Existence of Solutions

Lecture 2
Math 634
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Approximate Solutions

Consider the IVP

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = a, \end{cases} \tag{1}$$

where $f : \text{dom}(f) \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and $(t_0, a) \in \text{dom}(f)$ is constant. The Fundamental Theorem of Calculus implies that (1) is equivalent to the integral equation

$$x(t) = a + \int_{t_0}^{t} f(s, x(s))\, ds. \tag{2}$$

How does one go about proving that (2) has a solution if, unlike the case with so many IVPs studied in introductory courses, a formula for a solution cannot be found? One idea is to construct a sequence of "approximate" solutions, with the approximations becoming better and better, in some sense, as we move along the sequence. If we can show that this sequence, or a subsequence, converges to something, that limit might be an exact solution.

One way of constructing approximate solutions is *Picard iteration*. Here, we plug an initial guess in for x on the right-hand side of (2), take the resulting value of the right-hand side and plug that in for x again, etc. More precisely, we can set $x_1(t) := a$ and recursively define x_{k+1} in terms of x_k for $k > 1$ by

$$x_{k+1}(t) := a + \int_{t_0}^{t} f(s, x_k(s))\, ds.$$

Note that if, for some k, $x_k = x_{k+1}$ then we have found a solution.

Another approach is to construct a *Tonelli sequence*. For each $k \in \mathbb{N}$, let $x_k(t)$ be defined by

$$x_k(t) = \begin{cases} a, & \text{if } t_0 \le t \le t_0 + 1/k \\ a + \displaystyle\int_{t_0}^{t-1/k} f(s, x_k(s))\, dx, & \text{if } t \ge t_0 + 1/k \end{cases} \tag{3}$$

for $t \geq t_0$, and define $x_k(t)$ similarly for $t \leq t_0$.

We will use the Tonelli sequence to show that (2) (and therefore (1)) has a solution, and will use Picard iterates to show that, under an additional hypothesis on f, the solution of (2) is unique.

Existence

For the first result, we will need the following definitions and theorems.

Definition A sequence of functions $g_k : \mathcal{U} \subseteq \mathbb{R} \to \mathbb{R}^n$ is *uniformly bounded* if there exists $M > 0$ such that $|g_k(t)| \leq M$ for every $t \in \mathcal{U}$ and every $k \in \mathbb{N}$.

Definition A sequence of functions $g_k : \mathcal{U} \subseteq \mathbb{R} \to \mathbb{R}^n$ is *uniformly equicontinuous* if for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that $|g_k(t_1) - g_k(t_2)| < \varepsilon$ for every $k \in \mathbb{N}$ and every $t_1, t_2 \in \mathcal{U}$ satisfying $|t_1 - t_2| < \delta$.

Definition A sequence of functions $g_k : \mathcal{U} \subseteq \mathbb{R} \to \mathbb{R}^n$ *converges uniformly* to a function $g : \mathcal{U} \subseteq \mathbb{R} \to \mathbb{R}^n$ if for every $\varepsilon > 0$ there exists a number $N \in \mathbb{N}$ such that if $k \geq N$ and $t \in \mathcal{U}$ then $|g_k(t) - g(t)| < \varepsilon$.

Definition If $a \in \mathbb{R}^n$ and $\beta > 0$, then the *open ball of radius β centered at a*, denoted $\mathcal{B}(a, \beta)$, is the set

$$\left\{ x \in \mathbb{R}^n \;\middle|\; |x - a| < \beta \right\}.$$

Theorem (Arzela-Ascoli) *Every uniformly bounded, uniformly equicontinuous sequence of functions $g_k : \mathcal{U} \subseteq \mathbb{R} \to \mathbb{R}^n$ has a subsequence that converges uniformly on compact (closed and bounded) sets.*

Theorem (Uniform Convergence) *If a sequence of continuous functions $h_k : [b, c] \to \mathbb{R}^n$ converges uniformly to $h : [b, c] \to \mathbb{R}^n$, then*

$$\lim_{k \uparrow \infty} \int_b^c h_k(s)\, ds = \int_b^c h(s)\, ds.$$

We are now in a position to state and prove the Cauchy-Peano Existence Theorem.

Theorem (Cauchy-Peano) *Suppose $f : [t_0 - \alpha, t_0 + \alpha] \times \overline{\mathcal{B}(a, \beta)} \to \mathbb{R}^n$ is contin-*

uous and bounded by $M > 0$. Then (2) has a solution defined on $[t_0 - b, t_0 + b]$, where $b = \min\{\alpha, \beta/M\}$.

Proof. For simplicity, we will only consider $t \in [t_0, t_0 + b]$. For each $k \in \mathbb{N}$, let $x_k : [t_0, t_0 + b] \to \mathbb{R}^n$ be defined by (3). We will show that (x_k) converges to a solution of (1).

Step 1: Each x_k is well-defined.
Fix $k \in \mathbb{N}$. Note that the point (t_0, a) is in the interior of a set on which f is well-defined. Because of the formula for $x_k(t)$ and the fact that it is recursively defined on intervals of width $1/k$ moving steadily to the right, if x_k failed to be defined on $[t_0, t_0 + b]$ then there would be $t_1 \in [t_0 + 1/k, t_0 + b)$ for which $|x_k(t_1) - a| = \beta$. Pick the first such t_1. Using (3) and the bound on f, we see that

$$\begin{aligned} |x_k(t_1) - a| &= \left| \int_{t_0}^{t_1 - 1/k} f(s, x_k(s))\, ds \right| \le \int_{t_0}^{t_1 - 1/k} |f(s, x_k(s))|\, ds \\ &\le \int_{t_0}^{t_1 - 1/k} M\, ds = M(t_1 - t_0 - 1/k) < M(b - 1/k) \\ &\le \beta - M/k < \beta = |x_k(t_1) - a|, \end{aligned}$$

which is a contradiction.

Step 2: (x_k) is uniformly bounded.
Calculating as above, the formula (3) implies that

$$|x_k(t)| \le |a| + \int_{t_0}^{b + t_0 - 1/k} |f(s, x_k(s))|\, dx \le |a| + (b - 1/k)M \le |a| + \beta.$$

Step 3: (x_k) is uniformly equicontinuous.
If $t_1, t_2 \ge t_0 + 1/k$, then

$$|x_k(t_1) - x_k(t_2)| = \left| \int_{t_1}^{t_2} f(s, x_k(s))\, ds \right| \le \left| \int_{t_1}^{t_2} |f(s, x_k(s))|\, ds \right| \le M|t_2 - t_1|.$$

Since x_k is constant on $[t_0, t_0 + 1/k]$ and continuous at $t_0 + 1/k$, the estimate $|x_k(t_1) - x_k(t_2)| \le M|t_2 - t_1|$ holds for every $t_1, t_2 \in [t_0, t_0 + b]$. Thus, given $\varepsilon > 0$, we can set $\delta = \varepsilon/M$ and see that uniform equicontinuity holds.

Step 4: Some subsequence (x_{k_ℓ}) of (x_k) converges uniformly, say, to x on $[t_0, t_0 + b]$.
This follows directly from the previous steps and the Arzela-Ascoli Theorem.

Step 5: The sequence $(f(\cdot, x_{k_\ell}(\cdot)))$ converges uniformly to $f(\cdot, x(\cdot))$ on $[t_0, t_0 + b]$.
Let $\varepsilon > 0$ be given. Since f is continuous on a compact set, it is uniformly continuous. Thus, we can pick $\delta > 0$ such that $|f(s,p) - f(s,q)| < \varepsilon$ whenever $|p - q| < \delta$. Since (x_{k_ℓ}) converges uniformly to x, we can pick $N \in \mathbb{N}$ such that $|x_{k_\ell}(s) - x(s)| < \delta$ whenever $s \in [t_0, t_0 + b]$ and $\ell \geq N$. If $\ell \geq N$, then $|f(s, x_{k_\ell}(s)) - f(s, x(s))| < \varepsilon$.

Step 6: The function x is a solution of (1).
Fix $t \in [t_0, t_0 + b]$. If $t = t_0$, then clearly (2) holds. If $t > t_0$, then for ℓ sufficiently large

$$x_{k_\ell}(t) = a + \int_{t_0}^{t} f(s, x_{k_\ell}(s))\, ds - \int_{t-1/k_\ell}^{t} f(s, x_{k_\ell}(s))\, ds. \tag{4}$$

Obviously, the left-hand side of (4) converges to $x(t)$ as $\ell \uparrow \infty$. By the Uniform Convergence Theorem and the uniform convergence of $(f(\cdot, x_{k_\ell}(\cdot)))$, the first integral on the right-hand side of (4) converges to

$$\int_{t_0}^{t} f(s, x(s))\, ds,$$

and by the boundedness of f the second integral converges to 0. Hence, taking the limit of (4) as $\ell \uparrow \infty$ we see that x satisfies (2), and therefore (1), on $[t_0, t_0 + b]$. □

Note that this theorem guarantees existence, but not necessarily uniqueness, of a solution of (1).

Exercise 2 How many solutions of the IVP

$$\begin{cases} \dot{x} = 2\sqrt{|x|} \\ x(0) = 0, \end{cases}$$

on the interval $(-\infty, \infty)$ are there? Give formulas for all of them.