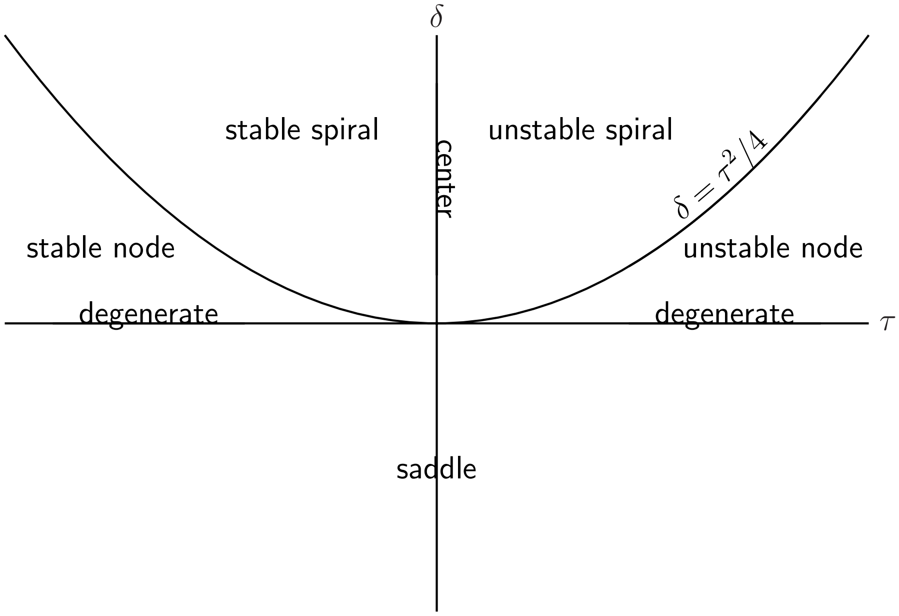
Qualitative Behavior of Linear Systems

Lecture 13
Math 634
9/29/99

Parameter Plane

Some of the information from the preceding phase portraits can be summarized in a parameter diagram. In particular, let $\tau = \operatorname{trace} A$ and let $\delta = \det A$, so the characteristic polynomial is $\lambda^2 - \tau\lambda + \delta$. Then the behavior of the trivial solution $x(t) \equiv 0$ is given by locating the corresponding point in the (τ, δ)-plane:



Growth and Decay Rates

Given $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, let

$$\mathcal{E}^u = \left\{ \bigoplus_{\lambda > 0} N(A - \lambda I) \right\} \oplus$$

$$\left\{ \bigoplus_{\substack{\operatorname{Re}\lambda > 0 \\ \operatorname{Im}\lambda \neq 0}} \{\operatorname{Re} u \mid u \in N(A - \lambda I)\} \right\} \oplus \left\{ \bigoplus_{\substack{\operatorname{Re}\lambda > 0 \\ \operatorname{Im}\lambda \neq 0}} \{\operatorname{Im} u \mid u \in N(A - \lambda I)\} \right\},$$

$$\mathcal{E}^s = \left\{ \bigoplus_{\lambda < 0} N(A - \lambda I) \right\} \oplus$$

$$\left\{ \bigoplus_{\substack{\operatorname{Re}\lambda < 0 \\ \operatorname{Im}\lambda \neq 0}} \{\operatorname{Re} u \mid u \in N(A - \lambda I)\} \right\} \oplus \left\{ \bigoplus_{\substack{\operatorname{Re}\lambda < 0 \\ \operatorname{Im}\lambda \neq 0}} \{\operatorname{Im} u \mid u \in N(A - \lambda I)\} \right\},$$

and

$$\mathcal{E}^c = N(A) \oplus$$

$$\left\{ \bigoplus_{\substack{\operatorname{Re}\lambda = 0 \\ \operatorname{Im}\lambda \neq 0}} \{\operatorname{Re} u \mid u \in N(A - \lambda I)\} \right\} \oplus \left\{ \bigoplus_{\substack{\operatorname{Re}\lambda = 0 \\ \operatorname{Im}\lambda \neq 0}} \{\operatorname{Im} u \mid u \in N(A - \lambda I)\} \right\}.$$

From our previous study of the real canonical form, we know that

$$\mathbb{R}^n = \mathcal{E}^u \oplus \mathcal{E}^s \oplus \mathcal{E}^c.$$

We call $\mathcal{E}^u$ the *unstable space* of A, $\mathcal{E}^s$ the *stable space* of A, and $\mathcal{E}^c$ the *center space* of A.

Each of these subspaces of $\mathbb{R}^n$ is invariant under the differential equation

$$\dot{x} = Ax. \tag{1}$$

In other words, if $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of (1) and $x(0)$ is in $\mathcal{E}^u$, $\mathcal{E}^s$, or $\mathcal{E}^c$, then $x(t)$ is in $\mathcal{E}^u$, $\mathcal{E}^s$, or $\mathcal{E}^c$, respectively, for all $t \in \mathbb{R}$. We shall see that each of these spaces is characterized by the growth or decay rates of the solutions it contains. Before doing so, we state and prove a basic fact about finite-dimensional normed vector spaces.

Theorem *All norms on $\mathbb{R}^n$ are equivalent.*

Proof. Since equivalence of norms is transitive, it suffices to prove that every norm $N : \mathbb{R}^n \to \mathbb{R}$ is equivalent to the standard Euclidean norm $|\cdot|$.

Given an arbitrary norm N, and letting x_i be the projection of $x \in \mathbb{R}^n$ onto the ith standard basis vector e_i, note that

$$\begin{aligned} N(x) = N\left(\sum_{i=1}^n x_i e_i\right) &\leq \sum_{i=1}^n |x_i| N(e_i) \leq \sum_{i=1}^n |x| N(e_i) \\ &\leq \left(\sum_{i=1}^n N(e_i)\right)|x|. \end{aligned}$$

This shows half of equivalence; it also shows that N is continuous, since, by the triangle inequality,

$$|N(x) - N(y)| \leq N(x-y) \leq \left(\sum_{i=1}^n N(e_i)\right)|x-y|.$$

The set $\mathcal{S} := \{x \in \mathbb{R}^n \mid |x| = 1\}$ is clearly closed and bounded and, therefore, compact, so by the extreme value theorem, N must achieve a minimum on $\mathcal{S}$. Since N is a norm (and is, therefore, positive definite), this minimum must be positive; call it k. Then for any $x \neq 0$,

$$N(x) = N\left(|x|\frac{x}{|x|}\right) = |x| N\left(\frac{x}{|x|}\right) \geq k|x|,$$

and the estimate $N(x) \geq k|x|$ obviously holds if $x = 0$, as well. □

Theorem *Given $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and the corresponding decomposition $\mathbb{R}^n = \mathcal{E}^u \oplus \mathcal{E}^s \oplus \mathcal{E}^c$, we have*

$$\mathcal{E}^u = \{x \in \mathbb{R}^n \mid \exists c > 0 \text{ s.t. } \lim_{t \downarrow -\infty} |e^{-ct} e^{tA} x| = 0\}, \tag{2}$$

$$\mathcal{E}^s = \{x \in \mathbb{R}^n \mid \exists c > 0 \text{ s.t. } \lim_{t \uparrow \infty} |e^{ct} e^{tA} x| = 0\}, \tag{3}$$

and

$$\mathcal{E}^c = \{x \in \mathbb{R}^n \mid \forall c > 0, \lim_{t \downarrow -\infty} |e^{ct} e^{tA} x| = 0 \text{ and } \lim_{t \uparrow \infty} |e^{-ct} e^{tA} x| = 0\}. \tag{4}$$

Proof. By equivalence of norms, instead of using the standard Euclidean norm on $\mathbb{R}^n$ we can use the norm

$$\|x\| := \sup\{|P_1 x|, \dots , |P_n x|\},$$

where $P_i : \mathbb{R}^n \to \mathbb{R}$ represents projection onto the ith basis vector corresponding to the real canonical form. Because of our knowledge of the structure of the real canonical form, we know that $P_i e^{tA} x$ is either of the form

$$p(t)e^{\lambda t}, \tag{5}$$

where $p(t)$ is a polynomial in t and $\lambda \in \mathbb{R}$ is an eigenvalue of A, or of the form

$$p(t)e^{at}(\alpha \cos bt + \beta \sin bt), \tag{6}$$

where $p(t)$ is a polynomial in t, $a + bi \in \mathbb{C} \setminus \mathbb{R}$ is an eigenvalue of A, and α and β are real constants. Furthermore, we know that if P_i corresponds to a vector in $\mathcal{E}^u$ then λ and a are positive, if P_i corresponds to a vector in $\mathcal{E}^s$ then λ and a are negative, and if P_i corresponds to a vector in $\mathcal{E}^c$ then λ and a are zero.

Now, suppose $x \in \mathcal{E}^s$. Then each $P_i e^{tA} x$ is either identically zero or has as a factor a negative exponential whose constant is the real part of an eigenvalue of A that is to the left of the imaginary axis in the complex plane. Let $\sigma(A)$ be the set of eigenvalues of A, and set

$$c = \frac{\bigl|\max\bigl\{\operatorname{Re}\lambda \bigm| \lambda \in \sigma(A) \text{ and } \operatorname{Re}\lambda < 0\bigr\}\bigr|}{2}.$$

Then $e^{ct} P_i e^{tA} x$ is either identically zero or decays exponentially to zero as $t \uparrow \infty$.

Conversely, suppose $x \notin \mathcal{E}^s$. Then $P_i x \neq 0$ for some P_i corresponding to a real canonical basis vector in $\mathcal{E}^u$ or in $\mathcal{E}^c$. In either case, $P_i e^{tA} x$ is not identically zero and is of the form (5) where $\lambda \geq 0$ or of the form (6) where $a \geq 0$. Thus, if $c > 0$ then

$$\limsup_{t\uparrow\infty} |e^{ct} P_i e^{tA} x| = \infty,$$

so

$$\limsup_{t\uparrow\infty} \|e^{ct} e^{tA} x\| = \infty.$$

The preceding two paragraphs showed that (3) is correct. By applying this fact to the time-reversed problem $\dot{x} = -Ax$, we find that (2) is correct, as well. We now consider (4).

If $x \in \mathcal{E}^c$, then for each i, $P_i e^{tA} x$ is either a polynomial or the product of a polynomial and a periodic function. If $c > 0$ and we multiply such a function of t by e^{ct} and let $t \downarrow -\infty$ or we multiply it by e^{-ct} and let $t \uparrow \infty$, then the result converges to zero.

If, on the other hand, $x \notin \mathcal{E}^c$ then for some i, $P_i e^{tA} x$ contains a nontrivial exponential term. If $c > 0$ is sufficiently small then either $e^{ct} P_i e^{tA} x$ diverges as $t \downarrow -\infty$ or $e^{-ct} P_i e^{tA} x$ diverges as $t \uparrow \infty$. This completes the verification of (4). $\square$