

# Invariant Sets and Limit Sets

Lecture 18

Math 634

10/11/99

We will now begin an intensive study of the continuously differentiable autonomous system

$$\dot{x} = f(x)$$

or, equivalently, of the corresponding dynamical system  $\varphi(t, x)$ . We will denote the phase space  $\Omega$  and assume that it is an open (not necessarily proper) subset of  $\mathbb{R}^n$ .

## Orbits

**Definition** Given  $x \in \Omega$ , the (*complete*) *orbit* through  $x$  is the set

$$\gamma(x) := \{\varphi(t, x) \mid t \in \mathbb{R}\},$$

the *positive semiorbit* through  $x$  is the set

$$\gamma^+(x) := \{\varphi(t, x) \mid t \geq 0\},$$

and the *negative semiorbit* through  $x$  is the set

$$\gamma^-(x) := \{\varphi(t, x) \mid t \leq 0\}.$$

## Invariant Sets

**Definition** A set  $\mathcal{M} \subseteq \Omega$  is *invariant* under  $\varphi$  if it contains the complete orbit of every point of  $\mathcal{M}$ . In other words, for every  $x \in \mathcal{M}$  and every  $t \in \mathbb{R}$ ,  $\varphi(t, x) \in \mathcal{M}$ .

**Definition** A set  $\mathcal{M} \subseteq \Omega$  is *positively invariant* under  $\varphi$  if it contains the positive semiorbit of every point of  $\mathcal{M}$ . In other words, for every  $x \in \mathcal{M}$  and every  $t \geq 0$ ,  $\varphi(t, x) \in \mathcal{M}$ .

**Definition** A set  $\mathcal{M} \subseteq \Omega$  is *negatively invariant* under  $\varphi$  if it contains the negative semiorbit of every point of  $\mathcal{M}$ . In other words, for every  $x \in \mathcal{M}$  and every  $t \leq 0$ ,  $\varphi(t, x) \in \mathcal{M}$ .

## Limit Sets

**Definition** Given  $x \in \Omega$ , the  $\omega$ -limit set of  $x$ , denoted  $\omega(x)$ , is the set

$$\begin{aligned} & \{y \in \Omega \mid \liminf_{t \uparrow \infty} |\varphi(t, x) - y| = 0\} \\ & = \{y \in \Omega \mid \exists t_1, t_2, \dots \rightarrow \infty \text{ s.t. } \varphi(t_k, x) \rightarrow y \text{ as } k \uparrow \infty\}. \end{aligned}$$

**Definition** Given  $x \in \Omega$ , the  $\alpha$ -limit set of  $x$ , denoted  $\alpha(x)$ , is the set

$$\begin{aligned} & \{y \in \Omega \mid \liminf_{t \downarrow -\infty} |\varphi(t, x) - y| = 0\} \\ & = \{y \in \Omega \mid \exists t_1, t_2, \dots \rightarrow -\infty \text{ s.t. } \varphi(t_k, x) \rightarrow y \text{ as } k \uparrow \infty\}. \end{aligned}$$

**Lemma** If, for each  $\mathcal{A} \in \Omega$ , we let  $\overline{\mathcal{A}}$  represent the topological closure of  $\mathcal{A}$  in  $\Omega$ , then

$$\omega(x) = \bigcap_{\tau \in \mathbb{R}} \overline{\gamma^+(\varphi(\tau, x))} \quad (1)$$

and

$$\alpha(x) = \bigcap_{\tau \in \mathbb{R}} \overline{\gamma^-(\varphi(\tau, x))}. \quad (2)$$

*Proof.* It suffices to prove (1); (2) can then be established by time reversal.

Let  $y \in \omega(x)$  be given. Pick a sequence  $t_1, t_2, \dots \rightarrow \infty$  such that  $\varphi(t_k, x) \rightarrow y$  as  $k \uparrow \infty$ . Let  $\tau \in \mathbb{R}$  be given. Pick  $K \in \mathbb{N}$  such that  $t_k \geq \tau$  for all  $k \geq K$ . Note that  $\varphi(t_k, x) \in \gamma^+(\varphi(\tau, x))$  for all  $k \geq K$ , so

$$y \in \overline{\gamma^+(\varphi(\tau, x))}.$$

Since this holds for all  $\tau \in \mathbb{R}$ , we know that

$$y \in \bigcap_{\tau \in \mathbb{R}} \overline{\gamma^+(\varphi(\tau, x))}. \quad (3)$$

Since (3) holds for each  $y \in \omega(x)$ , we know that

$$\omega(x) \subseteq \bigcap_{\tau \in \mathbb{R}} \overline{\gamma^+(\varphi(\tau, x))}. \quad (4)$$

Now, we prove the inverse inclusion. Let

$$y \in \bigcap_{\tau \in \mathbb{R}} \overline{\gamma^+(\varphi(\tau, x))}$$

be given. This implies, in particular, that

$$y \in \bigcap_{\tau \in \mathbb{N}} \overline{\gamma^+(\varphi(\tau, x))}.$$

For each  $k \in \mathbb{N}$ , we have

$$y \in \overline{\gamma^+(\varphi(k, x))}$$

so we can pick  $z_k \in \gamma^+(\varphi(k, x))$  such that  $|z_k - y| < 1/k$ . Since  $z_k \in \gamma^+(\varphi(k, x))$ , we can pick  $s_k \geq 0$  such that  $z_k = \varphi(s_k, \varphi(k, x))$ . If we set  $t_k = k + s_k$ , we see that  $t_k \geq k$ , so the sequence  $t_1, t_2, \dots$  goes to infinity. Also, since

$$|\varphi(t_k, x) - y| = |\varphi(s_k + k, x) - y| = |\varphi(s_k, \varphi(k, x)) - y| = |z_k - y| < 1/k,$$

we know that  $\varphi(t_k, x) \rightarrow y$  as  $k \uparrow \infty$ . Hence,  $y \in \omega(x)$ . Since this holds for every

$$y \in \bigcap_{\tau \in \mathbb{R}} \overline{\gamma^+(\varphi(\tau, x))},$$

we know that

$$\bigcap_{\tau \in \mathbb{R}} \overline{\gamma^+(\varphi(\tau, x))} \subseteq \omega(x).$$

Combining this with (4) gives (1). □

We now describe some properties of limit sets.

**Theorem** *Given  $x \in \Omega$ ,  $\omega(x)$  and  $\alpha(x)$  are closed (relative to  $\Omega$ ) and invariant. If  $\gamma^+(x)$  is contained in some compact subset of  $\Omega$ , then  $\omega(x)$  is nonempty, compact, and connected. If  $\gamma^-(x)$  is contained in some compact subset of  $\Omega$ , then  $\alpha(x)$  is nonempty, compact, and connected.*

*Proof.* Again, time-reversal arguments tell us that we only need to prove the statements about  $\omega(x)$ .

Step 1:  $\omega(x)$  is closed.

This is a consequence of the lemma and the fact that the intersection of closed sets is closed.

Step 2:  $\omega(x)$  is invariant.

Let  $y \in \omega(x)$  and  $t \in \mathbb{R}$  be given. Choose a sequence of times  $(t_k)$  converging to infinity such that  $\varphi(t_k, x) \rightarrow y$  as  $k \uparrow \infty$ . For each  $k \in \mathbb{N}$ , let  $s_k = t_k + t$ , and note that  $(s_k)$  converges to infinity and

$$\varphi(s_k, x) = \varphi(t_k + t, x) = \varphi(t, \varphi(t_k, x)) \rightarrow \varphi(t, y)$$

as  $k \uparrow \infty$  (by the continuity of  $\varphi(t, \cdot)$ ). Hence,  $\varphi(t, y) \in \omega(x)$ . Since  $t \in \mathbb{R}$  and  $y \in \omega(x)$  were arbitrary, we know that  $\omega(x)$  is invariant.

Now, suppose that  $\gamma^+(x)$  is contained in a compact subset  $\mathcal{K}$  of  $\Omega$ .

Step 3:  $\omega(x)$  is nonempty.

The sequence  $\varphi(1, x), \varphi(2, x), \dots$  is contained in  $\gamma^+(x) \subseteq \mathcal{K}$ , so by the Bolzano-Weierstrass Theorem, some subsequence  $\varphi(t_1, x), \varphi(t_2, x), \dots$  converges to a point  $y \in \mathcal{K}$ . By definition,  $y \in \omega(x)$ .

Step 4:  $\omega(x)$  is compact.

By the Heine-Borel Theorem,  $\mathcal{K}$  is closed (relative to  $\mathbb{R}^n$ ), so, by the choice of  $\mathcal{K}$ ,  $\omega(x) \subseteq \mathcal{K}$ . Since, by Step 1,  $\omega(x)$  is closed relative to  $\Omega$ , it is also closed relative to  $\mathcal{K}$ . Since  $\mathcal{K}$  is compact, this means  $\omega(x)$  is closed (relative to  $\mathbb{R}^n$ ). Also, by the Heine-Borel Theorem,  $\mathcal{K}$  is bounded so its subset  $\omega(x)$  is bounded, too. Thus,  $\omega(x)$  is closed (relative to  $\mathbb{R}^n$ ) and bounded and, therefore, compact.

Step 5:  $\omega(x)$  is connected.

Suppose  $\omega(x)$  were disconnected. Then there would be disjoint open subsets  $\mathcal{G}$  and  $\mathcal{H}$  of  $\Omega$  such that  $\mathcal{G} \cap \omega(x)$  and  $\mathcal{H} \cap \omega(x)$  are nonempty, and  $\omega(x)$  is contained in  $\mathcal{G} \cup \mathcal{H}$ . Then there would have to be a sequence  $s_1, s_2, \dots \rightarrow \infty$  and a sequence  $t_1, t_2, \dots \rightarrow \infty$  such that  $\varphi(s_k, x) \in \mathcal{G}$ ,  $\varphi(t_k, x) \in \mathcal{H}$ , and  $s_k < t_k < s_{k+1}$  for each  $k \in \mathbb{N}$ . Because (for each fixed  $k \in \mathbb{N}$ )

$$\{\varphi(t, x) \mid t \in [s_k, t_k]\}$$

is a (connected) curve going from a point in  $\mathcal{G}$  to a point in  $\mathcal{H}$ , there must be a time  $\tau_k \in (s_k, t_k)$  such that  $\varphi(\tau_k, x) \in \mathcal{K} \setminus \mathcal{G} \setminus \mathcal{H}$ . Pick such a  $\tau_k$  for each  $k \in \mathbb{N}$  and note that  $\tau_1, \tau_2, \dots \rightarrow \infty$  and, by the Bolzano-Weierstrass Theorem, some subsequence of  $(\varphi(\tau_k, x))$  must converge to a point  $y$  in  $\mathcal{K} \setminus \mathcal{G} \setminus \mathcal{H}$ . Note that  $y$ , being outside of  $\mathcal{G} \cup \mathcal{H}$ , cannot be in  $\omega(x)$ , which is a contradiction.  $\square$

Examples of empty  $\omega$ -limit sets are easy to find. Consider, for example, the one-dimensional dynamical system  $\varphi(t, x) := x + t$  (generated by the differential equation  $\dot{x} = 1$ ).

Examples of dynamical systems with nonempty, noncompact, disconnected  $\omega$ -limit sets are a little harder to find. Consider the planar autonomous system

$$\begin{cases} \dot{x} = -y(1 - x^2) \\ \dot{y} = x + y(1 - x^2). \end{cases}$$

Although it takes a little work to show it, this differential equation generates a dynamical system on  $\mathbb{R}^2$  (without the need for rescaling), and

$$\omega(x) = \{(-1, y) \mid y \in \mathbb{R}\} \cup \{(1, y) \mid y \in \mathbb{R}\}$$

for every  $x$  in the punctured strip

$$\{(x, y) \in \mathbb{R}^2 \mid |x| < 1 \text{ and } x^2 + y^2 > 0\}.$$