Nonautonomous Linear Systems

Lecture 15
Math 634
10/4/99

We now move from the constant coefficient equation $\dot{x} = Ax$ to the nonautonomous equation

$$\dot{x} = A(t)x. \tag{1}$$

For simplicity we will assume that the domain of A is $\mathbb{R}$.

Solution Formulas

In the scalar, or one-dimensional, version of (1)

$$\dot{x} = a(t)x \tag{2}$$

we can separate variables and arrive at the formula

$$x(t) = x_0 e^{\int_{t_0}^t a(\tau)\, d\tau}$$

for the solution of (2) that satisfies the initial condition $x(t_0) = x_0$.

It seems like the analogous formula for the solution of (1) with initial condition $x(t_0) = x_0$ should be

$$x(t) = e^{\int_{t_0}^t A(\tau)\, d\tau} x_0. \tag{3}$$

Certainly, the right-hand side of (3) makes sense (assuming that A is continuous). But does it give the correct answer?

Let's consider a specific example. Let

$$A(t) = \begin{bmatrix} 0 & 0 \\ 1 & t \end{bmatrix}$$

and $t_0 = 0$. Note that

$$\int_0^t A(\tau)\, d\tau = \begin{bmatrix} 0 & 0 \\ t & t^2/2 \end{bmatrix} = \frac{t^2}{2} \begin{bmatrix} 0 & 0 \\ 2/t & 1 \end{bmatrix},$$

and

$$e^{\begin{bmatrix}0 & 0\\ t & t^2/2\end{bmatrix}} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{t^2}{2}\begin{bmatrix}0 & 0\\ 2/t & 1\end{bmatrix} + \frac{\left(\frac{t^2}{2}\right)^2}{2!}\begin{bmatrix}0 & 0\\ 2/t & 1\end{bmatrix} + \frac{\left(\frac{t^2}{2}\right)^3}{3!}\begin{bmatrix}0 & 0\\ 2/t & 1\end{bmatrix} + \cdots$$
$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \left(e^{t^2/2} - 1\right)\begin{bmatrix}0 & 0\\ 2/t & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ \frac{2}{t}\left(e^{t^2/2} - 1\right) & e^{t^2/2}\end{bmatrix}.$$

On the other hand, we can solve the corresponding system

$$\dot{x}_1 = 0$$
$$\dot{x}_2 = x_1 + t x_2$$

directly. Clearly $x_1(t) = \alpha$ for some constant α. Plugging this into the equation for x_2, we have a first-order scalar equation which can be solved by finding an integrating factor. This yields

$$x_2(t) = \beta e^{t^2/2} + \alpha e^{t^2/2}\int_0^t e^{-s^2/2}\, ds$$

for some constant β. Since $x_1(0) = \alpha$ and $x_2(0) = \beta$, the solution of (1) is

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = \begin{bmatrix}1 & 0\\ e^{t^2/2}\int_0^t e^{-s^2/2}\, ds & e^{t^2/2}\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}.$$

Since

$$e^{t^2/2}\int_0^t e^{-s^2/2}\, ds \neq \frac{2}{t}\left(e^{t^2/2} - 1\right)$$

(3) doesn't work? What went wrong? The answer is that

$$\frac{d}{dt}e^{\int_0^t A(\tau)\, d\tau} = \lim_{h\to 0}\frac{e^{\int_0^{t+h} A(\tau)\, d\tau} - e^{\int_0^t A(\tau)\, d\tau}}{h} \neq \lim_{h\to 0}\frac{e^{\int_0^t A(\tau)\, d\tau}\left[e^{\int_t^{t+h} A(\tau)\, d\tau} - I\right]}{h},$$

in general, because of possible noncommutativity.

Structure of Solution Set

We abandon attempts to find a general formula for solving (1), and instead analyze the general structure of the solution set.

Definition If $x^{(1)}, x^{(2)}, \dots, x^{(n)}$ are linearly independent solutions of (1) (*i.e.*, no nontrivial linear combination gives the zero function) then the matrix

$$X(t) := \begin{bmatrix} x^{(1)}(t) & \cdots & x^{(n)}(t) \end{bmatrix}$$

is called a *fundamental matrix* for (1).

Theorem *The dimension of the vector space of solutions of* (1) *is* n.

Proof. Pick n linearly independent vectors $v^{(k)} \in \mathbb{R}^n$, $k = 1, \dots, n$, and let $x^{(k)}$ be the solution of (1) that satisfies the initial condition $x^{(k)}(0) = v^{(k)}$. Then these n solutions are linearly independent. Furthermore, we claim that any solution x of (1) is a linear combination of these n solutions. To see why this is so, note that $x(0)$ must be expressible as a linear combination of $\{v^{(1)}, \dots, v^{(n)}\}$. The corresponding linear combination of $\{x^{(1)}, \dots, x^{(n)}\}$ is, by linearity, a solution of (1) that agrees with x at $t = 0$. Since A is continuous, the Picard-Lindelöf Theorem applies to (1) to tell us that solutions of IVPs are unique, so this linear combination of $\{x^{(1)}, \dots, x^{(n)}\}$ must be identical to x. □

Definition If $X(t)$ is a fundamental matrix and $X(0) = I$, then it is called the *principal fundamental matrix*. (Uniqueness of solutions implies that there is only one such matrix.)

Definition Given n functions (in some order) from $\mathbb{R}$ to $\mathbb{R}^n$, their *Wronskian* is the determinant of the matrix that has these functions as its columns (in the corresponding order).

Theorem *The Wronskian of n solutions of* (1) *is identically zero if and only if the solutions are linearly dependent.*

Proof. Suppose $x^{(1)}, \dots, x^{(n)}$ are linearly dependent solutions; *i.e.*,

$$\sum_{k=1}^{n} \alpha_k x^{(k)} = 0$$

for some constants $\alpha_1, \dots, \alpha_n$ with $\sum_{k=1}^{n} \alpha_k^2 \neq 0$. Then $\sum_{k=1}^{n} \alpha_k x^{(k)}(t) = 0$ for every t, so the columns of the Wronskian $W(t)$ are linearly dependent for every t. This means $W \equiv 0$.

Conversely, suppose that the Wronskian W of n solutions $x^{(1)}, \dots, x^{(n)}$ is identically zero. In particular, $W(0) = 0$, so $x^{(1)}(0), \dots, x^{(n)}(0)$ are linearly dependent vectors. Pick constants $\alpha_1, \dots, \alpha_n$, with $\sum_{k=1}^n \alpha_k^2 \neq 0$, such that $\sum_{k=1}^n \alpha_k x^{(k)}(0) = 0$. The function $\sum_{k=1}^n \alpha_k x^{(k)}$ is a solution of (1) that is 0 when $t = 0$, but so is the function that is identically zero. By uniqueness of solutions, $\sum_{k=1}^n \alpha_k x^{(k)} = 0$; *i.e.*, $x^{(1)}, \dots, x^{(n)}$ are linearly dependent. □

Note that this proof also shows that if the Wronskian of n solutions of (1) is zero for some t, then it is zero for all t.

What if we're dealing with n arbitrary vector-valued functions (that are not necessarily solutions of (1))? If they are linearly dependent then their Wronskian is identically zero, but the converse is not true. For example,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} t \\ 0 \end{bmatrix}$$

have a Wronskian that is identically zero, but they are not linearly dependent. Also, n functions can have a Wronskian that is zero for some t and nonzero for other t. Consider, for example,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

Initial-Value Problems

Given a fundamental matrix $X(t)$ for (1), let $G(t, t_0) := X(t)[X(t_0)]^{-1}$. We claim that $x(t) := G(t, t_0)v$ solves the IVP

$$\begin{cases} \dot{x} = A(t)x \\ x(t_0) = v. \end{cases}$$

To verify this, note that

$$\frac{d}{dt}x = \frac{d}{dt}(X(t)[X(t_0)]^{-1}v) = A(t)X(t)[X(t_0)]^{-1}v = A(t)x,$$

and

$$x(t_0) = G(t_0, t_0)v = X(t_0)[X(t_0)]^{-1}v = v.$$

Inhomogeneous Equations

Consider the IVP

$$\begin{cases} \dot{x} = A(t)x + f(t) \\ x(t_0) = x_0. \end{cases} \tag{4}$$

In light of the results from the previous section when f was identically zero, it's reasonable to look for a solution x of (4) of the form $x(t) = G(t, t_0)y(t)$, where G is as before, and y is some vector-valued function.

Note that

$$\dot{x}(t) = A(t)X(t)[X(t_0)]^{-1}y(t) + G(t, t_0)\dot{y}(t) = A(t)x(t) + G(t, t_0)\dot{y}(t);$$

therefore, we need $G(t, t_0)\dot{y}(t) = f(t)$. Isolating, $\dot{y}(t)$, we need

$$\dot{y}(t) = X(t_0)[X(t)]^{-1}f(t) = G(t_0, t)f(t). \tag{5}$$

Integrating both sides of (5), we see that y should satisfy

$$y(t) - y(t_0) = \int_{t_0}^{t} G(t_0, s)f(s)\,ds.$$

If $x(t_0)$ is to be x_0, then, since $G(t_0, t_0) = I$, we need $y(t_0) = x_0$, so $y(t)$ should be

$$x_0 + \int_{t_0}^{t} G(t_0, s)f(s)\,ds,$$

or, equivalently, $x(t)$ should be

$$G(t, t_0)x_0 + \int_{t_0}^{t} G(t, s)f(s)\,ds,$$

since $G(t, t_0)G(t_0, s) = G(t, s)$. This is called the Variation of Constants formula or the Variation of Parameters formula.