Operators on Generalized Eigenspaces

Lecture 10
Math 634
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We've seen that the space on which a linear operator acts can be decomposed into the direct sum of generalized eigenspaces of that operator. The operator maps each of these generalized eigenspaces into itself, and, consequently, solutions of the differential equation starting in a generalized eigenspace stay in that generalized eigenspace for all time. Now we will see how the solutions within such a subspace behave by seeing how the operator behaves on this subspace.

It may seem like nothing much can be said in general since, given a finite-dimensional vector space $\mathcal{V}$, we can define a nilpotent operator S on $\mathcal{V}$ by

1. picking a basis $\{v_1, \dots, v_m\}$ for $\mathcal{V}$;
2. creating a graph by connecting the nodes $\{v_1, \dots, v_m, 0\}$ with directed edges in such a way that from each node there is a unique directed path to 0;
3. defining $S(v_j)$ to be the unique node v_k such that there is a directed edge from v_j to v_k;
4. extending S linearly to all of $\mathcal{V}$.

By adding any multiple of I to S we have an operator for which $\mathcal{V}$ is a generalized eigenspace. It turns out, however, that there are really only a small number of different possible structures that may arise from this seemingly general process.

To make this more precise, we first need a definition, some new notation, and a lemma.

Definition A subspace $\mathcal{Z}$ of a vector space $\mathcal{V}$ is a *cyclic subspace of S on $\mathcal{V}$* if $S\mathcal{Z} \subseteq \mathcal{Z}$ and there is some $x \in \mathcal{Z}$ such that $\mathcal{Z}$ is spanned by $\{x, Sx, S^2x, \dots\}$.

Given S, note that every vector $x \in \mathcal{V}$ generates a cyclic subspace. Call it $\mathcal{Z}(x)$ or $\mathcal{Z}(x, S)$. If S is nilpotent, write $\mathrm{nil}(x)$ or $\mathrm{nil}(x, S)$ for the smallest nonnegative integer k such that $S^k x = 0$.

Lemma *The set* $\{x, Sx, \ldots, S^{\mathrm{nil}(x)-1}x\}$ *is a basis for* $\mathcal{Z}(x)$.

Proof. Obviously these vectors span $\mathcal{Z}(x)$; the question is whether they are linearly independent. If they were not, we could write down a linear combination $\alpha_1 S^{p_1}x + \cdots + \alpha_k S^{p_k}x$, with $\alpha_j \neq 0$ and $0 \leq p_1 < p_2 < \cdots < p_k \leq \mathrm{nil}(x) - 1$, that added up to zero. Applying $S^{\mathrm{nil}(x)-p_1-1}$ to this linear combination would yield $\alpha_1 S^{\mathrm{nil}(x)-1}x = 0$, contradicting the definition of $\mathrm{nil}(x)$. $\square$

Theorem *If* $S : \mathcal{V} \to \mathcal{V}$ *is nilpotent then* $\mathcal{V}$ *can be written as the direct sum of cyclic subspaces of* S *on* $\mathcal{V}$. *The dimensions of these subspaces are determined by the operator* S.

Proof. The proof is inductive on the dimension of $\mathcal{V}$. It is clearly true if $\dim \mathcal{V} = 0$ or 1. Assume it is true for all operators on spaces of dimension less than $\dim \mathcal{V}$.

Step 1: The dimension of $S\mathcal{V}$ is less than the dimension of $\mathcal{V}$.
If this weren't the case, then S would be invertible and could not possibly be nilpotent.

Step 2: For some $k \in \mathbb{N}$ and for some nonzero $y_j \in S\mathcal{V}$, $j = 1, \ldots, k$,

$$S\mathcal{V} = \mathcal{Z}(y_1) \oplus \cdots \oplus \mathcal{Z}(y_k). \tag{1}$$

This is a consequence of Step 1 and the induction hypothesis.

Pick $x_j \in \mathcal{V}$ such that $Sx_j = y_j$, for $j = 1, \ldots, k$. Suppose that $z_j \in \mathcal{Z}(x_j)$ for each j and

$$z_1 + \cdots + z_k = 0. \tag{2}$$

We will show that $z_j = 0$ for each j. This will mean that the direct sum $\mathcal{Z}(x_1) \oplus \cdots \oplus \mathcal{Z}(x_k)$ exists.

Step 3: $Sz_1 + \cdots + Sz_k = 0$.
This follows from applying S to both sides of (2).

Step 4: For each j, $Sz_j \in \mathcal{Z}(y_j)$.
The fact that $z_j \in \mathcal{Z}(x_j)$ implies that

$$z_j = \alpha_0 x_j + \alpha_1 S x_j + \cdots + \alpha_{\mathrm{nil}(x_j)-1} S^{\mathrm{nil}(x_j)-1} x_j \tag{3}$$

for some α_i. Applying S to both sides of (3) gives

$$Sz_j = \alpha_0 y_j + \alpha_1 S y_j + \cdots + \alpha_{\mathrm{nil}(x_j)-2} S^{\mathrm{nil}(x_j)-2} y_j \in \mathcal{Z}(y_j).$$

Step 5: For each j, $Sz_j = 0$.
This is a consequence of Step 3, Step 4, and (1).

Step 6: For each j, $z_j \in \mathcal{Z}(y_j)$.
If

$$z_j = \alpha_0 x_j + \alpha_1 S x_j + \cdots + \alpha_{\mathrm{nil}(x_j)-1} S^{\mathrm{nil}(x_j)-1} x_j$$

then by Step 5

$$0 = Sz_j = \alpha_0 y_j + \alpha_1 S y_j + \cdots + \alpha_{\mathrm{nil}(x_j)-2} S^{\mathrm{nil}(x_j)-2} y_j.$$

Since $\mathrm{nil}(x_j) - 2 = \mathrm{nil}(y_j) - 1$, the vectors in this linear combination are linearly independent; thus, $\alpha_i = 0$ for $i = 0, \ldots, \mathrm{nil}(x_j) - 2$. In particular, $\alpha_0 = 0$, so

$$z_j = \alpha_1 y_j + \cdots + \alpha_{\mathrm{nil}(x_j)-1} S^{\mathrm{nil}(x_j)-2} y_j \in \mathcal{Z}(y_j).$$

Step 7: For each j, $z_j = 0$.
This is a consequence of Step 6, (1), and (2).

We now know that $\mathcal{Z}(x_1) \oplus \cdots \oplus \mathcal{Z}(x_k) =: \tilde{\mathcal{V}}$ exists, but it is not necessarily all of $\mathcal{V}$. Choose a subspace $\mathcal{W}$ of $\mathrm{Null}(S)$ such that $\mathrm{Null}(S) = (\tilde{\mathcal{V}} \cap \mathrm{Null}(S)) \oplus \mathcal{W}$. Choose a basis $\{w_1, \ldots, w_\ell\}$ for $\mathcal{W}$ and note that $\mathcal{W} = \mathcal{Z}(w_1) \oplus \cdots \oplus \mathcal{Z}(w_\ell)$.

Step 8: The direct sum $\mathcal{Z}(x_1) \oplus \cdots \oplus \mathcal{Z}(x_k) \oplus \mathcal{Z}(w_1) \oplus \cdots \oplus \mathcal{Z}(w_\ell)$ exists.
This is a consequence of the fact that the direct sums $\mathcal{Z}(x_1) \oplus \cdots \oplus \mathcal{Z}(x_k)$

and $\mathcal{Z}(w_1) \oplus \cdots \oplus \mathcal{Z}(w_\ell)$ exist and that $\tilde{\mathcal{V}} \cap \mathcal{W} = \{0\}$.

<u>Step 9:</u> $\mathcal{V} = \mathcal{Z}(x_1) \oplus \cdots \oplus \mathcal{Z}(x_k) \oplus \mathcal{Z}(w_1) \oplus \cdots \oplus \mathcal{Z}(w_\ell)$.
Let $x \in \mathcal{V}$ be given. Recall that $Sx \in S\mathcal{V} = \mathcal{Z}(y_1) \oplus \cdots \oplus \mathcal{Z}(y_k)$. Write $Sx = s_1 + \cdots + s_k$ with $s_j \in \mathcal{Z}(y_j)$. If

$$s_j = \alpha_0 y_j + \alpha_1 S y_j + \cdots + \alpha_{\mathrm{nil}(y_j)-1} S^{\mathrm{nil}(y_j)-1} y_j,$$

let

$$u_j = \alpha_0 x_j + \alpha_1 S x_j + \cdots + \alpha_{\mathrm{nil}(y_j)-1} S^{\mathrm{nil}(y_j)-1} x_j,$$

and note that $Su_j = s_j$ and that $u_j \in \mathcal{Z}(x_j)$. Setting $u = u_1 + \cdots + u_k$, we have

$$S(x-u) = Sx - Su = (s_1 + \cdots + s_k) - (s_1 + \cdots + s_k) = 0,$$

so $x - u \in \mathrm{Null}(S)$. By definition of $\mathcal{W}$, that means that

$$x - u \in \mathcal{Z}(x_1) \oplus \cdots \oplus \mathcal{Z}(x_k) \oplus \mathcal{Z}(w_1) \oplus \cdots \oplus \mathcal{Z}(w_\ell).$$

Since $u \in \mathcal{Z}(x_1) \oplus \cdots \oplus \mathcal{Z}(x_k)$, we have

$$x \in \mathcal{Z}(x_1) \oplus \cdots \oplus \mathcal{Z}(x_k) \oplus \mathcal{Z}(w_1) \oplus \cdots \oplus \mathcal{Z}(w_\ell).$$

This completes the proof of the first sentence in the theorem. The second sentence follows similarly by induction. $\square$