ODEs and Dynamical Systems

Lecture 1
Math 634
8/30/99

Ordinary Differential Equations

An ordinary differential equation (or ODE) is an equation involving derivatives of an unknown quantity with respect to a single variable. More precisely, suppose $j, k \in \mathbb{N}$, E is a Euclidean space, and

$$F : \mathrm{dom}(F) \subseteq \mathbb{R} \times \overbrace{E \times \cdots \times E}^{n+1 \text{ copies}} \to \mathbb{R}^j. \tag{1}$$

Then an *nth order ordinary differential equation* is an equation of the form

$$F(t, x(t), \dot{x}(t), \ddot{x}(t), x^{(3)}(t), \cdots, x^{(n)}(t)) = 0. \tag{2}$$

If $\mathcal{I} \subseteq \mathbb{R}$ is an interval, then $x : \mathcal{I} \to E$ is said to be *a solution of* (2) *on* $\mathcal{I}$ if x has derivatives up to order n at every $t \in \mathcal{I}$, and those derivatives satisfy (2). Often, we will suppress the dependence of x on t. Also, there will often be side conditions given that narrow down the set of solutions. In this class, we will concentrate on *initial conditions* which prescribe $x^{(\ell)}(t_0)$ for some fixed $t_0 \in \mathbb{R}$ (called the *initial time*) and some choices of $\ell \in \{0, 1, \ldots, n\}$. Some ODE classes study *two-point boundary-value problems*, in which the value of a function and its derivatives at two different points are required to satisfy given algebraic equations, but we won't focus on them in this one.

First-order Equations

Every ODE can be transformed into an equivalent first-order equation. In particular, given $x : \mathcal{I} \to E$, suppose we define

$$\begin{aligned} y_1 &:= x \\ y_2 &:= \dot{x} \\ y_3 &:= \ddot{x} \\ &\vdots \\ y_n &:= x^{(n-1)}, \end{aligned}$$

and let $y : \mathcal{I} \to E^n$ be defined by $y = (y_1, \dots, y_n)$. For $i = 1, 2, \dots, n-1$, define

$$G_i : \mathbb{R} \times E^n \times E^n \to \mathbb{R}$$

by

$$\begin{aligned} G_1(t, u, p) &:= p_1 - u_2 \\ G_2(t, u, p) &:= p_2 - u_3 \\ G_3(t, u, p) &:= p_3 - u_4 \\ &\vdots \\ G_{n-1}(t, u, p) &:= p_{n-1} - u_n, \end{aligned}$$

and, given F as in (1), define $G_n : \mathrm{dom}(G_n) \subseteq \mathbb{R} \times E^n \times E^n \to \mathbb{R}^j$ by

$$G_n(t, u, p) := F(t, u_1, \dots, u_n, p_1),$$

where

$$\mathrm{dom}(G_n) = \big\{(t, u, p) \in \mathbb{R} \times E^n \times E^n \;\big|\; (t, u_1, \dots, u_n, p_1) \in \mathrm{dom}(F)\big\}.$$

Letting $G : \mathrm{dom}(G_n) \subseteq \mathbb{R} \times E^n \times E^n \to \mathbb{R}^{n-1+j}$ be defined by

$$G := \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ \vdots \\ G_n \end{pmatrix},$$

we see that x satisfies (2) if and only if y satisfies $G(t, y(t), \dot{y}(t)) = 0$.

Equations Resolved w.r.t. the Derivative

Consider the first-order initial-value problem (or IVP)

$$\begin{cases} F(t, x, \dot{x}) = 0 \\ x(t_0) = x_0 \\ \dot{x}(t_0) = p_0, \end{cases} \tag{3}$$

where $F : \mathrm{dom}(F) \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and x_0, p_0 are given elements of $\mathbb{R}^n$ satisfying $F(t_0, x_0, p_0) = 0$. The Implicit Function Theorem says that typically the solutions (t, x, p) of the (algebraic) equation $F(t, x, p) = 0$ near (t_0, x_0, p_0) form an $(n+1)$-dimensional surface that can be parametrized by (t, x). In other words, locally the equation $F(t, x, p) = 0$ is equivalent to an equation of the form $p = f(t, x)$ for some $f : \mathrm{dom}(f) \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ with (t_0, x_0) in the interior of $\mathrm{dom}(f)$. Using this f, (3) is locally equivalent to the IVP

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0. \end{cases}$$

Autonomous Equations

Let $f : \mathrm{dom}(f) \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. The ODE

$$\dot{x} = f(t, x) \tag{4}$$

is *autonomous* if f doesn't really depend on t, *i.e.*, if $\mathrm{dom}(f) = \mathbb{R} \times \Omega$ for some $\Omega \subseteq \mathbb{R}^n$ and there is a function $g : \Omega \to \mathbb{R}^n$ such that $f(t, u) = g(u)$ for every $t \in \mathbb{R}$ and every $u \in \Omega$.

Every nonautonomous ODE is actually equivalent to an autonomous ODE. To see why this is so, given $x : \mathbb{R} \to \mathbb{R}^n$, define $y : \mathbb{R} \to \mathbb{R}^{n+1}$ by $y(t) = (t, x_1(t), \dots, x_n(t))$, and given $f : \mathrm{dom}(f) \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, define a new function $\tilde{f} : \mathrm{dom}(\tilde{f}) \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$\tilde{f}(p) = \begin{pmatrix} 1 \\ f_1(p_1, (p_2, \dots, p_{n+1})) \\ \vdots \\ f_n(p_1, (p_2, \dots, p_{n+1})) \end{pmatrix},$$

where $f = (f_1, \dots, f_n)^T$ and

$$\mathrm{dom}(\tilde{f}) = \big\{ p \in \mathbb{R}^{n+1} \;\big|\; (p_1, (p_2, \dots, p_{n+1})) \in \mathrm{dom}(f) \big\}.$$

Then x satisfies (4) if and only if y satisfies $\dot{y} = \tilde{f}(y)$.

Because of the discussion above, we will focus our study on first-order autonomous ODEs that are resolved w.r.t. the derivative. This decision is not completely without loss of generality, because by converting other

sorts of ODEs into an equivalent one of this form, we may be neglecting some special structure that might be useful for us to consider. This trade-off between abstractness and specificity is one that you will encounter (and have probably already encountered) in other areas of mathematics. Sometimes, when transforming the equation would involve too great a loss of information, we'll specifically study higher-order and/or nonautonomous equations.

Dynamical Systems

As we shall see, by placing conditions on the function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ and the point $x_0 \in \Omega$ we can guarantee that the autonomous IVP

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \tag{5}$$

has a solution defined on some interval I containing 0 in its interior, and this solution will be unique (up to restriction or extension). Furthermore, it is possible to "splice" together solutions of (5) in a natural way, and, in fact, get solutions to IVPs with different initial times. These considerations lead us to study a structure known as a *dynamical system*.

Given $\Omega \subseteq \mathbb{R}^n$, a continuous dynamical system (or a *flow*) on Ω is a function $\varphi : \mathbb{R} \times \Omega \to \Omega$ satisfying:

1. $\varphi(0, x) = x$ for every $x \in \Omega$;
2. $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$ for every $x \in \Omega$ and every $s, t \in \mathbb{R}$;
3. φ is continuous.

If f and Ω are sufficiently "nice" we will be able to define a function $\varphi : \mathbb{R} \times \Omega \to \Omega$ by letting $\varphi(\cdot, x_0)$ be the unique solution of (5), and this definition will make φ a dynamical system. Conversely, any continuous dynamical system $\varphi(t, x)$ that is differentiable w.r.t. t is generated by an IVP.

<u>Exercise 1</u> Suppose that:

- $\varphi : \mathbb{R} \times \Omega \to \Omega$ is a continuous dynamical system;
- $\dfrac{\partial \varphi(t,x)}{\partial t}$ exists for every $t \in \mathbb{R}$ and every $x \in \Omega$;
- $x_0 \in \Omega$ is given;
- $y : \mathbb{R} \to \Omega$ is defined by $y(t) := \varphi(t, x_0)$;
- $f : \Omega \to \Omega$ is defined by $f(p) := \left.\dfrac{\partial \varphi(s,p)}{\partial s}\right|_{s=0}$.

Show that y solves the IVP

$$\begin{cases} \dot{y} = f(y) \\ y(0) = x_0. \end{cases}$$

In this class (and Math 635) we will also study *discrete dynamical systems*. Given $\Omega \subseteq \mathbb{R}^n$, a discrete dynamical system on Ω is a function $\varphi : \mathbb{Z} \times \Omega \to \Omega$ satisfying:

1. $\varphi(0, x) = x$ for every $x \in \Omega$;
2. $\varphi(\ell, \varphi(m, x)) = \varphi(\ell + m, x)$ for every $x \in \Omega$ and every $\ell, m \in \mathbb{Z}$;
3. φ is continuous.

There is a one-to-one correspondence between discrete dynamical systems φ and *homeomorphisms* (continuous invertible functions) $F : \Omega \to \Omega$, this correspondence being given by $\varphi(1, \cdot) = F$. If we relax the requirement of invertibility and take a (possibly noninvertible) continuous function $F : \Omega \to \Omega$ and define $\varphi : \{0, 1, \dots\} \times \Omega \to \Omega$ by

$$\varphi(n, x) = \overbrace{F(F(\cdots(F}^{n \text{ copies}}(x))\cdots)),$$

then φ will almost meet the requirements to be a dynamical system, the only exception being that property 2, known as the *group property* may fail because $\varphi(n, x)$ is not even defined for $n < 0$. We may still call this

a dynamical system; if we're being careful we may call it a *semidynamical system*.

In a dynamical system, the set Ω is called the *phase space*. Dynamical systems are used to describe the evolution of physical systems in which the state of the system at some future time depends only on the initial state of the system and on the elapsed time. As an example, Newtonian mechanics permits us to view the earth-moon-sun system as a dynamical system, but the phase space is not physical space $\mathbb{R}^3$, but is instead an 18-dimensional Euclidean space in which the coordinates of each point reflect the position and momentum of each of the three objects. (Why isn't a 9-dimensional space, corresponding to the three spatial coordinates of the three objects, sufficient?)