principle of Duality proposed by Desargues.
      - · Note the duality in the equation $ax + by + ct = 0$.

Differential Geometry

- What is it?
  - The study of the properties of curves, surfaces, and their generalizations.
  - The study of the geometry of spaces in which the distance function is prescribed infinitesimally.
  - The early forms of differential geometry stemmed directly from the invention of the Calculus (and were practically indistinguishable from it).
  - Applications like cartography also played a large part in its development.
  - The term "differential geometry" was apparently not coined until 1894.
- Newton
  - Along with Huygens (but working independently and using different methods), he studied involutes, evolutes, and curvature of plane curves.
    - ⋆ An *involute* of a curve is the curve traced by the end of the cord that is originally placed along the first curve and then unwound.
    - ⋆ An *osculating circle* for a curve at a point on the curve is the circle passing through that point that best fits the curve.
    - ⋆ An *evolute* of a curve is the curve made up of the centers of the first curve's osculating circles.
    - ⋆ Fact: A curve is the evolute of any of its involutes.
    - ⋆ The curvature of a plane curve at a point on the curve is the reciprocal of the radius of the corresponding osculating circle.
- Euler
  - Along with Clairaut, he studied space curves.
    - ⋆ Euler represented these curves parametrically, with arc length being the parameter: $x = x(s)$, $y = y(s)$, $z = z(s)$.
    - ⋆ He defined and gave a formula for the curvature of space curves.
    - ⋆ There's a related quantity for space curves known as *torsion*. (The torsion of plane curves is zero.)
      - · At a point on a space curve, there's a plane (called the *osculating plane*) that the curve stays closest to near the given point.
      - · A perpendicular to this plane at that point is called the *binormal*.
      - · Torsion measures the rate at which the direction of the binormal changes as we move along the curve.
    - ⋆ From curvature and torsion, a curve can be reconstructed (if we know its starting position and direction).
  - Along with the Bernoullis, Euler also initiated the theory of surfaces.
    - ⋆ If we slice perpendicularly through a surface at a point, we have a curve whose curvature at the point we can measure.
    - ⋆ Of all the directions in which this slice could be made, there's one that gives maximum (signed) curvature and one that gives minimum (signed) curvature.
    - ⋆ These two curvatures are called the *principal curvatures* of the surface at the point.
    - ⋆ Euler showed how to calculate these.
    - ⋆ He also introduced parametric representations of surfaces: $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$.
- Gauss
  - Boyer: "Gauss advanced the totally new concept that a surface is a space in itself."
  - He gave a formula for the curvature of a surface at a point that uses only the parametric representation of the surface. It is the product of the principal curvatures.

- He called it the *total curvature*; we call it the *Gaussian curvature*.

- Riemann
  - He was a student of Gauss. His work extended Gauss's and was first delivered in his *Habilitationsvortrag*.
  - He was interested in the empirical question of exactly what kind of space we live in. How do we know its flat?
  - Introduced study of m-dimensional manifolds immersed in n-dimensional space.
  - Focused on the intrinsic nature of the manifold, not on specific coordinate system.
  - Riemannian metric
    - ⋆ Defined on an n-dimensional manifold parametrized by coordinates $x_1, x_2, \ldots, x_n$.
    - ⋆ An $n \times n$ matrix $G = (g_{ij})$ whose entries are functions of $x_1, x_2, \ldots, x_n$. This matrix must be *positive definite*, which means that if you multiply this matrix on the left by a nonzero row vector and on the right by the corresponding column vector the result is positive.
    - ⋆ The square of the length of an "infinitesimal" segment from $(x_1, x_2, \ldots, x_n)$ to $(x_1 + dx_1, x_2 + dx_2, \ldots, x_n + dx_n)$ is $\sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}(x_1, x_2, \ldots, x_n) dx_i dx_j$. If g_{ij} is identically 1 when $i = j$ and identically 0 when $i \neq j$, then this is the standard Euclidean metric, the generalization of the standard 2-dimensional distance formula (implied by the Pythagorean Theorem) to n-dimensions.
    - ⋆ The length of a parametric curve $(x_1(t), x_2(t), \ldots, x_n(t))$, $t \in [a, b]$ is given by

$$\int_a^b \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}(x_1(t), x_2(t), \ldots, x_n(t)) \frac{dx_i(t)}{dt} \frac{dx_j(t)}{dt}}\, dt$$

    - ⋆ A curve starting at $(x_1, x_2, \ldots, x_n)$ and ending at $(y_1, y_2, \ldots, y_n)$ that is shorter than or equal to every other curve starting at $(x_1, x_2, \ldots, x_n)$ and ending at $(y_1, y_2, \ldots, y_n)$ (in the previous sense) is a *geodesic*.

Topology

- What is it?
  - "Rubber-sheet geometry": The study of sets with a concept of nearness and continuity.
  - The word was invented in 1848 by Johann Listing.
  - Topology in $\mathbb{R}$
    - ⋆ A set of real numbers is *closed* if every convergent sequence from the set has a limit in the set. This terminology was introduced by Cantor.
    - ⋆ A set of real numbers is *open* if it is the union of open intervals. This terminology was introduced by W.H. Young in 1902.
    - ⋆ No point in an open set is the limit of a sequence from outside the set.
    - ⋆ Open sets and closed sets are complements of each other.
    - ⋆ Some sets are both open and closed.
    - ⋆ Some sets are neither open nor closed.
- Combinatorial and Algebraic
- Point-set
  - Some topological concepts
    - ⋆ Connectedness: Can you get from any point in the set to any other point in the set without leaving the set? Related to the Intermediate Value Theorem.

        - ⋆ Compactness: Does every sequence in the set get really close to some point in the set really often? Related to the Extreme Value Theorem.

- Fréchet
    - Introduced *metric spaces* in 1905. A metric space is a set on which a distance function is defined with the properties:
        - ⋆ The distance from x to y is the same as the distance from y to x.
        - ⋆ Distances are never negative, and are zero precisely when we're measuring the distance from a point to itself.
        - ⋆ The triangle inequality is satisfied.
    - Example: Taxi-cab geometry.
- Hausdorff
    - Introduced a topological space that consisted of a set and a collection of neighborhoods satisfying certain properties that generalize the concept of open intervals.
- Bourbaki
    - Not a person, but a pseudonym for a bunch of French mathematicians working together.
    - Introduced the most abstract modern definition of a topological space: A set X along with a collection of subsets of X called *open sets* that have the following properties:
        - ⋆ The intersection of two open sets is an open set.
        - ⋆ The union of any collection of open sets is an open set.
        - ⋆ X and the empty set are both open sets.

Algebra

Literal Algebra

- Algebra with letters.
- Kline: Use of symbolism far more significant than technical advances.
- Stages
  - Rhetorical: No abbreviations or symbols
  - Syncopated: Abbreviations for some frequent operations
  - Symbolic: Largely mathematical shorthand
- Babylonians had rhetorical algebra.
- Diophantus
  - Syncopated algebra
  - Introduced the symbol ς for the unknown, roughly the same as a sigma used at the end of a word.
  - Powers
    - ⋆ $x^2 \sim \Delta^\upsilon$ from the first two letters of dýnamis, meaning power.
    - ⋆ $x^3 \sim K^\upsilon$ from the first two letters of kýbos, meaning cube.
    - ⋆ $x^4 \sim \Delta^\upsilon\Delta$
    - ⋆ $x^5 \sim \Delta K^\upsilon$
    - ⋆ $x^6 \sim K^\upsilon K$
    - ⋆ x^{-k} was represented by appending a superscript χ to the symbol for x^k $(k > 0)$; e.g., $x^{-2} \sim \Delta^{\upsilon\chi}$.
    - ⋆ $x^0 \sim \overset{\circ}{M}$, from the first two letters of the Greek word for unity. (To balance units?)
  - Others
    - ⋆ $= \sim i\sigma$
    - ⋆ $- \sim$⫛ conjectured to be a condensation of $\lambda\iota$, the first two letters of the Greek root for "less".
  - Arabic numbers and Diophantine notation for $x^3 - 2x^2 + 10x - 1 = 5$:

$$K^\upsilon 1\varsigma 10 ⫛ \Delta^\upsilon 2\overset{\circ}{M}1 i\sigma \overset{\circ}{M}5$$

- Except for the Hindus, who syncopated their algebra, Diophantus' innovation was largely neglected for many hundreds of years.
- Al-Khowârizmî's work was rhetorical.
- In Western Europe, algebra was mainly rhetorical until the 15th century. Symbolic algebra was not prevalent there until 1650.
  - Vi'‘ete (late 1500s)
    - ⋆ The inventor of literal calculus.
    - ⋆ Used symbols for parameters as well as unknowns.
    - ⋆ Used vowels for unknowns, consonants for parameters.
    - ⋆ Introduced square brackets and braces as grouping symbols. (Parentheses appeared in 1544.)
  - Descartes (early 1600s)
    - ⋆ Dispensed with homogeniety.
    - ⋆ Used modern naming conventions.
    - ⋆ Made systematic use of positive integer exponents. (For $n = 2$, only occasionally.)
  - Miscellaneous

  - ⋆ Christoff Rudolff introduced $\sqrt{}$ in 1525.
  - ⋆ Fibonacci (1200) gave us the use of general letters in an argument.
  - ⋆ $+$ and $-$ introduced by Johann Widman in 1489. Used as full-blooded operations by Vander Hoecke in 1514.
  - ⋆ $=$ due to Robert Recorde in 1557. (Viete had used $\sim$; Descartes had used $\propto$.)
  - ⋆ $\times$ due to William Oughtred in early 1600s.
  - ⋆ Girard (1629) gave us $\sqrt[3]{}$ for cube root.
  - ⋆ Thomas Harriot introduced $<$ and $>$.
  - ⋆ Newton used rational exponents.
  - ⋆ John Hudde (1657) used letter for positive and negative numbers.
  - ⋆ x^2 not standard until adopted by Gauss in 1801.

Solving Polynomial Equations

- Subcategorizing equations of degree n
  - Equations having negative numbers standing alone or with negative or complex solutions weren't treated for many years.
  - For example, the quadratic equations solved in ancient and medieval time were of three types:
    - ⋆ $x^2 + px = q$
    - ⋆ $x^2 = px + q$
    - ⋆ $x^2 + q = px$
- Egyptians
  - Had trouble solving quadratic equations.
  - Are alleged to have used the "rule of false position" to solve linear equations. (This has been disputed by some.)
- Babylonians
  - Solved systems of two linear equations in two unknowns.
  - Solved systems of two equations with one linear and one quadratic.
  - Were solving quadratic equations 4000 years ago.
  - "Solved" cubic equations by table lookup.
  - Consistency in methods for solving quadratics but no explicit statement of quadratic formula, in general.
    - ⋆ With enough examples, such statements may have seemed unnecessary.
    - ⋆ Such statements are difficult to make without symbols for coefficients.
- Hindus
  - Realized that quadratics with positive discriminant had two formal roots.
- Arabs
  - Al-Khowârizmî ($\sim$ 800 AD)
    - ⋆ His name gave us our word "algorithm".
    - ⋆ Our word "algebra" comes from the Arabic word (al-jabr) for restoration that Al-Khowârizmî used as one of his rules for manipulating equations to solve them.
      - · The restoration is the compensating change made to the other side of the equation in which one side has been changed.
      - · In *Don Quixote* there is a bone-setter called an algebrist.
    - ⋆ Explained the four basic operations and solved linear and quadratic equations.
    - ⋆ Did not make any great algebraic breakthrough.
    - ⋆ His work was thorough and practical.

    * Did not consider cubic equations.
    * His work seems to show Hindu, Greek, and Babylonian influences.
  - Omar Khayyam ($\sim$ 1100 AD)
    * Famous poet: *The Rubaiyat*
    * "Solved" the cubic equation $x^3 + ax^2 + b^2x = b^2c$ by drawing a circle of radius $(a + c)/2$ centered at $((c - a)/2, 0)$, the rectangular hyperbola $y = bc/x - b$, and taking the x-coordinate of the upper intersection.
      · Of course, did not use analytic geometry.
      · It's not clear how this amounts to anything more than a restatement of the problem.

- Descartes
  - Recommendation: Move everything to one side.
  - Stated the Factor Theorem.
- Cubics and Quartics
  - 16th century Italy
  - Names
    * del Ferro
    * Fior
    * Tartaglia
    * Cardano
    * Ferrari
    * Bombelli
  - Techniques for solving general cubic and quartic equations are found in Cardano's *Ars Magna*.
  - Although there was and is substantial controversy over who deserves credit for these (general) solutions, there is a strong case that Cardano should be credited with solving the cubic and that Ferrari should be credited with solving the quartic.
  - Bombelli helped make sense of formulas (for real roots) involving imaginary numbers.

Fundamental Theorem of Algebra

- The Theorem: Every polynomial equation (with real or complex coefficients) of degree 1 or higher has a solution (in $\mathbb{C}$).
- A Corollary: Every polynomial (of degree 1 or higher) with real coefficients can be factored into linear and quadratic factors with real coefficients.
- Leibniz didn't believe the Corollary; Euler did. Euler had proved the Corollary up to degree 6.
- The Theorem was first stated by Girard in 1629.
- Gauss gave the first proof of the Theorem in 1799. He is credited as the prover, but from a modern perspective it was insufficiently rigorous.
- Gauss's Proof
  - Plugs in z in polar form, collects real and imaginary terms, and sets the real part and imaginary part to zero, giving us two equations in two real unknowns.
  - Each equation should have a solution set that is a curve in the complex plane. Do the curves intersect?
  - These curves should intersect a large circle in points that are interlaced.
  - If the parts of the curves inside the circle are connected, then there should be an intersection point.

Unsolvability of the Quintic

- The Theorem: There is no formula involving the four basic arithmetic operations and root extraction applied the coefficients of a quintic (or higher degree) polynomial that is guaranteed to produce a zero of the polynomial.
- What the theorem doesn't rule out.
  - Other sorts of formulas.
  - Formulas that work on specific types of polynomials.
  - Different formulas for different polynomials. (Although this also was eventually ruled out in general.)
- First explicitly conjectured by Euler in 1749.
- The majority opinion credits Abel with proving this in 1826.
- Others (Cauchy, Ruffini) had made substantial progress before Abel.
- Abel filled in gaps in previous work. Many felt (and many still feel) that Abel's proof itself is incomplete.
- Ideas of the Proof?
  - The coefficients of a polynomial are symmetric functions of the zeros of the polynomial.
  - *Every* symmetric function of the zeros of the polynomial is a function of its coefficients.
  - The properties of permutations should be studied.

Vector Algebra (Math 214)

- Hamilton gave us the symbol ∇ and the name "nabla".
- Josiah Willard Gibbs is credited with inventing vector analysis, essentially in the form we know it, in the 1880s.
- Relies on both Hamilton and Grassmann.

Linear Algebra (Math 343)

- Leibniz initiated the systematic study of systems of linear equations in the 1600s.
- The study of determinants *preceded* the study of matrices.
- Maclaurin gave formulas for solutions for systems of up to 4 equations in 4 unknowns in 1729, using determinants.
- Vandermonde is credited with being the founder of the theory of determinants because of the systematic nature of his work.
- Cauchy gave us the word "determinant" in this context, the arrangement of numbers in a square array, and the standard double subscript notation. Cayley gave us the vertical lines around the array in 1841.
- Grassmann's *Ausdehnungslehre* (1844) is arguably the foundation document for linear algebra (and more).
- Sylvester gave us the word "matrix" in this context.
- Cayley is credited with creating the theory of matrices, based in large part on the study of linear transformations in $\mathbb{R}^2$.
- Who invented the Cayley-Hamilton Theorem? Jordan canonical form?

Abstract Algebra (Math 371–372)

- Abstraction too early repeatedly proved unfruitful.

- In 1830, George Peacock tried to systematize the study of algebra (like Euclid did for geometry). He had a Principle of Permanence of Form that was analogous to the geometric Principle of Continuity. Boyer: “The beginnings of postulational thinking in arithmetic and algebra”.
- Abel and Galois laid the groundwork for these abstract structures but didn’t explicitly discuss the structures themselves.
- Van der Waerden’s *Modern Algebra* in 1930 really solidified the field.
- Separate British/American and Continental European threads.
- Groups
    - Definition: A group is a set G with a binary operation $* : G \times G \to G$ such that:
        - ⋆ $*$ is associative: $a * (b * c) = (a * b) * c$ for every $a, b, c \in G$.
        - ⋆ there is an identity $e \in G$ such that $e * g = g * e = e$ for every $g \in G$.
        - ⋆ Every $g \in G$ has an inverse $g^{-1} \in G$ with the property that $g * g^{-1} = g^{-1} * g = e$.
    - Motivation: Solvability of Equations, Classical Geometric Constructions problems.
    - Cayley abstracted the notion of a group in 1849.
    - von Dyck gave us our current set of axioms in 1883.
    - A big project that ended in the late 20th century was the classification of finite *simple* groups. A simple group plays roughly the same role w.r.t. groups that a prime number plays w.r.t. numbers. Its work is chronicled in 500 articles with 100 authors. There are 3 main categories and 26 other groups that don’t fit those other categories. These 26 are called *sporadic* groups. The largest of the sporadic groups is the *Monster group*, which has approximately $8 \cdot 10^{53}$ elements.
- Rings
    - Definition: A Ring is a set R and two binary operations $+$ and $\times$ s.t.:
        - ⋆ $(R, +)$ is a group that is abelian (commutative).
        - ⋆ $\times$ is associative.
        - ⋆ There is a multiplicative identity. (Optional)
        - ⋆ $\times$ distributes over $+$ (on both sides).
    - Motivation: Examples like the integers, polynomials, and matrices.
    - Hilbert talked about “Number Rings” in 1897.
    - Fraenkel talked about abstract rings in 1914.
    - Bell introduced the English term in 1930.
- Fields
    - Definition: A Field is a set F and two binary operations $+$ and $\times$ s.t.:
        - ⋆ $(F, +)$ is a group that is abelian (commutative) and has identity 0.
        - ⋆ $(F - \{0\}, \times)$ is an abelian group.
        - ⋆ $\times$ distributes over $+$.
    - Motivation: Solvability of Equations, Classical Geometric Constructions problems
    - In 1879 Dedekind gave the first explicit definition of a number field.
    - In 1893, Weber gives us a definition of an abstract field.
    - The word “field” was introduced by E.H. Moore in 1893.

Calculus, Part 1

Predecessors of Newton/Leibniz

- Area by Method of Exhaustion in antiquity
  - Eudoxus
  - Archimedes
- India
- Pedersen: “Many methods were developed to solve calculus problems; common to most of them was their *ad hoc* character. It is possible to find examples from the time before Newton and Leibniz which, when translated into modern mathematical language, show that differentiation and integration are inverse procedures; however, these examples are all related to specific problems and not to general theories.”
- Descartes’ Method for finding tangents
- Fermat’s Method of Maxima and Minima
- Cavalieri’s Principle: If cross-sections are in a constant ratio, areas are in that same ratio.
- Roberval’s quadrature of the cycloid
- Wallis’s “arithmetic” integration
- Fermat’s “logarithmic” integration

Newton and Leibniz

- Pedersen: “The special merit of Newton and Leibniz was that they both worked out a general theory of the infinitesimal calculus. However, it cannot be said that either Newton or Leibniz gave to his calculus a higher degree of mathematical rigour than their predecessors had done.”
- Priority dispute
  - Newton did work in early 1670s, published in 1687
  - Leibniz did his work later but published earlier (1684).
  - Allegiance to Leibniz hamstrung British analysts and kept them isolated for over a century.
- Notation
  - Newton: variable quantity = “fluent”; its rate of change = “fluxion” The fluxion of x is $\dot{x}$. The fluent whose fluxion is x was $\acute{x}$.
  - Leibniz: infinitesimal = dx; antiderivative = $\int y\,dx$. The integral sign is an elongated S for “sum”.
  - Leibniz’ notation prevailed.
- Leibniz believed in the reality of infinitesimals; Newton didn’t.

Power Series

- Newton’s derivation of Maclaurin sine series [using binomial series, power series inversion, and integration of power series]
- Euler’s sums [using Maclaurin sine and cosine series, identities for coefficients of infinite sums and products, and differentiation of logarithms]

Calculus, Part 2

Dealing with the Infinitely Small

- Newton: Vanishing quantities. "Errors, no matter how small, are not be considered in mathematics." (Even infinitesimals can't just be dropped from the final answer.)
- Leibniz: Infinitesimal quantities
- Berkeley
  - Theologian, Philosopher, Bishop of Cloyne, near Cork, in the Church of Ireland.
  - In 1734, published *The Analyst; or a Discourse Addressed to an Infidel Mathematician, wherein It is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith*. Its epigraph was Matthew 7:5: "First cast out the beam out of thine own Eye; and then shalt thou see clearly to cast out the mote out of thy brother's eye."
  - His general points was that the foundations of calculus were no clearer or self-evident than the mysteries of revealed religion.
  - In the calculation of derivatives, he objected to conflicting assumptions within the same argument: dx is nonzero so we can divide by it; dx is zero so that we can neglect it: "And what are these Fluxions? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither finite Quantities, nor Quantities infinitely small nor yet nothing. May we not call them the Ghosts of departed Quantities?" This is in response to Newton's assertion: "Perhaps it may be objected, that there is no ultimate proportion of evanescent quantities; because the proportion, before the quantities have vanished, is not the ultimate; and when they are vanished, is [not defined]."
  - Of Leibniz's infinitesimals, Berkeley wrote: "Now to conceive a quantity infinitely small, that is, infinitely less than any sensible or imaginable quantity or than any the least finite magnitude is, I confess, above my capacity. But to conceive a part of such infinitely small quantity [$dx\,dx$] that shall be still infinitely less than it, and consequently though multiplied infinitely shall never equal the minutest finite quantity, is, I suspect, an infinite difficulty to any man whatsoever."
- D'Alembert said derivatives are limits of difference quotients, but didn't really say what limits are.
- Lagrange said derivatives are just (pieces of) power series coefficients (and therefore was restricted to working with functions that have power series expansions).
- Cauchy (1820s)
  - Gave the following definition of limit: "When the values successively attribted to a variable approach indefinitely to a fixed value, in a manner so as to end by differing from it by as little as one wishes, this last is called the limit of all the others."
  - Said: "[W]hen the successive numerical values of a variable decrease indefinitely (so as to become less than any given number), this variable will be called ... an infinitely small quantity".
- Weierstrass (late 1800s)
  - Said: "$\lim_{x\to a} f(x) = L$ if and only if, for every $\varepsilon > 0$, there exists a $\delta > 0$ so that, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$."
  - Some (but not all) of Cauchy's proofs had been δ-ϵ proofs, but he didn't use δ's and ε's in his definitions.
- Robinson (1960s)
  - Invented Nonstandard Analysis, in which there are infinitely small positive quantities and derivatives can be taken without limits.

- It hasn't caught on.

Definite Integration

- Definition
  - Leibniz (late 1600s): It was a sum of infinitesimals.
  - Fourier (1822): Gave us our current notation of definite integral and defined it to be an area.
  - Cauchy (1820s)
    - ⋆ He began with a function f continuous on the interval from x_0 to X. He then took $x_0 < x_1 < \cdots < x_{n-1} < X$, called $x_1 - x_0, x_2 - x_1, x_3 - x_2, \ldots, X - x_{n-1}$ the "elements" of the interval, and set
    $$S = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + (x_3 - x_2)f(x_2) + \cdots + (X - x_{n-1})f(x_{n-1}).$$
    - ⋆ He then said: "If we decrease indefinitely the numerical values of these elements while augmenting their number, the value of S ... ends by attaining a certain limit that depends uniquely on the form of the function $f(x)$ and the extreme values x_0 and X attained by the variable x. This limit is what we call a definite integral."
    - ⋆ This is what we today call the "left-hand rule".
  - Riemann (1854)
    - ⋆ He began with a function bounded on $[a, b]$, partitioned the interval $a < x_1 < x_2 < \cdots < x_{n-1} < b$, set $\delta_1 = x_1 - a$, $\delta_2 = x_2 - x_1$, $\delta_3 = x_3 - x_2$ on up to $\delta_n = b - x_{n-1}$, took numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ between 0 and 1, and set
    $$S = \delta_1 f(a + \varepsilon_1\delta_1) + \delta_2 f(x_1 + \varepsilon_2\delta_2) + \delta_3 f(x_2 + \varepsilon_3\delta_3) + \cdots + \delta_n f(x_{n-1} + \varepsilon_n\delta_n).$$
    - ⋆ He then said: "If this sum has the property that, however the δ_k and ε_k are chosen, it becomes infinitely close to a fixed value of A as the δ_k become infinitely small, then we call this fixed value $\int_a^b f(x)\,dx$. If the sum does not have this property, then $\int_a^b f(x)\,dx$ has no meaning."
    - ⋆ This is the standard integral of introductory Calculus.
    - ⋆ For continuous functions, Riemann's integral gives the same results as Cauchy's integral.
  - Lebesgue (1904)
    - ⋆ His definition was based on the idea of the *measure* of a set of real numbers, which is an extension of our natural notion of length. Not all sets are measurable, but everyone you can construct of is. He denoted the measure of the set E by $m(E)$.
    - ⋆ A function is measurable if $\{x \in \mathbb{R} : f(x) > a\}$ is a measurable set for every real number a. Essentially every function you can think of is measurable.
    - ⋆ Given a bounded, measurable function f, Lebesgue took an interval $[\ell, L]$ containing the *range* of the function, and for a given $\varepsilon > 0$, partitioned it by points $\ell = \ell_0 < \ell_1 < \ell_2 < \cdots < \ell_n = L$ so that the maximal gap is less than ε. He set $E_k = \{x : \ell_k \leq f(x) < \ell_{k+1}$ for $k = 0, 1, 2, \ldots, n-1$ and $E_n = \{x : f(x) = \ell_n\}$.
    - ⋆ He then set
    $$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \left[\sum_{k=0}^{n} \ell_k \cdot m(E_k) \right].$$
    - ⋆ This is the standard integral of advanced analysis.
    - ⋆ If f is a bounded, Riemann-integrable function, then Lebesgue's integral gives the same results as Riemann's integral.

- Characterizations of Riemann-integrability
  - Riemann
    - The *norm* of a partition is the width of its largest subinterval.
    - A function f is integrable if and only if, for any $\sigma > 0$ no matter how small, we can find a norm so that, for all partitions of $[a, b]$ having a norm that small or smaller, the total length of the subintervals where the function oscillates more than σ is negligible.
  - Lebesgue
    - A set has *measure zero* if it "can be enclosed in a finite or a denumerable infinitude of intervals whose total length is as small as we wish". Examples of sets of measure zero are $\mathbb{N}$ and $\mathbb{Q}$.
    - A bounded function is integrable if and only if the set of its points of discontinuity has measure zero.

Calculus, Part 3

What's a Function?

- Euler (1748): "[A] Function of a variable quantity is an analytical expression composed in whatever way of that variable and of numbers and constant quantities." This roughly corresponds today to what we call an *analytic* function, which is a function locally representable by a power series.
- Euler (1755): "Those quantities that depend on others ..., namely, those that undergo a change when others change, are called functions of these quantities. This definition applies rather widely and includes all ways in which one quantity can be determined by others."
- Fourier (1822)
  - "The function $f(x)$ represents a succession of values or ordinates each of which is arbitrary. ...We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as if it were a single quantity."
- Dirichlet
  - 1829: "One supposes that $\phi(x)$ equals a determined constant c when the variable x takes a rational value and equals another constant d when the variable is irrational."
    - ⋆ Such a function (with c and d being 0 and 1) is now called "Dirichlet's function".
    - ⋆ It is a bounded function that is not Riemann-integrable.
    - ⋆ It is Lebesgue-integrable.
  - 1837: "y is a function of the variable x, defined on the interval $a < x < b$, if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way the correspondence is established."
    - ⋆ This is essentially the modern concept of function.
- Riemann (1854)
  - He constructed a function that is Riemann-integrable but has infinitely many discontinuities scattered throughout the interval of integration.
  - He let $(x) := x - n$, where n is the integer nearest x; if x is halfway between two integers, he set $(x) = 0$. His function was

$$f(x) = \frac{(x)}{1} + \frac{(2x)}{4} + \frac{(3x)}{9} + \frac{(4x)}{16} + \cdots = \sum_{k=1}^{\infty} \frac{(kx)}{k^2}.$$

  - It is discontinuous at every number of the form $\frac{m}{2n}$ where m and n are relatively prime. At such a point, there is a jump of height $\frac{\pi^2}{8n^2}$.
- Weierstrass (late 1800s)
  - He proved: If $a \geq 3$ is an odd integer and if b is a constant strictly between 0 and 1 such that $ab > 1 + 3\pi/2$, then the function

$$f(x) = \sum_{k=0}^{\infty} b^k \cos(\pi a^k x)$$

    is everywhere continuous and nowhere differentiable.
- Volterra (1881)
  - Differentiable functions must be continuous, but their derivatives don't have to be. Example:

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

  - Volterra constructed a function that was differentiable everywhere but whose derivative was not Riemann-integrable.

Textbooks

- L'Hôpital (1696)
  - *Analyse des infiniment petits pour l'intelligence des lignes courbes* (Analysis of the Infinitely Small for the Understanding of Curved Lines)
  - The first Calculus textbook.
  - The things he learned from Johann Bernoulli.
  - "L'Hôpital's Rule" is really Johann Bernoulli's rule.
- Euler (1755 and 1768)
  - *Institutiones calculi differentialis* in 1755.
  - *Institutiones calculi integralis* in 1768 (3 volumes).
  - A total of over 2000 pages.
  - Studied by many prominent mathematicians of the 19th century.
- Cauchy
  - *Course d'analyse de l'École Royale Polytechnique* in 1821.
  - *Résumeé des leçons données a l'École Royale Polytechnique, sur le calcul infinitésimal* in 1823.
  - Stated and proved the Intermediate Value Theorem, the Mean Value Theorem, and the Fundamental Theorem of Calculus.

Weierstrass's 4 Uniformity Theorems

- Cauchy had confused pointwise convergence of functions with uniform convergence of functions.
  - The functions $f_1, f_2, f_3, \ldots$ converge *pointwise* to f on $[a,b]$ if for each $x \in [a,b]$ and each $\varepsilon > 0$ there is a natural number N such that $|f_n(x) - f(x)| < \varepsilon$ for every $n \geq N$.
  - The functions $f_1, f_2, f_3, \ldots$ converge *uniformly* to f on $[a,b]$ if for each $\varepsilon > 0$ there is a natural number N such that $|f_n(x) - f(x)| < \varepsilon$ for every $n \geq N$ and every $x \in [a,b]$.
  - Example: Let $f_k(x) = x^k$ and let

    $$f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases}$$

    On $[0,1]$, (f_k) converges pointwise, but not uniformly, to f.
- Weierstrass clarified the distinction between these two types of convergence and proved the following 4 related theorems.
  - Theorem: If (f_k) is a sequence of continuous functions converging uniformly to f on $[a,b]$, then f is continuous.
    - ⋆ A counterexample in the case of pointwise convergence is provided above.
  - Theorem: If (f_k) is a sequence of bounded, Riemann-integrable functions converging uniformly to f on $[a,b]$, then f is Riemann-integrable on $[a,b]$ and

    $$\lim_{k\to\infty} \left[\int_a^b f_k(x)\,dx \right] = \int_a^b \left[\lim_{k\to\infty} f_k(x) \right] dx = \int_a^b f(x)\,dx.$$

    - ⋆ Counterexample in the case of pointwise convergence: $[a,b] = [0,1]$ and

      $$f_k(x) = \begin{cases} k & \text{if } 0 < x < 1/k, \\ 0 & \text{otherwise.} \end{cases}$$

- The Weierstrass Approximation Theorem: If f is continuous on $[a, b]$, then there is a sequence of polynomials that converges uniformly to f on $[a, b]$.
- The Weierstrass M-test: If $|f_k(x)| \leq M_k$ on $[a, b]$ for $k = 1, 2, 3, \ldots$, and $\sum_{k=1}^{\infty} M_k$ converges, then $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on $[a, b]$.

Set Theory in the 19th Century

Its Genesis

- Terminology
  - Bolzano gave us the word "set" in his 1851 publication of *Paradoxes of the Infinite*.
- Notation
  - Peano introduced the symbols $\cap$ for intersection and $\cup$ for union in 1888.
  - Peano introduced the symbol ε for membership in 1889. Bertrand Russell gave us the stylized $\in$ in 1903.
  - The symbol $\varnothing$ for the empty set was introduced by André Weil in 1939.
  - Cantor introduced the enclosure of the elements of a set in curly braces $\{\}$ in 1895.
  - The history of the symbol $\subseteq$ for "is a subset of" is unclear.
- Theory
  - Cantor is generally credited as the inventor of set theory. Particular innovations included:
    - ⋆ His work with infinite sets.
    - ⋆ His break from the "Part-Whole" theory of collections that was used by his predecessors and that had no distinction between a singleton $\{s\}$ and s itself. The fundamental relationship for Cantorian set theory was that of membership.

Infinite Sets

- Starting with ancient Greeks like Aristotle and extending into the 19th century, a distinction was commonly made between a "potential infinity" and an "actual infinity", with the former being in some way being too indefinite to form a set, and the existence of the latter being vocally doubted, even by mathematicians who seemed to be working with such entities.
  - Gauss (1831): "I protest against the use of an infinite quantity as an actual entity; this is never allowed in mathematics. The infinite is only a manner of speaking, in which one properly speaks of limits to which certain ratios can come as near as desired, while others are permitted to increase without bound."
- Cantor was among the first to take the view that there is nothing logically wrong with working with actual infinities. This view is held by the overwhelming majority of modern mathematicians.
  - Cantor (1887): "According to Herbart, the concept of the infinite should 'rest on the idea of a *movable boundary* that can be pushed ever further back in the blink of an eye'. ...Do the gentlemen recall that, apart from the journeys that take place in fantasies or dreams, for safe hiking or travel it is absolutely necessary to have *stable ground* and a *smooth path*, a path that never breaks off but remains and is passable wherever the journey leads? ...The long journey that Herbart prescribes for his '*movable boundary*' is avowedly not on a finite path, so its path must be an infinite path, indeed an *actual infinite path* since it itself doesn't move but is stable everywhere. Therefore, every potential infinity (*the moving boundary*) demands an *actual infinity* (the secure path for travel) *and can't be thought of otherwise*. ...However, since we, through our work, have secured the broad highway of the Transfinite and have carefully graded and paved it, we open it to traffic and make it an iron foundation, usable by all friends of the potential infinite, in particular to the wanderlustig Herbartian 'boundary'; gladly and calmly we leave to the restless the monotony of their unenviable fate; no matter how far they wander, the ground will never again disappear under their feet. Have a nice trip!"
- Cantor's road to investigating infinite sets:

    - Along with other mathematicians of the late 1800s he was interested in the question of representing functions with trigonometric series of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx].$$

    - In particular, he was interested in the question: Of what sets S is it true that if $f(x) = 0$ on S, then $0 = a_0 = b_1 = a_1 = b_2 = a_2 = \cdots$?
    - He knew it was true of $S = \mathbb{R}$ and of $S = \mathbb{R}$ minus a finite number of points.
    - He knew of an operation H that turned one set into another and that was such that if it was true of S, then it was true of $H(S)$.
    - Applying this operation repeatedly gives more and more sets, which Cantor needed to index somehow.
    - His indices ended up being the ordinal numbers, including the transfinite ones.
- Cantor *defined* a set to be infinite if its elements could be put in a one-to-one correspondence with the elements of a proper subset of itself.
- Two sets had the same *size* or *power* or *cardinality* if there was a one-to-one correspondence matching up their elements.
- A set is *uncountable* if their is no one-to-correspondence matching its elements with the elements of a subset of $\mathbb{N}$.
- In 1873, Cantor proved that the set of real numbers is uncountable.
    - His more famous “diagonalization” proof of the same fact came in 1891.
    - In his 1873 work, this was only presented as a stepping stone to a nonconstructive proof of the existence of transcendental numbers.
- In 1877, Cantor showed lines, planes, and higher-dimensional objects all have the same cardinality.
    - The idea: Interweave decimal expansions.
- In 1891, Cantor not only proved that the infinite set of real numbers was “bigger” than the infinite set of natural numbers but also proved that there is no “biggest” set.
    - Definition: The *power set* of a set X is the set whose elements are the subsets of X.
    - Example: The power set of $\{1, 2, 3\}$ is the set

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

    - Theorem: There is no one-to-one correspondence between the elements of the power set of X and the elements of a subset of X.
    - Proof: Suppose, to the contrary, that there were such a correspondence from the power set of X to a subset S of X. For each $s \in S$, let $f(s)$ be the subset of X with which it corresponds (under this correspondence). Every subset of X is supposed to be f of something, but we'll show that that's impossible. In particular, consider the following subset of X:

$$T := \{s \in S \mid s \notin f(s)\}.$$

      Can T be f of something? Suppose $T = f(t)$. Either $t \in T$ or $t \notin T$. If $t \in T$, then $t \in f(t)$, so t doesn't satisfy the condition for being in T, which means that $t \notin T$, which means that the first of the two options doesn't hold. If, on the other hand, $t \notin T$, then $t \notin f(t)$, so t *does* satisfy the condition for being in T, which means that $t \in T$, so the second of the two options doesn't hold. This contradiction means that our original supposition must have been wrong, so the theorem holds.
    - Among other things, this means that there is no universal set that has everything as a member.

Reaction

- Kronecker, Cantor's advisor, was extremely conservative on these issues, at times denying the existence of irrational numbers. Cantor felt that Kronecker hindered his work on sets.
- Poincaré (1908): "Later generations will regard set theory as a disease from which one has recovered."
- Hilbert (1926): "[Cantor's transfinite arithmetic is] the most astonishing product of mathematical thought, one of the most beautiful realizations of human activity in the domain of the purely intelligible. ... *No one shall expel us from the paradise which Cantor created for us.*"
- It can be plausibly argued that set theory is the basis for the vast majority of modern mathematics.

Logical Issues in the 20th Century

The Paradoxes of Naive Set Theory

- Most boil down to sophisticated versions of the question: Is the sentence "This sentence is false." true or false?
- Recently, careful students of Cantor have made a strong case that Cantor knew of these and used careful phrasing to side-step them.
  - Deiser (2004): "Long before presenting his definition of a 'set' [in 1895], Cantor was well aware that not every multitude could be considered a set."
- Russell's Paradox
  - If there is a Universal Set U, it is an element of itself. We can say it is *self-contained*. Consider the set S of all sets that are not self-contained. Is S self-contained?

Responses to the Paradoxes

- Rejection of an absolute Principle of Comprehension
  - The Principle of Comprehension says that given any property there is a set consisting precisely of those things having that property.
  - Taking the property of being identical with oneself we would get the set of everything.
- Distinction between sets and classes
  - Any collection qualifies as a class. Only certain well-behaved classes qualify as sets.
  - Classes that are not sets are *proper classes*.
  - Usage of the word "class" to mean a collection preceded usage of the word "set" to mean a collection. Usage of it in this particular sense, however, was introduced by Von Neumann in the 1920s.
- Theory of types
  - Different sorts of objects are assigned different *types*, and a set of one type can only have elements that are of a lower type.
  - Used in Russell and Whitehead's *Principia Mathematica* in the early 1910s.
    - ⋆ It is an example of *logicism*: the attempt to reduce all of mathematics to basic logic.
    - ⋆ It is over 2000 pages long in 3 volumes.
    - ⋆ It is practically unreadable.
    - ⋆ On page 360, the authors are finally able to say: "From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$."
- Axiomatic Set Theory
  - The standard axiomatization was introduced by Zermelo in 1908 and refined by Fraenkel in the 1920s.
    - ⋆ It is called "Zermelo-Fraenkel Set Theory with the Axiom of Choice" and abbreviated ZFC.
    - ⋆ If the Axiom of Choice is dropped, it is abbreviated ZF.

The Axiom of Choice

- Statement: If S is a collection of nonempty pairwise disjoint sets, there is a set T consisting of exactly one element from each of the sets in S.
- The name comes from the idea of choosing one element from each set in the collection.
- It was introduced by Zermelo as one of his axioms for set theory in 1904.
- Zermelo used it to prove that every set can be well-ordered; i.e., that the elements of a set can be ordered in such a way that every nonempty subset has a least element with respect to that order.

- It implies the Banach-Tarski Paradox (presented by Banach and Tarski in 1924): A ball in 3-dimensional space can be decomposed into finitely many pieces (5, actually) in such a way that those 5 pieces can be reassembled through rigid motions (translation and rotation) into two balls of the same size as the original.
- If the axiom were false, then other strange things would result. For example, there would be two sets with the property that neither could be put in one-to-one correspondence with a subset of the other. That is, there would be two sets of incomparable sizes.
- In 1940, Gödel proved that if there is no logical contradiction in the axioms of ZF then there is no logical contradiction in the axioms of ZFC.
- In 1963, Cohen proved that if there is no logical contradiction in the axioms of ZF then no logical contradiction is introduced by assuming the *negation* of the Axiom of Choice.
- Together, Gödel's and Cohen's work showed that the Axiom of Choice is independent of ZF: ZF can't be used to prove or disprove it.
- The status of the Axiom of Choice with respect to ZFC is analogous to the status of the Parallel Postulate with respect to Euclid's axioms.
- Most modern mathematicians accept and use the Axiom of Choice freely.

The Continuum Hypothesis

- Statement: Every infinite subset of $\mathbb{R}$ has the same cardinality as $\mathbb{N}$ or the same cardinality as $\mathbb{R}$; there is no cardinality between $\aleph_0$ and c.
- It was conjectured by Cantor in the 1870s.
- On August 26, 1884, he wrote a letter to Mittag-Leffler (editor of *Acta Mathematica*) to say he had a simple proof of it.
- On October 20, 1884, he wrote to Mittag-Leffler again to say that his previously announced proof was wrong.
- On November 14, 1884, he wrote to Mittag-Leffler again to say that he had a proof that the Continuum Hypothesis was false.
- On November 15, 1884, he wrote to Mittag-Leffler again to say that the latest proof was wrong.
- Hilbert's 1st Problem was to resolve the Continuum Hypothesis.
- In 1938, Gödel proved that if ZFC was logically consistent, then no contradiction would be introduced by assuming the Continuum Hypothesis.
- In 1963, Cohen proved that if ZFC was logically consistent, then no contradiction would be introduced by assuming the *negation* of the Continuum Hypothesis.
- Thus, ZFC can't prove or disprove the Continuum Hypothesis.

Constructivism and Intuitionism

- Kronecker: "Definitions must contain the means of reaching a decision in a finite number of steps, and existence proofs must be conducted so that the quantity in question can be calculated with any required degree of accuracy."
- Kronecker's position was an extreme form of *constructivism*.
- In the early 1900s, a school of mathematics known as *Intuitionism* attracted some leading mathematicians, with Brouwer being the leader of the movement.
- Before getting involved with Intuitionism, Brouwer had proved *The Brouwer Fixed Point Theorem*, which stated that every continuous function from a three-dimensional ball to itself has a fixed point, *i.e.*, a point in the ball that is mapped to itself. His proof was nonconstructive.
- Such proofs were eventually rejected by Brouwer and were rejected by other Intuitionists.

- Intuitionists rejected the Law of the Excluded Middle, which says that statements are true or false. Thus, while they accepted $A \Rightarrow \sim (\sim A)$ as a logical identity, they didn't accept $\sim (\sim A) \Rightarrow A$ as a logical identity.
- Hilbert: "Taking the law of the excluded middle from mathematicians is the same as prohibiting the astronomer his telescope or the boxer the use of his fists."
- Intuitionists rejected the Axiom of Choice.
- Intuitionists did not accept functions like

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0. \end{cases}$$

  It is a theorem of Intuitionistic analysis that all functions are continuous.
- Weyl, a student of Hilbert's who "converted" to Intuitionism: "In this book, I am not concerned to disguise the 'solid rock' on which the house of analysis is built with a wooden platform of formalism, in order to talk the reader into believing at the end that this platform is the true foundation. What will be proposed is rather the view that this house is largely built on sand."
- Hilbert (1922): "What Weyl and Brouwer are doing is mainly following in the path of Kronecker; they are trying to establish mathematics by throwing everything overboard that does not suit them and dictatorially promulgating an embargo. The effect of this is to dismember and cripple our science and to run the risk of losing a large part of our most valuable possessions ... Brouwer's program is not, as Weyl believes it to be, the Revolution, but only the repetition of a vain Putsch, which then was undertaken with greater dash, yet failed utterly."
- Only a tiny minority of mathematicians today have Intuitionist sympathies.

Consistency and Completeness of Formal Systems

- Formal systems are systems that attempt to reduce mathematical reasoning to a mechanical process of applying a set of transformation rules (corresponding to logical inferences) to finite strings of characters. A proof is a finite list of strings such that every string in the list follows from applying a transformation rule to previous strings in the list. The statement being proved is the last string in the list.
- Introduced by Hilbert.
- A formal system is *consistent* if it is not possible to prove some statement and its logical negation.
- Hilbert's 2nd Problem was to prove that the axioms of arithmetic are consistent.
- A formal system is *complete* if every true statement expressible in the system can be proved in the system.
- Gödel's First Incompleteness Theorem (1931): Any consistent formal system S within which a certain amount of elementary arithmetic can be carried out is incomplete with regard to statements of elementary arithmetic: there are such statements which can neither be proved, nor disproved, in S.
- Gödel's Second Incompleteness Theorem (1931): For any consistent formal system S within which a certain amount of elementary arithmetic can be carried out, the consistency of S cannot be proved in S itself.
- The essence of the proof was the encoding of the sentence "This sentence cannot be proved." in the language of the system.
- Many *interpretations* of Gödel's work are overblown. See Torkel Franzén's *Gödel's Theorem: An Incomplete Guide to its Use and Abuse.*