Definitions of Stability

Lecture 20
Math 634
10/15/99

In the previous lecture, we saw that all the "interesting" local behavior of flows occurs near equilibrium points. One important aspect of the behavior of flows has to do with whether solutions that start near a given solution stay near it for all time and/or move closer to it as time elapses. This question, which is the subject of *stability theory*, is not just of interest when the given solution corresponds to an equilibrium solution, so we study it–initially, at least–in a fairly broad context.

Definitions

First, we define some types of stability for solutions of the (possibly nonautonomous) equation

$$\dot{x} = f(t, x). \tag{1}$$

Definition A solution $\overline{x}(t)$ of (1) is *(Lyapunov) stable* if for each $\varepsilon > 0$ and $t_0 \in \mathbf{R}$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that if $x(t)$ is a solution of (1) and $|x(t_0) - \overline{x}(t_0)| < \delta$ then $|x(t) - \overline{x}(t)| < \varepsilon$ for all $t \geq t_0$.

Definition A solution $\overline{x}(t)$ of (1) is *asymptotically stable* if it is (Lyapunov) stable and if for every $t_0 \in \mathbf{R}$ there exists $\delta = \delta(t_0) > 0$ such that if $x(t)$ is a solution of (1) and $|x(t_0) - \overline{x}(t_0)| < \delta$ then $|x(t) - \overline{x}(t)| \to 0$ as $t \uparrow \infty$.

Definition A solution $\overline{x}(t)$ of (1) is *uniformly stable* if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $x(t)$ is a solution of (1) and $|x(t_0) - \overline{x}(t_0)| < \delta$ for some $t_0 \in \mathbb{R}$ then $|x(t) - \overline{x}(t)| < \varepsilon$ for all $t \geq t_0$.

Some authors use a weaker definition of uniform stability that turns out to be equivalent to Lyapunov stability for autonomous equations. Since our main interest is in autonomous equations and this alternative definition is somewhat more complicated than the definition given above, we will not use it here.

Definition A solution $\overline{x}(t)$ of (1) is *orbitally stable* if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $x(t)$ is a solution of (1) and $|x(t_1) - \overline{x}(t_0)| < \delta$ for some $t_0, t_1 \in \mathbb{R}$ then

$$\bigcup_{t \geq t_1} x(t) \subseteq \bigcup_{t \geq t_0} \mathcal{B}(\overline{x}(t), \varepsilon).$$

Next, we present a couple of definitions of stability for subsets of the (open) phase space $\Omega \subseteq \mathbb{R}^n$ of a dynamical system $\varphi(t, x)$. (In these definitions, a *neighborhood* of a set $\mathcal{A} \subseteq \Omega$ is an open subset of Ω that contains $\mathcal{A}$.)

Definition The set $\mathcal{A}$ is *stable* if every neighborhood of $\mathcal{A}$ contains a positively invariant neighborhood of $\mathcal{A}$.

Note that the definition implies that stable sets are positively invariant.

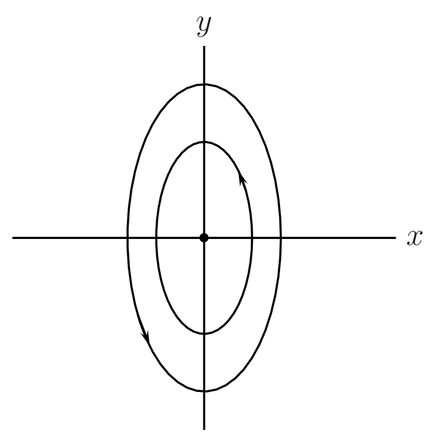
Definition The set $\mathcal{A}$ is *asymptotically stable* if it is stable and there is some neighborhood $\mathcal{V}$ of $\mathcal{A}$ such that $\omega(x) \subseteq \mathcal{A}$ for every $x \in \mathcal{V}$. (If $\mathcal{V}$ can be chosen to be the entire phase space, then $\mathcal{A}$ is *globally asymptotically stable*.)

Examples

We now consider a few examples that clarify some properties of these definitions.

$\boxed{1}$

$$\begin{cases} \dot{x} = -y/2 \\ \dot{y} = 2x. \end{cases}$$



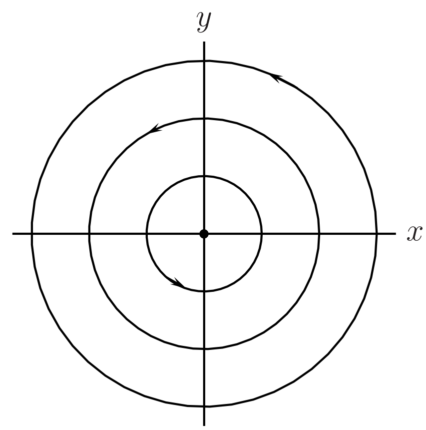
Orbits are ellipses with major axis along the y-axis. The equilibrium solution at the origin is Lyapunov stable even though nearby orbits sometimes move away from it.

$\boxed{\mathbf{2}}$

$$\begin{cases} \dot{r} = 0 \\ \dot{\theta} = r^2, \end{cases}$$

or, equivalently,

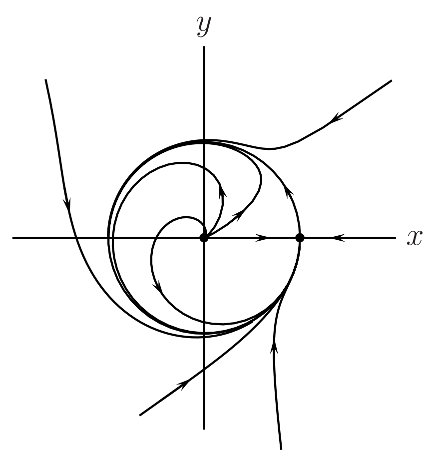
$$\begin{cases} \dot{x} = -(x^2 + y^2)y \\ \dot{y} = (x^2 + y^2)x. \end{cases}$$



The solution moving around the unit circle is not Lyapunov stable, since nearby solutions move with different angular velocities. It is, however, orbitally stable. Also, the set consisting of the unit circle is stable.

$\boxed{\mathbf{3}}$

$$\begin{cases} \dot{r} = r(1 - r) \\ \dot{\theta} = \sin^2(\theta/2). \end{cases}$$



The constant solution $(x, y) = (1, 0)$ is not Lyapunov stable and the set $\{(1, 0)\}$ is not stable. However, every solution beginning near $(1, 0)$ converges to $(1, 0)$ as $t \uparrow \infty$. This shows that it is not redundant to require Lyapunov stability (or stability) in the definition of asymptotic stability of a solution (or a set).

Stability in Autonomous Equations

When we are dealing with a smooth autonomous differential equation

$$\dot{x} = f(x) \tag{2}$$

on an open set $\Omega \subseteq \mathbb{R}^n$, all of the varieties of stability can be applied to essentially the same object. In particular, let $\overline{x}$ be a function that solves (2), and let

$$\mathcal{A}(\overline{x}) := \left\{ \overline{x}(t) \;\middle|\; t \in \mathbb{R} \right\}$$

be the corresponding orbit. Then it makes sense to talk about the Lyapunov, asymptotic, orbital, or uniform stability of $\overline{x}$, and it makes sense to talk about the stability or asymptotic stability of $\mathcal{A}(\overline{x})$.

In this context, certain relationships between the various types of stability follow from the definitions without too much difficulty.

Theorem *Let $\overline{x}$ be a function that solves* (2)*, and let $\mathcal{A}(\overline{x})$ be the corresponding orbit. Then:*

1. *If $\overline{x}$ is asymptotically stable then $\overline{x}$ is Lyapunov stable;*
2. *If $\overline{x}$ is uniformly stable then $\overline{x}$ is Lyapunov stable;*
3. *If $\overline{x}$ is uniformly stable then $\overline{x}$ is orbitally stable;*
4. *If $\mathcal{A}(\overline{x})$ is asymptotically stable then $\mathcal{A}(\overline{x})$ is stable;*
5. *If $\mathcal{A}(\overline{x})$ contains only a single point, then Lyapunov stability of $\overline{x}$, orbital stability of $\overline{x}$, uniform stability of $\overline{x}$, and stability of $\mathcal{A}(\overline{x})$ are equivalent.*

We will not prove this theorem, but we will note that parts 1 and 2 are immediate results of the definitions (even if we were dealing with a nonautonomous equation) and part 4 is also an immediate result of the definitions (even if $\mathcal{A}$ were an arbitrary set).

<u>Exercise 13</u> In items 1–18, an autonomous differential equation, a phase space Ω (that is an open subset of $\mathbb{R}^n$), and a particular solution $\overline{x}$ of the equation are specified. For each of these items, state which of the following statements is/are true:

(a) $\overline{x}$ is Lyapunov stable;

(b) $\overline{x}$ is asymptotically stable;

(c) $\overline{x}$ is orbitally stable;

(d) $\overline{x}$ is uniformly stable;

(e) $\mathcal{A}(\overline{x})$ is stable;

(f) $\mathcal{A}(\overline{x})$ is asymptotically stable.

You do *not* need to justify your answers or show your work. It may convenient to express your answers in a concise form (*e.g.*, in a table of some sort). Use of variables r and θ signifies that the equation (as well as the particular solution) is to be interpreted as in polar form.
(The exercise is continued in the next box.)

Exercise 13 (continued)

1. $\dot{x} = x$, $\Omega = \mathbb{R}$, $\overline{x}(t) := 0$
2. $\dot{x} = x$, $\Omega = \mathbb{R}$, $\overline{x}(t) := e^t$
3. $\{\dot{x}_1 = 1 + x_2^2, \dot{x}_2 = 0\}$, $\Omega = \mathbb{R}^2$, $\overline{x}(t) := (t, 0)$
4. $\{\dot{r} = 0, \dot{\theta} = r^2\}$, $\Omega = \mathbb{R}^2$, $\overline{x}(t) := (1, t)$
5. $\dot{x} = x$, $\Omega = (0, \infty)$, $\overline{x}(t) := e^t$
6. $\{\dot{x}_1 = 1, \dot{x}_2 = -x_1 x_2\}$, $\Omega = \mathbb{R}^2$, $\overline{x}(t) := (t, 0)$
7. $\dot{x} = \tanh x$, $\Omega = \mathbb{R}$, $\overline{x}(t) := \sinh^{-1}(e^t)$
8. $\{\dot{x}_1 = \tanh x_1, \dot{x}_2 = 0\}$, $\Omega = (0, \infty) \times \mathbb{R}$, $\overline{x}(t) := (\sinh^{-1}(e^t), 0)$
9. $\dot{x} = \tanh x$, $\Omega = (0, \infty)$, $\overline{x}(t) := \sinh^{-1}(e^t)$
10. $\{\dot{x}_1 = \operatorname{sech} x_1, \dot{x}_2 = -x_1 x_2 \operatorname{sech} x_1\}$, $\Omega = \mathbb{R}^2$,
    $\overline{x}(t) := (\sinh^{-1}(t), 0)$
11. $\dot{x} = x^2/(1 + x^2)$, $\Omega = \mathbb{R}$, $\overline{x}(t) := -2/(t + \sqrt{t^2 + 4})$
12. $\{\dot{x}_1 = \operatorname{sech} x_1, \dot{x}_2 = -x_2\}$, $\Omega = \mathbb{R}^2$, $\overline{x}(t) := (\sinh^{-1}(t), 0)$
13. $\dot{x} = \operatorname{sech} x$, $\Omega = \mathbb{R}$, $\overline{x}(t) := \sinh^{-1}(t)$
14. $\{\dot{x}_1 = 1, \dot{x}_2 = 0\}$, $\Omega = \mathbb{R}^2$, $\overline{x}(t) := (t, 0)$
15. $\dot{x} = 0$, $\Omega = \mathbb{R}$, $\overline{x}(t) := 0$
16. $\dot{x} = 1$, $\Omega = \mathbb{R}$, $\overline{x}(t) := t$
17. $\{\dot{x}_1 = -x_1, \dot{x}_2 = -x_2\}$, $\Omega = \mathbb{R}^2$, $\overline{x}(t) := (e^{-t}, 0)$
18. $\dot{x} = -x$, $\Omega = \mathbb{R}$, $\overline{x}(t) := 0$