Hartman-Grobman Theorem: Part 1

Lecture 24
Math 634
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Let $\Omega \subset \mathbb{R}^n$ be open and let $f : \Omega \to \mathbb{R}^n$ be continuously differentiable. Suppose that $x_0 \in \Omega$ is a hyperbolic equilibrium point of the autonomous equation

$$\dot{x} = f(x). \tag{1}$$

Let $B = Df(x_0)$, and let φ be the (local) flow generated by (1). The version of the Hartman-Grobman Theorem we're primarily interested in is the following.

Theorem (Local Hartman-Grobman Theorem for Flows) *Let Ω, f, x_0, B, and φ be as described above. Then there are neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of x_0 and a homeomorphism $h : \mathcal{U} \to \mathcal{V}$ such that*

$$\varphi(t, h(x)) = h(x_0 + e^{tB}(x - x_0))$$

whenever $x \in \mathcal{U}$ and $x_0 + e^{tB}(x - x_0) \in \mathcal{U}$.

It will be easier to derive this theorem as a consequence of a global theorem for maps than to prove it directly. In order to state this version of the theorem, we will need to introduce a couple of function spaces and a definition.

Let

$$C_b^0(\mathbb{R}^n) = \left\{ w \in C(\mathbb{R}^n, \mathbb{R}^n) \;\middle|\; \sup_{x \in \mathbb{R}^n} |w(x)| < \infty \right\}.$$

When equipped with the norm

$$\|w\|_0 := \sup_{x \in \mathbb{R}^n} \|w(x)\|,$$

where $\|\cdot\|$ is some norm on $\mathbb{R}^n$, $C_b^0(\mathbb{R}^n)$ is a Banach space. (We shall pick a particular norm $\|\cdot\|$ later.)

Let

$$C_b^1(\mathbb{R}^n) = \left\{ w \in C^1(\mathbb{R}^n, \mathbb{R}^n) \cap C_b^0(\mathbb{R}^n) \;\middle|\; \sup_{x \in \mathbb{R}^n} \|Dw(x)\| < \infty \right\},$$

where $\|\cdot\|$ is the operator norm corresponding to some norm on $\mathbb{R}^n$. Note that the functional

$$\operatorname{Lip}(w) := \sup_{\substack{x_1, x_2 \in \mathbb{R}^n \\ x_1 \neq x_2}} \frac{\|w(x_1) - w(x_2)\|}{\|x_1 - x_2\|}$$

is defined on $C_b^1(\mathbb{R}^n)$. We will not define a norm on $C_b^1(\mathbb{R}^n)$, but will often use Lip, which is not a norm, to describe the size of elements of $C_b^1(\mathbb{R}^n)$.

Definition If $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and none of the eigenvalues of A lie on the unit circle, then A is *hyperbolic.*

Note that if x_0 is a hyperbolic equilibrium point of (1) and $A = e^{Df(x_0)}$, then A is hyperbolic.

Theorem (Global Hartman-Grobman Theorem for Maps) *Suppose that the map $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is hyperbolic and invertible. Then there exists a number $\varepsilon > 0$ such that for every $g \in C_b^1(\mathbb{R}^n)$ satisfying* $\operatorname{Lip}(g) < \varepsilon$ *there exists a unique function $v \in C_b^0(\mathbb{R}^n)$ such that*

$$F(h(x)) = h(Ax)$$

for every $x \in \mathbb{R}^n$, where $F = A + g$ and $h = I + v$. Furthermore, $h : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism.