Nearly Autonomous Linear Systems
Lecture 16
Math 634
10/6/99

Suppose $A(t)$ is, in some sense, close to a constant matrix A. The question we wish to address in this section is the extent to which solutions of the nonautonomous system

$$\dot{x} = A(t)x \tag{1}$$

behave like solutions of the autonomous system

$$\dot{x} = Ax. \tag{2}$$

Before getting to our main results, we present a pair of lemmas.

Lemma *The following are equivalent:*

1. *Each solution of* (2) *is bounded as* $t \uparrow \infty$*.*
2. *The function* $t \mapsto \|e^{tA}\|$ *is bounded as* $t \uparrow \infty$ *(where* $\|\cdot\|$ *is the usual operator norm).*
3. $\operatorname{Re}\lambda \leq 0$ *for every eigenvalue* λ *of* A *and the algebraic multiplicity of each imaginary eigenvalue matches its geometric multiplicity.*

Proof. That statement 2 implies statement 1 is a consequence of the definition of the operator norm, since, for each solution x of (2),

$$|x(t)| = |e^{tA}x(0)| \leq \|e^{tA}\| \cdot |x(0)|.$$

That statement 1 implies statement 3, and statement 3 implies statement 2 are consequences of what we have learned about the real canonical form of A, along with the equivalence of norms on $\mathbb{R}^n$. □

Lemma (Generalized Gronwall Inequality) *Suppose* X *and* Φ *are non-negative, continuous, real-valued functions on* $[t_0, T]$ *for which there is a nonnegative constant* C *such that*

$$X(t) \leq C + \int_{t_0}^{t} \Phi(s)X(s)\,ds,$$

for every $t \in [t_0, T]$*. Then*

$$X(t) \leq Ce^{\int_{t_0}^{t} \Phi(s)\,ds}.$$

Proof. The proof is very similar to the proof of the standard Gronwall inequality. The details are left to the reader. □

The first main result deals with the case when $A(t)$ converges to A sufficiently quickly as $t \uparrow \infty$.

Theorem *Suppose that each solution of* (2) *remains bounded as* $t \uparrow \infty$ *and that, for some* $t_0 \in \mathbb{R}$*,*

$$\int_{t_0}^{\infty} \|A(t) - A\|\,dt < \infty, \tag{3}$$

where $\|\cdot\|$ *is the standard operator norm. Then each solution of* (1) *remains bounded as* $t \uparrow \infty$*.*

Proof. Let t_0 be such that (3) holds. Given a solution x of (1), let $f(t) = (A(t) - A)x(t)$, and note that x satisfies the constant-coefficient inhomogeneous problem

$$\dot{x} = Ax + f(t). \tag{4}$$

Since the matrix exponential provides a fundamental matrix solution to constant-coefficient linear systems, applying the variation of constants formula to (4) yields

$$x(t) = e^{(t-t_0)A}x(t_0) + \int_{t_0}^{t} e^{(t-s)A}(A(s) - A)x(s)\,ds. \tag{5}$$

Now, by the first lemma, the boundedness of solutions of (2) in forward time tells us that there is a constant $M > 0$ such that $\|e^{tA}\| \leq M$ for every $t \geq t_0$. Taking norms and estimating, gives (for $t \geq t_0$)

$$\begin{aligned} |x(t)| &\leq \|e^{(t-t_0)A}\| \cdot |x(t_0)| + \int_{t_0}^{t} \|e^{(t-s)A}\| \cdot \|A(s) - A\| \cdot |x(s)|\,ds \\ &\leq M|x(t_0)| + \int_{t_0}^{t} M\|A(s) - A\| \cdot |x(s)|\,ds. \end{aligned}$$

Setting $X(t) = |x(t)|$, $\Phi(t) = M\|A(t) - A\|$, and $C = M|x(t_0)|$, and applying the generalized Gronwall inequality, we find that

$$|x(t)| \leq M|x(t_0)|e^{M\int_{t_0}^{t} \|A(s)-A\|\,ds}.$$

By (3), the right-hand side of this inequality is bounded on $[t_0, \infty)$, so $x(t)$ is bounded as $t \uparrow \infty$. □

The next result deals with the case when the origin is a sink for (2). Will all the solutions of (1) also all converge to the origin as $t \uparrow \infty$? Yes, if $\|A(t) - A\|$ is sufficiently small.

Theorem *Suppose all the eigenvalues of A have negative real part. Then there is a constant $\varepsilon > 0$ such that if $\|A(t) - A\| \leq \varepsilon$ for all t sufficiently large then every solution of* (1) *converges to* 0 *as $t \uparrow \infty$.*

Proof. Since the origin is a sink, we know that we can choose constants $k, b > 0$ such that $\|e^{tA}\| \leq ke^{-bt}$ for all $t \geq 0$. Pick a constant $\varepsilon \in (0, b/k)$, and assume that there is a time $t_0 \in \mathbb{R}$ such that $\|A(t) - A\| \leq \varepsilon$ for every $t \geq t_0$.

Now, given a solution x of (1) we can conclude, as in the proof of the previous theorem, that

$$|x(t)| \leq \|e^{(t-t_0)A}\| \cdot |x(t_0)| + \int_{t_0}^{t} \|e^{(t-s)A}\| \cdot \|A(s) - A\| \cdot |x(s)|\,ds$$

for all $t \geq t_0$. This implies that

$$|x(t)| \leq ke^{-b(t-t_0)}|x(t_0)| + \int_{t_0}^{t} ke^{-b(t-s)}\varepsilon \cdot |x(s)|\,ds$$

for all $t \geq t_0$. Multiplying through by $e^{b(t-t_0)}$ and setting $y(t) := e^{b(t-t_0)}|x(t)|$ yield

$$y(t) \leq k|x(t_0)| + k\varepsilon \int_{t_0}^{t} y(s)\,ds$$

for all $t \geq t_0$. The standard Gronwall inequality applied to this estimate gives

$$y(t) \leq k|x(t_0)|e^{k\varepsilon(t-t_0)}$$

for all $t \geq t_0$, or, equivalently,

$$|x(t)| \leq k|x(t_0)|e^{(k\varepsilon - b)(t-t_0)}$$

for all $t \geq t_0$. Since $\varepsilon < b/k$, this inequality implies that $x(t) \to 0$ as $t \uparrow \infty$. □

Thus, the origin remains a "sink" even when we perturb A by a small time-dependent quantity. Can we perhaps just look at the (possibly, time-dependent) eigenvalues of $A(t)$ itself and conclude, for example, that if all of those eigenvalues have negative real part for all t then all solutions of (1) converge to the origin as $t \uparrow \infty$? The following example of Markus and Yamabe shows that the answer is "No".

<u>Exercise 11</u> Show that if

$$A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\sin t \cos t \\ -1 - \frac{3}{2}\sin t \cos t & -1 + \frac{3}{2}\sin^2 t \end{bmatrix}$$

then the eigenvalues of $A(t)$ both have negative real part for every $t \in \mathbb{R}$, but

$$x(t) := \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} e^{t/2},$$

which becomes unbounded as $t \to \infty$, is a solution to (1).