Principle of Linearized Stability

Lecture 21
Math 634
10/18/99

Suppose f is a continuously differentiable function such that

$$\dot{x} = f(x) \tag{1}$$

generates a continuous dynamical system φ on $\Omega \subseteq \mathbb{R}^n$. Suppose, moreover, that $x_0 \in \Omega$ is a singular point of φ. If x solves (1) and we set $u := x - x_0$ and $A := Df(x_0)$, we see that, by the definition of derivative,

$$\dot{u} = f(u + x_0) = f(x_0) + Df(x_0)u + R(u) = Au + R(u), \tag{2}$$

where $R(u)/|u| \to 0$ as $|u| \downarrow 0$. Because $R(u)$ is small when u is small, it is reasonable to believe that solutions of (2) behave similarly to solutions of

$$\dot{u} = Au \tag{3}$$

for u near 0. Equivalently, it is reasonable to believe that solutions of (1) behave like solutions of

$$\dot{x} = A(x - x_0) \tag{4}$$

for x near x_0. Equation (3) (or sometimes (4)) is called the *linearization* of (1) at x_0.

Now, we've defined (several types of) stability for equilibrium solutions of (1) (as well as for other types of solutions and sets), but we haven't really given any tools for determining stability. In this lecture we present one such tool, using the linearized equation(s) discussed above.

Definition An equilibrium point x_0 of (1) is *hyperbolic* if none of the eigenvalues of $Df(x_0)$ have zero real part.

If x_0 is hyperbolic, then either all the eigenvalues of $A := Df(x_0)$ have negative real part or at least one has positive real part. In the former case, we know that 0 is an asymptotically stable equilibrium solution of (3); in the latter case, we know that 0 is an unstable solution of (3). The following theorem says that similar things can be said about the nonlinear equation (1).

Theorem (Principle of Linearized Stability) *If* x_0 *is a hyperbolic equilibrium solution of* (1)*, then* x_0 *is either unstable or asymptotically stable, and its stability type (w.r.t.* (1)*) matches the stability type of* 0 *as an equilibrium solution of* (3) *(where* $A := Df(x_0)$*).*

This theorem is an immediate consequence of the following two propositions.

Proposition (Asymptotic Stability) *If* x_0 *is an equilibrium point of* (1) *and all the eigenvalues of* $A := Df(x_0)$ *have negative real part, then* x_0 *is asymptotically stable.*

Proposition (Instability) *If* x_0 *is an equilibrium point of* (1) *and some eigenvalue of* $A := Df(x_0)$ *has positive real part, then* x_0 *is unstable.*

Before we prove these propositions, we state and prove a lemma to which we have referred before in passing.

Lemma *Let* $\mathcal{V}$ *be a finite-dimensional real vector space and let* $L \in \mathcal{L}(\mathcal{V}, \mathcal{V})$*. If all the eigenvalues of* L *have real part larger than* c*, then there is an inner product* $\langle \cdot, \cdot \rangle$ *and an induced norm* $\| \cdot \|$ *on* $\mathcal{V}$ *such that*

$$\langle v, Lv \rangle \geq c\|v\|^2$$

for every $v \in \mathcal{V}$*.*

Proof. Let $n = \dim \mathcal{V}$, and pick $\varepsilon > 0$ so small that all the eigenvalues of L have real part greater than $c + n\varepsilon$. Choose a basis $\{v_1, \dots, v_n\}$ for $\mathcal{V}$ that puts L in "modified" real canonical form with the off-diagonal 1's replaced by ε's, and let $\langle \cdot, \cdot \rangle$ be the inner product associated with this basis (*i.e.* $\langle v_i, v_j \rangle = \delta_{ij}$) and let $\| \cdot \|$ be the induced norm on $\mathcal{V}$.

Given $v = \sum_{i=1}^n \alpha_i v_i \in \mathcal{V}$, note that (if $L = (\ell_{ij})$)

$$\begin{aligned}\langle v, Lv \rangle &= \sum_{i=1}^n \ell_{ii}\alpha_i^2 + \sum_{i=1}^n \sum_{j \neq i} \ell_{ij}\alpha_i\alpha_j \geq \sum_{i=1}^n \ell_{ii}\alpha_i^2 - \sum_{i=1}^n \sum_{j \neq i} \varepsilon \left(\frac{\alpha_i^2 + \alpha_j^2}{2} \right) \\ &\geq \sum_{i=1}^n \ell_{ii}\alpha_i^2 - \sum_{i=1}^n n\varepsilon\alpha_i^2 = \sum_{i=1}^n (\ell_{ii} - n\varepsilon)\alpha_i^2 \geq \sum_{i=1}^n c\alpha_i^2 = c\|v\|^2.\end{aligned}$$

□

Note that applying this theorem to $-L$ also tells us that, for some inner product,

$$\langle v, Lv \rangle \leq c\|v\|^2 \tag{5}$$

if all the eigenvalues of L have real part less than c.

Proof of Proposition on Asymptotic Stability. Without loss of generality, assume that $x_0 = 0$. Pick $c < 0$ such that all the eigenvalues of A have real part strictly less than c. Because of equivalence of norms and because of the lemma, we can work with a norm $\|\cdot\|$ and a corresponding inner product $\langle \cdot, \cdot \rangle$ for which (5) holds, with $L = A$. Let $r > 0$ be small enough that $\|R(x)\| \leq -c/2\|x\|$ for all x satisfying $\|x\| \leq r$, and let

$$\mathcal{B}_r := \{x \in \Omega \mid \|x\| < r\}.$$

If $x(t)$ is a solution of (1) that starts in $\mathcal{B}_r$ at time $t = 0$, then as long as $x(t)$ remains in $\mathcal{B}_r$

$$\begin{aligned}\frac{d}{dt}\|x(t)\|^2 &= 2\langle x(t), \dot{x}(t)\rangle = 2\langle x(t), f(x(t))\rangle \\ &= 2\langle x(t), Ax(t)\rangle + 2\langle x(t), R(x(t))\rangle \\ &\leq 2c\|x(t)\|^2 + 2\|x(t)\| \cdot \|R(x(t))\| \\ &\leq 2c\|x(t)\|^2 - c\|x(t)\|^2 = c\|x(t)\|^2.\end{aligned}$$

This means that $x(t) \in \mathcal{B}_r$ for all $t \geq 0$, and $x(t)$ converges to 0 (exponentially quickly) as $t \uparrow \infty$. □

The proof of the second proposition will be geometric and will contain ideas that will be used to prove stronger results later in this course.

Proof of Proposition on Instability. We assume again that $x_0 = 0$. If $\mathcal{E}^u$,$\mathcal{E}^s$, and $\mathcal{E}^c$ are, respectively, the unstable, stable, and center spaces corresponding to (3), set $\mathcal{E}^- := \mathcal{E}^s \oplus \mathcal{E}^c$ and $\mathcal{E}^+ := \mathcal{E}^u$. Then $\mathbb{R}^n = \mathcal{E}^+ \oplus \mathcal{E}^-$, all of the eigenvalues of $A^+ := A|_{\mathcal{E}^+}$ have positive real part, and all of the eigenvalues of $A^- := A|_{\mathcal{E}^-}$ have nonpositive real part. Pick constants $a > b > 0$ such that all of the eigenvalues of A^+ have real part larger than a and all of the eigenvalues of A^- have real part less than b. Define an inner product $\langle \cdot, \cdot \rangle_+$ (and induced norm $\|\cdot\|_+$) on $\mathcal{E}^+$ such that

$$\langle v, Av \rangle_+ \geq a\|v\|_+^2$$

for all $v \in \mathcal{E}^+$, and define an inner product $\langle \cdot, \cdot \rangle_-$ (and induced norm $\| \cdot \|_-$) on $\mathcal{E}^-$ such that

$$\langle w, Aw \rangle_- \leq b \|w\|_-^2$$

for all $w \in \mathcal{E}^-$. Define $\langle \cdot, \cdot \rangle$ on $\mathcal{E}^+ \oplus \mathcal{E}^-$ to be the *direct sum* of $\langle \cdot, \cdot \rangle_+$ and $\langle \cdot, \cdot \rangle_-$; *i.e.*, let

$$\langle v_1 + w_1, v_2 + w_2 \rangle := \langle v_1, v_2 \rangle_+ + \langle w_1, w_2 \rangle_-$$

for all $(v_1, w_1), (v_2, w_2) \in \mathcal{E}^+ \times \mathcal{E}^-$. Let $\| \cdot \|$ be the induced norm, and note that

$$\|v + w\|^2 = \|v\|_+^2 + \|w\|_-^2 = \|v\|^2 + \|w\|^2$$

for all $(v, w) \in \mathcal{E}^+ \times \mathcal{E}^-$.

Now, take (1) and project it onto $\mathcal{E}^+$ and $\mathcal{E}^-$ to get the corresponding system for $(v, w) \in \mathcal{E}^+ \times \mathcal{E}^-$

$$\begin{cases} \dot{v} = A^+ v + R^+(v, w) \\ \dot{w} = A^- w + R^-(v, w), \end{cases} \tag{6}$$

with $\|R^\pm(v, w)\| / \|v + w\|$ converging to 0 as $\|v + w\| \downarrow 0$. Pick $\varepsilon > 0$ small enough that $a - b - 2\sqrt{2}\varepsilon > 0$, and pick $r > 0$ small enough that $\|R^\pm(v, w)\| \leq \varepsilon \|v + w\|$ whenever

$$v + w \in \mathcal{B}_r := \{ v + w \in \mathcal{E}^+ \oplus \mathcal{E}^- \mid \|v + w\| < r \}.$$

Consider the truncated cone

$$\mathcal{K}_r := \{ v + w \in \mathcal{E}^+ \oplus \mathcal{E}^- \mid \|v\| > \|w\| \} \cap \mathcal{B}_r.$$

(See Figure 1.) Suppose $x = v + w$ is a solution of (6) that starts in $\mathcal{K}_r$ at time $t = 0$. For as long as the solution remains in $\mathcal{K}_r$,

$$\begin{aligned} \frac{d}{dt} \|v\|^2 &= 2\langle v, \dot{v} \rangle = 2\langle v, A^+ v \rangle + 2\langle v, R^+(v, w) \rangle \\ &\geq 2a\|v\|^2 - 2\|v\| \cdot \|R^+(v, w)\| \geq 2a\|v\|^2 - 2\varepsilon \|v\| \cdot \|v + w\| \\ &= 2a\|v\|^2 - 2\varepsilon \|v\| \left(\|v\|^2 + \|w\|^2 \right)^{1/2} \geq 2a\|v\|^2 - 2\sqrt{2}\varepsilon \|v\|^2 \\ &= 2(a - \sqrt{2}\varepsilon)\|v\|^2, \end{aligned}$$

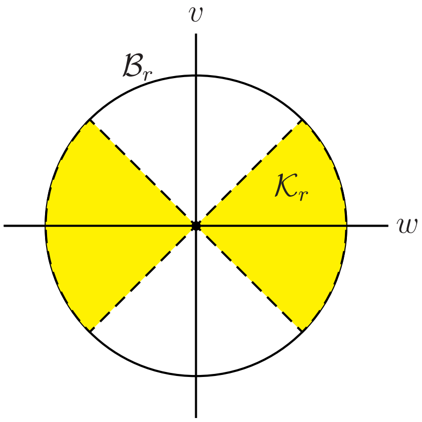

Figure 1: The truncated cone.

and

$$\begin{aligned}\frac{d}{dt}\|w\|^2 &= 2\langle w, \dot{w}\rangle = 2\langle w, A^- w\rangle + 2\langle w, R^-(v,w)\rangle \\ &\le 2b\|w\|^2 + 2\|w\| \cdot \|R^-(v,w)\| \le 2b\|w\|^2 + 2\varepsilon\|w\| \cdot \|v+w\| \\ &= 2b\|w\|^2 + 2\varepsilon\|w\| \left(\|v\|^2 + \|w\|^2\right)^{1/2} \le 2b\|v\|^2 + 2\sqrt{2}\varepsilon\|v\|^2 \\ &= 2(b+\sqrt{2}\varepsilon)\|v\|^2.\end{aligned}$$

The first estimate says that as long as the solution stays in $\mathcal{K}_r$, $\|v\|$ grows exponentially; this means that the solution must eventually leave $\mathcal{K}_r$. Combining the first and second estimates, we have

$$\frac{d}{dt}(\|v\|^2 - \|w\|^2) \ge 2(a - b - 2\sqrt{2}\varepsilon)\|v\|^2 > 0,$$

so $g(v+w) := \|v\|^2 - \|w\|^2$ increases as t increases. But g is 0 on the lateral surface of $\mathcal{K}_r$ and is strictly positive in $\mathcal{K}_r$, so the solution cannot leave $\mathcal{K}_r$ through its lateral surface. Thus, the solution leaves $\mathcal{K}_r$ by leaving $\mathcal{B}_r$. Since this holds for all solutions starting in $\mathcal{K}_r$, we know that x_0 must be an unstable equilibrium point for (1). ☐