Real Canonical Form

Lecture 11
Math 634
9/24/99

Real Canonical Form

We now use the information contained in the previous theorems to find simple matrices representing linear operators. Clearly, a nilpotent operator S on a cyclic space $\mathcal{Z}(x)$ can be represented by the matrix

$$\begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix},$$

with the corresponding basis being $\{x, Sx, \ldots, S^{\mathrm{nil}(x)-1}x\}$. Thus, an operator T on a generalized eigenspace $N(T - \lambda I)$ can be represented by a matrix of the form

$$\begin{bmatrix} \lambda & 0 & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \lambda \end{bmatrix}. \tag{1}$$

If $\lambda = a + bi \in \mathbb{C} \setminus \mathbb{R}$ is an eigenvalue of an operator $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, and $\mathcal{Z}(x, T - \lambda I)$ is one of the cyclic subspaces whose direct sum is $N(T - \lambda I)$, then $\mathcal{Z}(\overline{x}, T - \overline{\lambda} I)$ can be taken to be one of the cyclic subspaces whose direct sum is $N(T - \overline{\lambda} I)$. If we set $k = \mathrm{nil}(x, T - \lambda I) - 1$ and $y_j = \mathrm{Re}((T - \lambda I)^j x)$ and $z_j = \mathrm{Im}((T - \lambda I)^j x)$ for $j = 0, \ldots, k$, then we have $Ty_j = ay_j - bz_j + y_{j+1}$ and $Tz_j = by_j + az_j + z_{j+1}$ for $j = 0, \ldots, k-1$, and $Ty_k = ay_k - bz_k$ and $Tz_k = by_k + az_k$. The $2k + 2$ real vectors $\{z_0, y_0, \ldots, z_k, y_k\}$ span $Z(x, T - \lambda I) \oplus Z(\overline{x}, T - \overline{\lambda} I)$ over $\mathbb{C}$ and also span a $(2k+2)$-dimensional space over $\mathbb{R}$ that is invariant under T. On this real vector space, the action of T can be represented by the matrix

$$\left[\begin{array}{cc|cc|cc|cc|cc}
a & -b & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
b & a & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
\hline
1 & 0 & \ddots & \ddots & \ddots & \ddots & & & \vdots & \vdots \\
0 & 1 & \ddots & \ddots & \ddots & \ddots & & & \vdots & \vdots \\
\hline
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\hline
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & a & -b \\
0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 1 & b & a
\end{array}\right]. \tag{2}$$

The restriction of an operator to one of its generalized eigenspaces has a matrix representation like

$$\begin{bmatrix}
\begin{bmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{bmatrix} & & & & \\
& \begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix} & & & \\
& & \begin{bmatrix} \lambda \end{bmatrix} & & \\
& & & \begin{bmatrix} \lambda \end{bmatrix} & \\
& & & & \ddots
\end{bmatrix} \tag{3}$$

if the eigenvalue λ is real, with blocks of the form (1) running down the diagonal. If the eigenvalue is complex, then the matrix representation is similar to (3) but with blocks of the form (2) instead of the form (1) on the diagonal.

Finally, the matrix representation of the entire operator is block diagonal, with blocks of the form (3) (or its counterpart for complex eigenvalues). This is called the *real canonical form*. If we specify the order in which blocks should appear, then matrices are similar if and only if they have the same real canonical form.