Periodic Linear Systems

Lecture 17
Math 634
10/8/99

We now consider

$$\dot{x} = A(t)x \tag{1}$$

when A is a continuous periodic $n \times n$ matrix function of t; *i.e.*, when there is a constant $T > 0$ such that $A(t+T) = A(t)$ for every $t \in \mathbb{R}$. When that condition is satisfied, we say, more precisely, that A is T*-periodic*. If T is the smallest positive number for which this condition holds, we say that T is the *minimal period* of A.

Let A be T-periodic, and let $X(t)$ be a fundamental matrix for (1). Define $\tilde{X} : \mathbb{R} \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ by $\tilde{X}(t) = X(t+T)$. Clearly, the columns of $\tilde{X}$ are linearly independent functions of t. Also,

$$\frac{d}{dt}\tilde{X}(t) = \frac{d}{dt}X(t+T) = X'(t+T) = A(t+t)X(t+T) = A(t)\tilde{X}(t),$$

so $\tilde{X}$ solves the matrix equivalent of (1). Hence, $\tilde{X}$ is a fundamental matrix for (1).

Because the dimension of the solution space of (1) is n, this means that there is a nonsingular (constant) matrix C such that $X(t+T) = X(t)C$ for every $t \in \mathbb{R}$. C is called a *monodromy* matrix.

Lemma *There exists* $B \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ *such that* $C = e^{TB}$.

Proof. Without loss of generality, we assume that $T = 1$, since if it isn't we can just rescale B by a scalar constant. We also assume, without loss of generality, that C is in Jordan canonical form. (If it isn't, then use the fact that $P^{-1}CP = e^B$ implies that $C = e^{PBP^{-1}}$.) Furthermore, because of the way the matrix exponential acts on a block diagonal matrix, it suffices to show that for each $p \times p$ Jordan block

$$\tilde{C} := \begin{bmatrix} \lambda & 0 & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \lambda \end{bmatrix},$$

$\tilde{C} = e^{\tilde{B}}$ for some $\tilde{B} \in \mathcal{L}(\mathbb{C}^p, \mathbb{C}^p)$.

Now, an obvious candidate for $\tilde{B}$ is the natural logarithm of $\tilde{C}$, defined in some reasonable way. Since the matrix exponential was defined by a power series, it seems reasonable to use a similar definition for a matrix logarithm. Note that $\tilde{C} = \lambda I + N = \lambda I(I + \lambda^{-1}N)$, where N is nilpotent. (Since C is invertible, we know that all of the eigenvalues λ are nonzero.) We guess

$$\tilde{B} = (\log \lambda)I + \log(I + \lambda^{-1}N), \tag{2}$$

where

$$\log(I + M) := -\sum_{k=1}^{\infty} \frac{(-M)^k}{k},$$

in analogy to the Maclaurin series for $\log(1 + x)$. Since N is nilpotent, this series terminates in our application of it to (2). Direct substitution shows that $e^{\tilde{B}} = \tilde{C}$, as desired. □

The eigenvalues ρ of C are called the *Floquet multipliers* (or characteristic multipliers) of (1). The corresponding numbers λ satisfying $\rho = e^{\lambda T}$ are called the *Floquet exponents* (or characteristic exponents) of (1). Note that the Floquet exponents are only determined up to a multiple of $(2\pi i)/T$. Given B for which $C = e^{TB}$, the exponents can be chosen to be the eigenvalues of B.

Theorem *There exists a T-periodic function $P : \mathbb{R} \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that*

$$X(t) = P(t)e^{tB}.$$

Proof. Let $P(t) = X(t)e^{-tB}$. Then

$$\begin{aligned} P(t+T) &= X(t+T)e^{-(t+T)B} = X(t+T)e^{-TB}e^{-tB} = X(t)Ce^{-TB}e^{-tB} \\ &= X(t)e^{TB}e^{-TB}e^{-tB} = X(t)e^{-tB} = P(t). \end{aligned}$$

□

The decomposition of $X(t)$ given in this theorem shows that the behavior of solutions can be broken down into the composition of a part that is periodic in time and a part that is exponential in time. Recall, however, that B may have entries that are not real numbers, so $P(t)$ may be complex, also. If we want to decompose $X(t)$ into a *real* periodic matrix times a matrix of the form e^{tB} where B is *real*, we observe that $X(t+2T) = X(t)C^2$, where C is the same monodromy matrix as before. It can be shown that the *square* of a real matrix can be written as the exponential of a *real* matrix. Write $C^2 = e^{TB}$ with B real, and let $P(t) = X(t)e^{-tB}$ as before. Then, $X(t) = P(t)e^{tB}$ where P is now $2T$-periodic, and everything is real.

The Floquet multipliers and exponents do not depend on the particular fundamental matrix chosen, even though the monodromy matrix does. They depend only on $A(t)$. To see this, let $X(t)$ and $Y(t)$ be fundamental matrices with corresponding monodromy matrices C and D. Because $X(t)$ and $Y(t)$ are fundamental matrices, there is a nonsingular constant matrix S such that $Y(t) = X(t)S$ for all $t \in \mathbb{R}$. In particular, $Y(0) = X(0)S$ and $Y(T) = X(T)S$. Thus,

$$C = [X(0)]^{-1}X(T) = S[Y(0)]^{-1}Y(T)S^{-1} = S[Y(0)]^{-1}Y(0)DS^{-1} = SDS^{-1}.$$

This means that the monodromy matrices are similar and, therefore, have the same eigenvalues.

Interpreting Floquet Multipliers and Exponents

Theorem *If ρ is a Floquet multiplier of* (1) *and λ is a corresponding Floquet exponent, then there is a nontrivial solution x of* (1) *such that $x(t+T) = \rho x(t)$ for every $t \in \mathbb{R}$ and $x(t) = e^{\lambda t}p(t)$ for some T-periodic vector function p.*

Proof. Pick x_0 to be an eigenvector of B corresponding to the eigenvalue λ, where $X(t) = P(t)e^{tB}$ is the decomposition of a fundamental matrix $X(t)$.

Let $x(t) = X(t)x_0$. Then, clearly, x solves (1). The power series formula for the matrix exponential implies that x_0 is an eigenvector of e^{tB} with eigenvalue $e^{\lambda t}$. Hence,

$$x(t) = X(t)x_0 = P(t)e^{tB}x_0 = P(t)e^{\lambda t}x_0 = e^{\lambda t}p(t),$$

where $p(t) = P(t)x_0$. Also,

$$x(t+T) = e^{\lambda T}e^{\lambda t}p(t+T) = \rho e^{\lambda t}p(t) = \rho x(t).$$

□

Time-dependent Change of Variables

Let x solve (1), and let $y(t) = [P(t)]^{-1}x(t)$, where P is as defined previously. Then

$$\frac{d}{dt}[P(t)y(t)] = \frac{d}{dt}x(t) = A(t)x(t) = A(t)P(t)y(t) = A(t)X(t)e^{-tB}y(t).$$

But

$$\begin{aligned}\frac{d}{dt}[P(t)y(t)] &= P'(t)y(t) + P(t)y'(t) \\ &= [X'(t)e^{-tB} - X(t)e^{-tB}B]y(t) + X(t)e^{-tB}y'(t) \\ &= A(t)X(t)e^{-tB}y(t) - X(t)e^{-tB}By(t) + X(t)e^{-tb}y'(t),\end{aligned}$$

so

$$X(t)e^{-tB}y'(t) = X(t)e^{-tB}By(t),$$

which implies that $y'(t) = By(t)$; *i.e.*, y solves a constant coefficient linear equation. Since P is periodic and, therefore, bounded, the growth and decay of x and y are closely related. Furthermore, the growth or decay of y is determined by the eigenvalues of B, *i.e.*, by the Floquet exponents of (1). For example, we have the following results.

Theorem *If all the Floquet exponents of* (1) *have negative real parts then all solutions of* (1) *converge to 0 as* $t \uparrow \infty$.

Theorem *If there is a nontrivial* T*-periodic solution of* (1) *then there must be a Floquet multiplier of modulus 1.*

Computing Floquet Multipliers and Exponents

Although Floquet multipliers and exponents are determined by $A(t)$, it is not obvious how to calculate them. As a previous exercise illustrated, the eigenvalues of $A(t)$ don't seem to be extremely relevant. The following result helps a little bit.

Theorem *If* (1) *has Floquet multipliers* $\rho_1, \ldots, \rho_n$ *and corresponding Floquet exponents* $\lambda_1, \ldots, \lambda_n$*, then*

$$\rho_1 \cdots \rho_n = \exp\left(\int_0^T \operatorname{trace} A(t)\, dt\right) \tag{3}$$

and

$$\lambda_1 + \cdots + \lambda_n \equiv \frac{1}{T}\int_0^T \operatorname{trace} A(t)\, dt \bmod \frac{2\pi i}{T} \tag{4}$$

Proof. We focus on (3). The formula (4) will follow immediately from (3).

Let $W(t)$ be the determinant of the principal fundamental matrix $X(t)$. Let S_n be the set of permutations of $\{1, 2, \ldots, n\}$ and let $\epsilon : S_n \to \{-1, 1\}$ be the parity map. Then

$$W(t) = \sum_{\sigma \in S_n} \epsilon(\sigma) X_{1,\sigma(1)} X_{2,\sigma(2)} \cdots X_{n,\sigma(n)},$$

where $X_{i,j}$ is the (i,j)-th entry of $X(t)$.

Differentiating yields

$$\begin{aligned}
\frac{dW(t)}{dt} &= \sum_{\sigma \in S_n} \epsilon(\sigma) \frac{d}{dt}\left[X_{1,\sigma(1)} X_{2,\sigma(2)} \cdots X_{n,\sigma(n)}\right] \\
&= \sum_{i=1}^{n} \sum_{\sigma \in S_n} \epsilon(\sigma) X_{1,\sigma(1)} \cdots X_{i-1,\sigma(i-1)} \left[\frac{d}{dt} X_{i,\sigma(i)}\right] X_{i+1,\sigma(i+1)} \cdots X_{n,\sigma(n)} \\
&= \sum_{i=1}^{n} \sum_{\sigma \in S_n} \epsilon(\sigma) X_{1,\sigma(1)} \cdots X_{i-1,\sigma(i-1)} \left[\sum_{j=1}^{n} A_{i,j}(t) X_{j,\sigma(i)}\right] X_{i+1,\sigma(i+1)} \cdots X_{n,\sigma(n)} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j}(t) \left(\sum_{\sigma \in S_n} \epsilon(\sigma) X_{1,\sigma(1)} \cdots X_{i-1,\sigma(i-1)} X_{j,\sigma(i)} X_{i+1,\sigma(i+1)} \cdots X_{n,\sigma(n)}\right).
\end{aligned}$$

If $i \neq j$, the inner sum is the determinant of the matrix obtained by replacing the ith row of $X(t)$ by its jth row. This new matrix, having two identical rows, must necessarily have determinant 0. Hence,

$$\frac{dW(t)}{dt} = \sum_{i=1}^{n} A_{i,i}(t) \det X(t) = [\operatorname{trace} A(t)] W(t).$$

Thus,

$$W(t) = e^{\int_0^t \operatorname{trace} A(s)\, ds} W(0) = e^{\int_0^t \operatorname{trace} A(s)\, ds}.$$

In particular,

$$\begin{aligned} e^{\int_0^T \operatorname{trace} A(s)\, ds} &= W(T) = \det X(T) = \det(P(T)e^{TB}) = \det(P(0)e^{TB}) \\ &= \det e^{TB} = \det C = \rho_1 \rho_2 \cdots \rho_n. \end{aligned}$$

□

<u>Exercise 12</u> Consider (1) where

$$A(t) = \begin{bmatrix} \frac{1}{2} - \cos t & b \\ a & \frac{3}{2} + \sin t \end{bmatrix}$$

and a and b are constants. Show that there is a solution of (1) that becomes unbounded as $t \uparrow \infty$.