

Regular and Singular Points

Lecture 19

Math 634

10/13/99

Consider the differential equation $\dot{x} = f(x)$ and its associated dynamical system $\varphi(t, x)$ on a phase space Ω .

Definition We say that a point $x \in \Omega$ is an *equilibrium point* or a *singular point* or a *critical point* if $f(x) = 0$. For such a point, $\varphi(t, x) = x$ for all $t \in \mathbb{R}$.

Definition A point $x \in \Omega$ that is not a singular point is called a *regular point*.

We shall show that all of the interesting local behavior of a continuous dynamical system takes place close to singular points. We shall do this by showing that in the neighborhood of each regular point, the flow is very similar to unidirectional, constant-velocity flow.

One way of making the notion of similarity of flows precise is the following.

Definition Two dynamical systems $\varphi : \mathbb{R} \times \Omega \rightarrow \Omega$ and $\psi : \mathbb{R} \times \Theta \rightarrow \Theta$ are *topologically conjugate* if there exists a homeomorphism (*i.e.*, a continuous bijection with continuous inverse) $h : \Omega \rightarrow \Theta$ such that

$$h(\varphi(t, x)) = \psi(t, h(x)) \tag{1}$$

for every $t \in \mathbb{R}$ and every $x \in \Omega$. In other words, $\psi = h \circ \varphi(t, \cdot) \circ h^{-1}$, or, equivalently, the diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{\varphi(t, \cdot)} & \Omega \\ h \downarrow & & \downarrow h \\ \Theta & \xrightarrow{\psi(t, \cdot)} & \Theta \end{array}$$

commutes for each $t \in \mathbb{R}$. The function h is called a *topological conjugacy*. If, in addition, h and h^{-1} are r -times continuously differentiable, we say that φ and ψ are *C^r -conjugate*.

A weaker type of similarity is the following.

Definition Two dynamical systems $\varphi : \mathbb{R} \times \Omega \rightarrow \Omega$ and $\psi : \mathbb{R} \times \Theta \rightarrow \Theta$ are *topologically equivalent* if there exists a homeomorphism $h : \Omega \rightarrow \Theta$ and a time reparametrization function $\alpha : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ such that, for each $x \in \Omega$, $\alpha(\cdot, x) : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing surjection and

$$h(\varphi(\alpha(t, x), x)) = \psi(t, h(x))$$

for every $t \in \mathbb{R}$ and every $x \in \Omega$. If, in addition, h is r -times continuously differentiable, we say that φ and ψ are C^r -equivalent.

A topological equivalence maps orbits to orbits and preserves the orientation of time but may reparametrize time on each individual orbit.

As an example of the difference between these two concepts, consider the two planar dynamical systems

$$\varphi(t, x) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} x$$

and

$$\psi(t, y) = \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} y,$$

generated, respectively, by the differential equations

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x$$

and

$$\dot{y} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} y.$$

The functions $h(x) = x$ and $\alpha(t, x) = 2t$ show that these two flows are topologically equivalent. But these two flows are not topologically conjugate, since, by setting $t = \pi$ we see that any function $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying (1) would have to satisfy $h(x) = h(-x)$ for all x , which would mean that h is not invertible.

Because of examples like this, topological equivalence seems to be the preferred concept when dealing with flows. The following theorem, however, shows that in a neighborhood of a regular point, a smooth flow satisfies a local version of C^r -conjugacy with respect to a unidirectional, constant-velocity flow.

Theorem (C^r Rectification Theorem) Suppose $f : \Omega \rightarrow \mathbb{R}^n$ is r -times continuously differentiable (with $r \geq 1$) and x_0 is a regular point of the flow generated by

$$\dot{x} = f(x). \tag{2}$$

Then there is a neighborhood \mathcal{V} of x_0 , a neighborhood \mathcal{W} of the origin in \mathbb{R}^n , and a C^r invertible map $g : \mathcal{V} \rightarrow \mathcal{W}$ such that, for each solution x of (2) in \mathcal{V} , $y(t) := g(x(t))$ satisfies the equation

$$\dot{y} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3}$$

in \mathcal{W} .

Proof. Without loss of generality, we shall assume that $x_0 = 0$ and $f(x_0) = f(0) = \alpha e_1$ for some $\alpha > 0$. Let \mathcal{W} be a small ball centered at 0 in \mathbb{R}^n , and define $G(y) := G((y_1, \dots, y_n)^T) = \varphi(y_1, (0, y_2, \dots, y_n)^T)$, where φ is the flow generated by (2). (While φ might not be a genuine dynamical system because it might not be defined for all time, we know that it is at least defined long enough that G is well-defined if \mathcal{W} is sufficiently small.)

In words, $G(y)$ is the solution obtained by projecting y onto the plane through the origin perpendicular to $f(0)$ and locating the solution of (2) that starts at this projected point after y_1 units of time have elapsed.

Step 1: $\varphi(\cdot, p)$ is C^{r+1} .

We know that

$$\frac{d}{dt}\varphi(t, p) = f(\varphi(t, p)). \tag{4}$$

If f is continuous then, since $\varphi(\cdot, p)$ is continuous, (4) implies that $\varphi(\cdot, p)$ is C^1 . If f is C^1 , then the previous observation implies that $\varphi(\cdot, p)$ is C^1 . Then (4) implies that $\frac{d}{dt}\varphi(t, p)$ is the composition of C^1 functions and is, therefore, C^1 ; this means that $\varphi(\cdot, p)$ is C^2 . Continuing inductively, we see that, since f is C^r , $\varphi(\cdot, p)$ is C^{r+1} .

Step 2: $\varphi(t, \cdot)$ is C^r .

This is a consequence of applying differentiability with respect to parameters

inductively.

Step 3: G is C^r .

This is a consequence of Steps 1 and 2 and the formula for G in terms of φ .

Step 4: $DG(0)$ is an invertible matrix.

Since

$$\left. \frac{\partial G(y)}{\partial y_1} \right|_{y=0} = \left. \frac{\partial}{\partial t} \varphi(t, 0) \right|_{t=0} = f(0) = \alpha e_1$$

and

$$\left. \frac{\partial G(y)}{\partial y_k} \right|_{y=0} = \left. \frac{\partial}{\partial p} \varphi(0, p) \right|_{p=0} e_k = \left. \frac{\partial p}{\partial p} \right|_{p=0} e_k = e_k,$$

for $k \neq 1$, we have

$$DG(0) = \begin{bmatrix} \alpha e_1 & e_2 & \cdots & e_n \end{bmatrix},$$

which is invertible since $\alpha \neq 0$.

Step 5: If \mathcal{W} is sufficiently small, then G is invertible.

This is a consequence of Step 4 and the Inverse Function Theorem.

Set g equal to the (locally defined) inverse of G . Since G is C^r , so is g . The only thing remaining to check is that if x satisfies (2) then $g \circ x$ satisfies (3). Equivalently, we can check that if y satisfies (3) then $G \circ y$ satisfies (2).

Step 6: If y satisfies (3) then $G \circ y$ satisfies (2).

By the chain rule,

$$\begin{aligned} \frac{d}{dt} G(y(t)) &= \left. \frac{\partial}{\partial s} \varphi(s, (0, y_2, \dots, y_n)) \right|_{s=y_1} \dot{y}_1 + \left. \frac{\partial}{\partial p} \varphi(y_1, p) \right|_{p=(0, y_2, \dots, y_n)} \begin{bmatrix} 0 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_n \end{bmatrix} \\ &= f(\varphi(y_1, (0, y_2, \dots, y_n))) = f(G(y)). \end{aligned}$$

□