Hartman-Grobman Theorem: Part 2

Lecture 25
Math 634
10/27/99

Subspaces and Norms

We start off with a lemma that is analogous to the lemma in Lecture 21, except this one will deal with the magnitude, rather than the real part, of eigenvalues.

Lemma *Let $\mathcal{V}$ be a finite-dimensional real vector space and let $L \in \mathcal{L}(\mathcal{V}, \mathcal{V})$. If all the eigenvalues of L have magnitude less than c, then there is a norm $\|\cdot\|$ on $\mathcal{V}$ such that*

$$\|Lv\| \leq c\|v\|$$

for every $v \in \mathcal{V}$.

Proof. As in the previous lemma, the norm will be the Euclidean norm corresponding to the modification of the real canonical basis that yields a matrix representation of L that has ε's in place of the off-diagonal 1's. With respect to this basis, it can be checked that

$$L^T L = D + R(\varepsilon),$$

where D is a diagonal matrix, each of whose diagonal entries is less than c^2, and $R(\varepsilon)$ is a matrix whose entries converge to 0 as $\varepsilon \downarrow 0$. Hence, as in the proof of the earlier lemma, we can conclude that if ε is sufficiently small then

$$\|Lv\|^2 = \langle v, L^T L v\rangle \leq c^2\|v\|^2$$

for every $v \in \mathcal{V}$ (where $\langle \cdot, \cdot \rangle$ is the inner product that induces $\|\cdot\|$). $\square$

Note that if L is a linear operator, all of whose eigenvalues have magnitude *greater* than c, then by applying the lemma to L^{-1} (which exists, since 0 is not an eigenvalue of L), we see that

$$\|Lv\| \geq c\|v\|$$

for some norm $\|\cdot\|$.

Now, suppose that $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is hyperbolic. Then, since A has only finitely many eigenvalues, there is a number $a \in (0,1)$ such that none of the eigenvalues of A are in the closed annulus

$$\overline{B(0,a^{-1})} \setminus B(0,a).$$

Using the notation developed when we were deriving the real canonical form, let

$$\mathcal{E}^- = \left\{ \bigoplus_{\lambda \in (-a,a)} N(A - \lambda I) \right\} \oplus$$
$$\left\{ \bigoplus_{\substack{|\lambda|<a \\ \operatorname{Im}\lambda \neq 0}} \left\{ \operatorname{Re} u \;\middle|\; u \in N(A-\lambda I) \right\} \right\} \oplus \left\{ \bigoplus_{\substack{|\lambda|<a \\ \operatorname{Im}\lambda \neq 0}} \left\{ \operatorname{Im} u \;\middle|\; u \in N(A-\lambda I) \right\} \right\},$$

and let

$$\mathcal{E}^+ = \left\{ \bigoplus_{\lambda \in (-\infty,-a^{-1}) \cup (a^{-1},\infty)} N(A - \lambda I) \right\} \oplus$$
$$\left\{ \bigoplus_{\substack{|\lambda|>a^{-1} \\ \operatorname{Im}\lambda \neq 0}} \left\{ \operatorname{Re} u \;\middle|\; u \in N(A-\lambda I) \right\} \right\} \oplus \left\{ \bigoplus_{\substack{|\lambda|>a^{-1} \\ \operatorname{Im}\lambda \neq 0}} \left\{ \operatorname{Im} u \;\middle|\; u \in N(A-\lambda I) \right\} \right\}.$$

Then $\mathbb{R}^n = \mathcal{E}^- \oplus \mathcal{E}^+$, and $\mathcal{E}^-$ and $\mathcal{E}^+$ are both invariant under A. Define $P^- \in \mathcal{L}(\mathbb{R}^n, \mathcal{E}^-)$ and $P^+ \in \mathcal{L}(\mathbb{R}^n, \mathcal{E}^+)$ to be the linear operators that map each $x \in \mathbb{R}^n$ to the unique elements $P^-x \in \mathcal{E}^-$ and $P^+x \in \mathcal{E}^+$ such that $P^-x + P^+x = x$.

Let $A^- \in \mathcal{L}(\mathcal{E}^-, \mathcal{E}^-)$ and $A^+ \in \mathcal{L}(\mathcal{E}^+, \mathcal{E}^+)$ be the restrictions of A to $\mathcal{E}^-$ and $\mathcal{E}^+$, respectively. By the lemma (and the discussion thereafter) we can find a norm $\|\cdot\|_-$ for $\mathcal{E}^-$ and a norm $\|\cdot\|_+$ for $\mathcal{E}^+$ such that

$$\|A^-x\|_- \leq a\|x\|_-$$

for every $x \in \mathcal{E}^-$, and

$$\|A^+x\|_+ \geq a^{-1}\|x\|_+$$

for every $x \in \mathcal{E}^+$. Define a norm $\|\cdot\|$ on $\mathbb{R}^n$ by the formula

$$\|x\| = \max\{\|P^- x\|_-, \|P^+ x\|_+\}. \tag{1}$$

This is the norm on $\mathbb{R}^n$ that we will use throughout our proof of the (global) Hartman-Grobman Theorem (for maps). Note that $\|x\| = \|x\|_-$ if $x \in \mathcal{E}^-$, and $\|x\| = \|x\|_+$ if $x \in \mathcal{E}^+$.

Recall that we equipped $C_b^0(\mathbb{R}^n)$ with the norm $\|\cdot\|_0$ defined by the formula

$$\|w\|_0 := \sup_{x\in\mathbb{R}^n} \|w(x)\|.$$

The norm on $\mathbb{R}^n$ on the right-hand side of this formula is that given in (1). Recall also that we will use the functional Lip defined by the formula

$$\mathrm{Lip}(w) := \sup_{\substack{x_1,x_2\in\mathbb{R}^n \\ x_1\neq x_2}} \frac{\|w(x_1) - w(x_2)\|}{\|x_1 - x_2\|}$$

The norm on $\mathbb{R}^n$ on the right-hand side of this formula is also that given in (1).

Let

$$C_b^0(\mathcal{E}^-) = \bigl\{w \in C(\mathbb{R}^n, \mathcal{E}^-) \bigm| \sup_{x\in\mathbb{R}^n} \|w(x)\|_- < \infty\bigr\},$$

and let

$$C_b^0(\mathcal{E}^+) = \bigl\{w \in C(\mathbb{R}^n, \mathcal{E}^+) \bigm| \sup_{x\in\mathbb{R}^n} \|w(x)\|_+ < \infty\bigr\}.$$

Since $\mathbb{R}^n = \mathcal{E}^- \oplus \mathcal{E}^+$, it follows that

$$C_b^0(\mathbb{R}^n) = C_b^0(\mathcal{E}^-) \oplus C_b^0(\mathcal{E}^+),$$

and the corresponding decomposition of an element $w \in C_b^0(\mathbb{R}^n)$ is

$$w = P^- \circ w + P^+ \circ w.$$

We equip $C_b^0(\mathcal{E}^-)$ and $C_b^0(\mathcal{E}^+)$ with the same norm $\|\cdot\|_0$ that we used on $C_b^0(\mathbb{R}^n)$, thereby making each of these two spaces a Banach space. It is not hard to see that

$$\|w\|_0 = \max\{\|P^- \circ w\|_0, \|P^+ \circ w\|_0\}.$$