Solving Linear Systems

Lecture 12
Math 634
9/27/99

Exercise 7 Classify all the real canonical forms for operators on $\mathbb{R}^4$. In other words, find a collection of 4×4 matrices, possibly with (real) variable entries and possibly with constraints on those variables, such that

1. Only matrices in real canonical form match one of the matrices in your collection.
2. Each operator on $\mathbb{R}^4$ has a matrix representation matching one of the matrices in your collection.
3. No matrix matching one of your matrices is similar to a matrix matching one of your other matrices.

For example, a suitable collection of matrices for operators on $\mathbb{R}^2$ would be:

$$\begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}; \qquad \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}; \qquad \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad (b \neq 0).$$

Computing e^{tA}

Given an operator $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, let M be its real canonical form. Write $M = S + N$, where S has M's diagonal elements λ_k and diagonal blocks

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

and 0's else, and N has M's off-diagonal 1's and 2×2 identity matrices. If you consider the restrictions of S and N to each of the cyclic subspaces of $A-\lambda I$ into which the generalized eigenspace $N(A-\lambda I)$ of A is decomposed, you'll probably be able to see that these restrictions commute. As a consequence of this fact (and the way $\mathbb{R}^n$ can be represented in terms of these cyclic subspaces), S and N commute. Thus $e^{tM} = e^{tS}e^{tN}$.

Now, e^{tS} has $e^{\lambda_k t}$ where S has λ_k, and has

$$\begin{bmatrix} e^{a_k t}\cos b_k t & -e^{a_k t}\sin b_k t \\ e^{a_k t}\sin b_k t & e^{a_k t}\cos b_k t \end{bmatrix}$$

where S has

$$\begin{bmatrix} a_k & -b_k \\ b_k & a_k \end{bmatrix}.$$

The series definition can be used to compute e^{tN}, since the fact that N is nilpotent implies that the series is actually a finite sum. The entries of e^{tN} will be polynomials in t. For example,

$$\begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & & & \\ t & \ddots & & \\ \vdots & \ddots & \ddots & \\ t^m & \cdots & t & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & & & \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \ddots & & \\ & \ddots & \ddots & \\ & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \mapsto \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & & & \\ \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} & \ddots & & \\ \vdots & \ddots & \ddots & \\ \begin{bmatrix} t^m/m! & 0 \\ 0 & t^m/m! \end{bmatrix} & \cdots & \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}.$$

Identifying A with its matrix representation with respect to the standard basis, we have $A = PMP^{-1}$ for some invertible matrix P. Consequently, $e^{tA} = Pe^{tM}P^{-1}$. Thus, the entries of e^{tA} will be linear combinations of polynomials times exponentials or polynomials times exponentials times trigonometric functions.

Exercise 8 Compute e^{tA} (and justify your computations) if

1. $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$

Linear Planar Systems

A thorough understanding of constant coefficient linear systems $\dot{x} = Ax$ in the plane is very helpful in understanding systems that are nonlinear and/or higher-dimensional.

There are 3 main categories of real canonical forms for an operator A in $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$:

- $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$
- $\begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$
- $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \qquad (b \neq 0)$

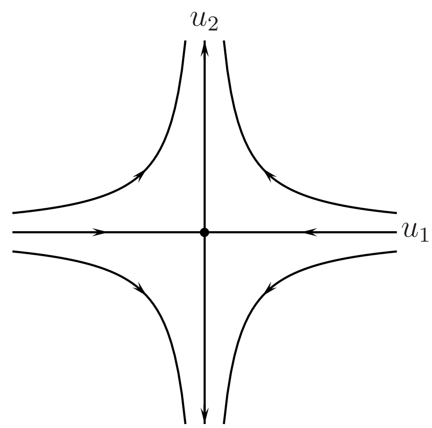
We will subdivide these 3 categories further into a total of 14 categories and consider the corresponding *phase portraits*, *i.e.*, sketches of some of the *trajectories* or parametric curves traced out by solutions in phase space.

$\boxed{\mathbf{1}}$

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

$(\lambda < 0 < \mu)$

saddle

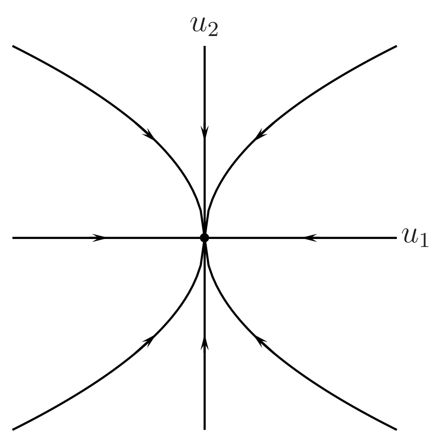


$\boxed{\mathbf{2}}$

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

$(\lambda < \mu < 0)$

stable node

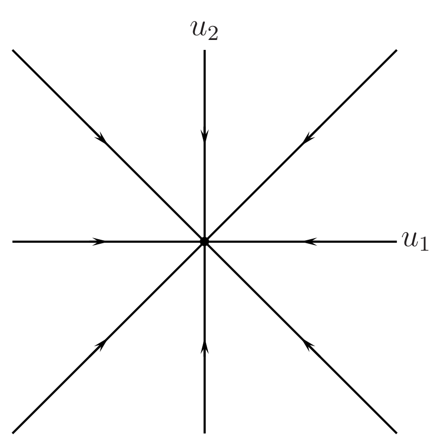


$\boxed{\mathbf{3}}$

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

$(\lambda = \mu < 0)$

stable node

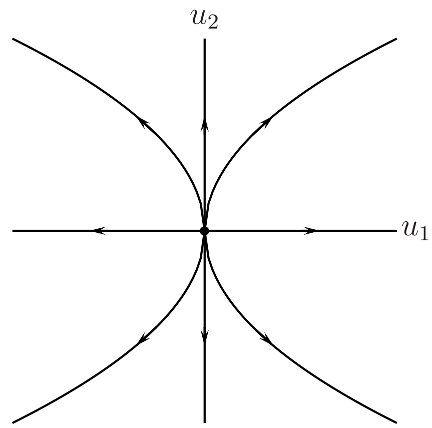


$\boxed{\mathbf{4}}$

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

$(0 < \mu < \lambda)$

unstable node

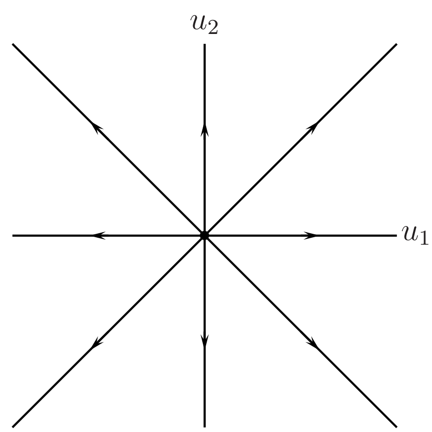


$\boxed{\mathbf{5}}$

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

$(0 < \lambda = \mu)$

unstable node

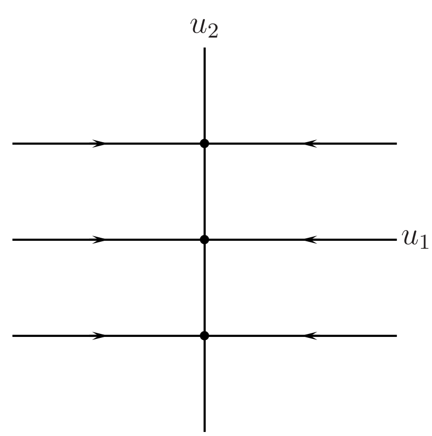


$\boxed{\mathbf{6}}$

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

$(\lambda < \mu = 0)$

degenerate

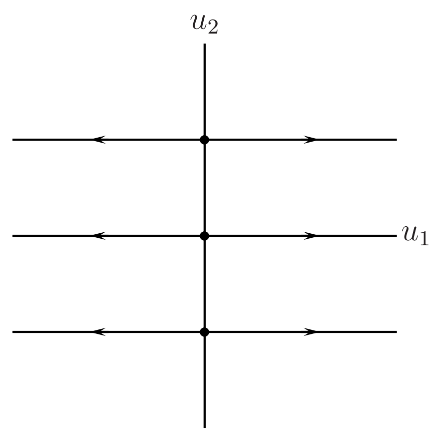


$\boxed{\mathbf{7}}$

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

$(0 = \mu < \lambda)$

degenerate

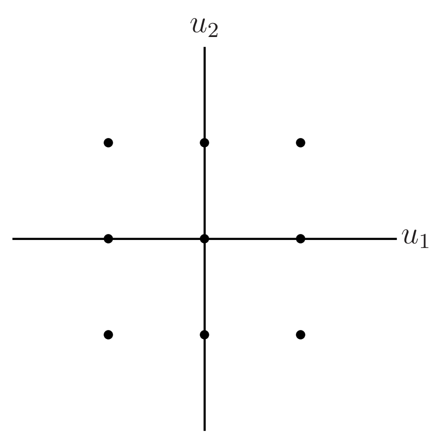


$\boxed{\mathbf{8}}$

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

$(0 = \mu = \lambda)$

degenerate

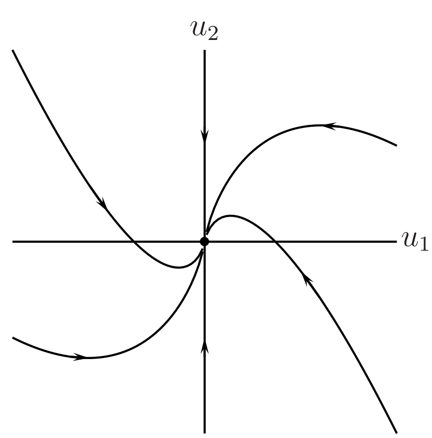


$\boxed{\mathbf{9}}$

$$A = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$

$(\lambda < 0)$

stable node

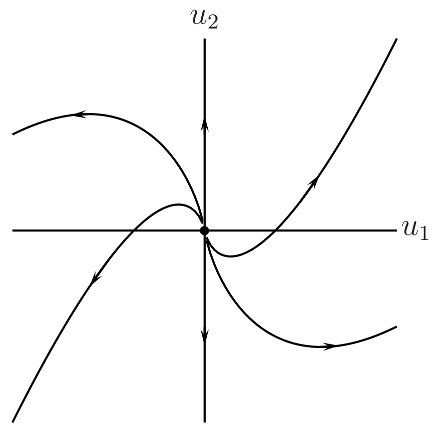


$\boxed{\mathbf{10}}$

$$A = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$

$(0 < \lambda)$

unstable node

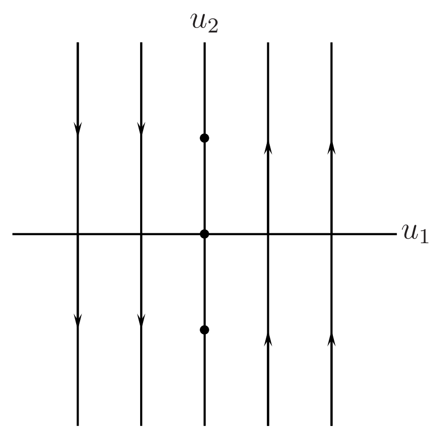


$\boxed{\mathbf{11}}$

$$A = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$

$(\lambda = 0)$

degenerate

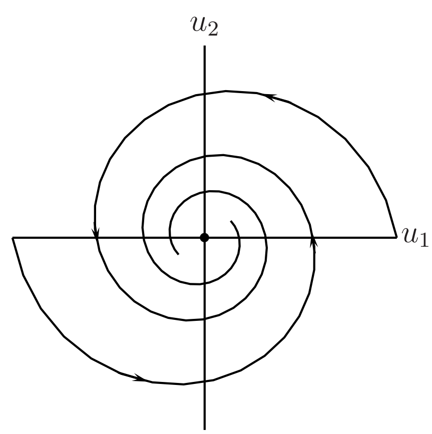


$\boxed{\mathbf{12}}$

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$(a < 0 < b)$

stable spiral

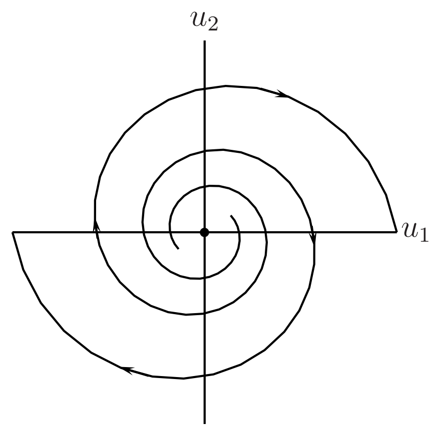


$\boxed{\mathbf{13}}$

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$(b < 0 < a)$

unstable spiral

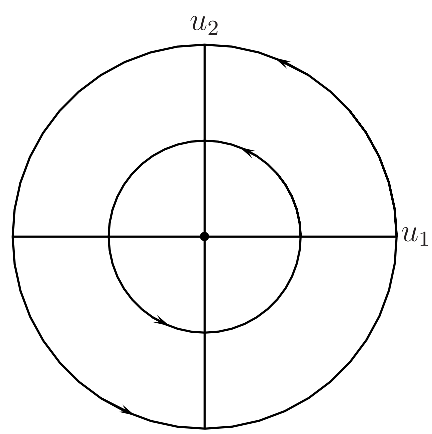


$\boxed{\mathbf{14}}$

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$(a = 0, b > 0)$

center



If A is not in real canonical form, then the phase portrait should look similar but may be rotated, flipped, stretched, skewed, etc.