Hartman-Grobman Theorem: Part 3

Lecture 26
Math 634
10/29/99

Linear and Nonlinear Maps

Now, assume that A is invertible, so that

$$\inf_{x\neq 0} \frac{\|Ax\|}{\|x\|} > 0.$$

Choose, and fix, a positive constant

$$\varepsilon < \min\left\{1 - a, \inf_{x\neq 0} \frac{\|Ax\|}{\|x\|}\right\}.$$

Choose, and fix, a function $g \in \mathbb{C}_b^1(\mathbb{R}^n)$ for which $\operatorname{Lip}(g) < \varepsilon$. The (global) Hartman-Grobman Theorem (for maps) will be proved by constructing a map Θ from $C_b^0(\mathbb{R}^n)$ to $C_b^0(\mathbb{R}^n)$ whose fixed points would be precisely those objects v which, when added to the identity I, would yield solutions h to the conjugacy equation

$$(A + g) \circ h = h \circ A, \tag{1}$$

and then showing that Θ is a contraction (and that h is a homeomorphism).

Plugging $h = I + v$ into (1) and manipulating the result, we can see that that equation is equivalent to the equation

$$\mathcal{L}v = \Psi(v), \tag{2}$$

where $\Psi(v) := g \circ (I + v) \circ A^{-1}$ and

$$\mathcal{L}v = v - A \circ v \circ A^{-1} =: (\operatorname{id} - \mathcal{A})v.$$

Since the composition of continuous functions is continuous, and the composition of functions is bounded if the outer function in the composition is bounded, it is clear that Ψ is a (nonlinear) map from $C_b^0(\mathbb{R}^n)$ to $C_b^0(\mathbb{R}^n)$. Similarly, $\mathcal{A}$ and, therefore, $\mathcal{L}$ are linear maps from $C_b^0(\mathbb{R}^n)$ to $C_b^0(\mathbb{R}^n)$. We will show that $\mathcal{L}$ can be inverted and then apply $\mathcal{L}^{-1}$ to both sides of (2) to get

$$v = \mathcal{L}^{-1}(\Psi(v)) =: \Theta(v), \tag{3}$$

as our fixed point equation.

Inverting $\mathcal{L}$

Since A behaves significantly differently on $\mathcal{E}^-$ than it does on $\mathcal{E}^+$, $\mathcal{A}$ and, therefore, $\mathcal{L}$ behave significantly differently on $C_b^0(\mathcal{E}^-)$ than they do on $C_b^0(\mathcal{E}^+)$. For this reason, we will analyze $\mathcal{L}$ by looking at its restrictions to $C_b^0(\mathcal{E}^-)$ and to $C_b^0(\mathcal{E}^+)$. Note that $C_b^0(\mathcal{E}^-)$ and $C_b^0(\mathcal{E}^+)$ are invariant under $\mathcal{A}$ and, therefore, under $\mathcal{L}$. Let $\mathcal{A}^- \in \mathcal{L}(C_b^0(\mathcal{E}^-), C_b^0(\mathcal{E}^-))$ and $\mathcal{A}^+ \in \mathcal{L}(C_b^0(\mathcal{E}^+), C_b^0(\mathcal{E}^+))$ be the restrictions of $\mathcal{A}$ to $C_b^0(\mathcal{E}^-)$ and $C_b^0(\mathcal{E}^+)$, respectively, and let $\mathcal{L}^- \in \mathcal{L}(C_b^0(\mathcal{E}^+), C_b^0(\mathcal{E}^+))$ and $\mathcal{L}^+ \in \mathcal{L}(C_b^0(\mathcal{E}^+), C_b^0(\mathcal{E}^+))$ be the corresponding restrictions of $\mathcal{L}$. Then $\mathcal{L}$ will be invertible if and only if $\mathcal{L}^-$ and $\mathcal{L}^+$ are each invertible. To invert $\mathcal{L}^-$ and $\mathcal{L}^+$ we use the following general result about the invertibility of operators on Banach spaces.

Lemma *Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|_{\mathcal{X}}$ and corresponding operator norm $\|\cdot\|_{\mathcal{L}(\mathcal{X},\mathcal{X})}$. Let G be a linear map from $\mathcal{X}$ to $\mathcal{X}$, and let $c < 1$ be a constant. Then:*

(a) *If $\|G\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \le c$, then $\operatorname{id} - G$ is invertible and*

$$\|(\operatorname{id} - G)^{-1}\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \le \frac{1}{1-c}.$$

(b) *If G is invertible and $\|G^{-1}\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \le c$, then $\operatorname{id} - G$ is invertible and*

$$\|(\operatorname{id} - G)^{-1}\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \le \frac{c}{1-c}.$$

Proof. The space of bounded linear maps from $\mathcal{X}$ to $\mathcal{X}$ is a Banach space using the operator norm. In case **(a)**, the bound on $\|G\|_{\mathcal{L}(\mathcal{X},\mathcal{X})}$, along with the Cauchy convergence criterion, implies that the series

$$\sum_{k=0}^{\infty} G^k$$

converges to a bounded linear map from $\mathcal{X}$ to $\mathcal{X}$; call it H. In fact, we see that (by the formula for the sum of a geometric series)

$$\|H\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \le \frac{1}{1-c}.$$

It is not hard to check that $H \circ (\operatorname{id} - G) = (\operatorname{id} - G) \circ H = \operatorname{id}$, so $H = (\operatorname{id} - G)^{-1}$.

In case **(b)**, we can apply the results of **(a)** with G^{-1} in place of G to deduce that $\operatorname{id} - G^{-1}$ is invertible and that

$$\|(\operatorname{id} - G^{-1})^{-1}\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \leq \frac{1}{1-c}.$$

Since $\operatorname{id} - G = -G(\operatorname{id} - G^{-1}) = -(\operatorname{id} - G^{-1})G$, it is not hard to check that $-(\operatorname{id} - G^{-1})^{-1}G^{-1}$ is the inverse of $\operatorname{id} - G$ and that

$$\| - (\operatorname{id} - G^{-1})^{-1}G^{-1}\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \leq \frac{c}{1-c}.$$

□

The first half of this lemma is useful for inverting small perturbations of the identity, while the second half is useful for inverting large perturbations of the identity. It should, therefore, not be too surprising that we will apply the first half with $G = \mathcal{A}^-$ and the second half with $G = \mathcal{A}^+$ (since A compresses things in the $\mathcal{E}^-$ direction and stretches things in the $\mathcal{E}^+$ direction).

First, consider $\mathcal{A}^-$. If $w \in C_b^0(\mathcal{E}^-)$, then

$$\begin{aligned}\|\mathcal{A}^- w\|_0 = \|A \circ w \circ A^{-1}\|_0 &= \sup_{x\in\mathbb{R}^n} \|Aw(A^{-1}x)\| = \sup_{y\in\mathbb{R}^n} \|Aw(y)\| \\ &\leq a \sup_{y\in\mathbb{R}^n} \|w(y)\| = a\|w\|_0,\end{aligned}$$

so the operator norm of $\mathcal{A}^-$ is bounded by a. Applying the first half of the lemma with $\mathcal{X} = C_b^0(\mathcal{E}^-)$, $G = \mathcal{A}^-$, and $c = a$, we find that $\mathcal{L}^-$ is invertible, and its inverse has operator norm bounded by $(1-a)^{-1}$.

Next, consider $\mathcal{A}^+$. It is not hard to see that $\mathcal{A}^+$ is invertible, and $(\mathcal{A}^+)^{-1}w = A^{-1} \circ w \circ A$. If $w \in C_b^0(\mathcal{E}^+)$, then (because the eigenvalues of the restriction of A^{-1} to $\mathcal{E}^+$ all have magnitude less than a)

$$\begin{aligned}\|(\mathcal{A}^+)^{-1} w\|_0 = \|A^{-1} \circ w \circ A\|_0 &= \sup_{x\in\mathbb{R}^n} \|A^{-1}w(Ax)\| = \sup_{y\in\mathbb{R}^n} \|A^{-1}w(y)\| \\ &\leq a \sup_{y\in\mathbb{R}^n} \|w(y)\| = a\|w\|_0,\end{aligned}$$

so the operator norm of $(\mathcal{A}^+)^{-1}$ is bounded by a. Applying the second half of the lemma with $\mathcal{X} = C_b^0(\mathcal{E}^+)$, $G = \mathcal{A}^+$, and $c = a$, we find that $\mathcal{L}^+$ is invertible, and its inverse has operator norm bounded by $a(1-a)^{-1}$.

Putting these two facts together, we see that $\mathcal{L}$ is invertible, and, in fact,

$$\mathcal{L}^{-1} = (\mathcal{L}^-)^{-1} \circ P^- + (\mathcal{L}^+)^{-1} \circ P^+.$$

If $w \in C_b^0(\mathbb{R}^n)$, then

$$\begin{aligned}
\|\mathcal{L}^{-1}w\|_0 &= \sup_{x\in\mathbb{R}^n} \|\mathcal{L}^{-1}w(x)\| = \sup_{x\in\mathbb{R}^n} \max\{\|P^-\mathcal{L}^{-1}w(x)\|, \|P^+\mathcal{L}^{-1}w(x)\|\} \\
&= \sup_{x\in\mathbb{R}^n} \max\{\|(\mathcal{L}^-)^{-1}P^-w(x)\|, \|(\mathcal{L}^+)^{-1}P^+w(x)\|\} \\
&\le \sup_{x\in\mathbb{R}^n} \max\left\{\frac{1}{1-a}\|w(x)\|, \frac{a}{1-a}\|w(x)\|\right\} = \frac{1}{1-a}\sup_{x\in\mathbb{R}^n}\|w(x)\| \\
&= \frac{1}{1-a}\|w\|_0,
\end{aligned}$$

so the operator norm of $\mathcal{L}^{-1}$ is bounded by $(1-a)^{-1}$.