Exponential Decay

Lecture 14
Math 634
10/1/99

Definition If $\mathcal{E}^u = \mathbb{R}^n$, we say that the origin is a *source* and e^{tA} is an *expansion.*

Definition If $\mathcal{E}^s = \mathbb{R}^n$, we say that the origin is a *sink* and e^{tA} is a *contraction.*

Theorem

(a) *The origin is a source for the equation $\dot{x} = Ax$ if and only if for a given norm $\|\cdot\|$ there are constants $k, b > 0$ such that*

$$\|e^{tA}x\| \leq ke^{tb}\|x\|$$

for every $t \leq 0$ and $x \in \mathbb{R}^n$.

(b) *The origin is a sink for the equation $\dot{x} = Ax$ if and only if for a given norm $\|\cdot\|$ there are constants $k, b > 0$ such that*

$$\|e^{tA}x\| \leq ke^{-tb}\|x\|$$

for every $t \geq 0$ and $x \in \mathbb{R}^n$.

Proof. The "if" parts are a consequence of the previous theorem. The "only if" parts follow from the proof of the previous theorem. □

Note that a contraction does not always "contract" things immediately; *i.e.*, $|e^{tA}x| \not\leq |x|$, in general. For example, consider

$$A = \begin{bmatrix} -1/4 & 0 \\ 1 & -1/4 \end{bmatrix}.$$

If

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

is a solution of $\dot{x} = Ax$, then

$$\frac{d}{dt}|x(t)|^2 = 2\langle x, \dot{x}\rangle = 2\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} -1/4 & 0 \\ 1 & -1/4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{2}x_1^2 + 2x_1x_2 - \frac{1}{2}x_2^2$$
$$= x_1x_2 - \frac{1}{2}(x_1 - x_2)^2,$$

which is greater than zero if, for example, $x_1 = x_2 > 0$. However, we have the following:

Theorem

(a) *If e^{tA} is an expansion then there is* some *norm $\|\cdot\|$ and some constant $b > 0$ such that*

$$\|e^{tA}x\| \leq e^{tb}\|x\|$$

for every $t \leq 0$ and $x \in \mathbb{R}^n$.

(b) *If e^{tA} is a contraction then there is* some *norm $\|\cdot\|$ and some constant $b > 0$ such that*

$$\|e^{tA}x\| \leq e^{-tb}\|x\|$$

for every $t \geq 0$ and $x \in \mathbb{R}^n$.

Proof. The idea of the proof is to pick a basis with respect to which A is represented by a matrix like the real canonical form but with some small constant $\varepsilon > 0$ in place of the off-diagonal 1's. (This can be done by rescaling.) If the Euclidean norm with respect to this basis is used, the desired estimates hold. The details of the proof may be found in Chapter 7, §1, of Hirsch and Smale. □

Exercise 9

(a) Show that if e^{tA} and e^{tB} are both contractions on $\mathbb{R}^n$, and $BA = AB$, then $e^{t(A+B)}$ is a contraction.

(b) Give a concrete example that shows that **(a)** can fail if the assumption that $AB = BA$ is dropped.

<u>Exercise 10</u> Problem 5 on page 137 of Hirsch and Smale reads:
"For any solution to $\dot{x} = Ax$, $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, show that exactly one of the following alternatives holds:

(a) $\lim_{t\uparrow\infty} x(t) = 0$ and $\lim_{t\downarrow-\infty} |x(t)| = \infty$;

(b) $\lim_{t\uparrow\infty} |x(t)| = \infty$ and $\lim_{t\downarrow-\infty} x(t) = 0$;

(c) there exist constants $M, N > 0$ such that $M < |x(t)| < N$ for all $t \in \mathbb{R}$."

Is what they ask you to prove true? If so, prove it. If not, determine what other possible alternatives exist, and prove that you have accounted for all possibilities.