Hartman-Grobman Theorem: Part 4

Lecture 27
Math 634
11/1/99

The Contraction Map

Recall that we are looking for fixed points v of the map $\Theta := \mathcal{L}^{-1} \circ \Psi$, where $\Psi(v) := g \circ (I + v) \circ A^{-1}$. We have established that $\mathcal{L}^{-1}$ is a well-defined linear map from $C_b^0(\mathbb{R}^n)$ to $C_b^0(\mathbb{R}^n)$ and that its operator norm is bounded by $(1-a)^{-1}$. This means that Θ is a well-defined (nonlinear) map from $C_b^0(\mathbb{R}^n)$ to $C_b^0(\mathbb{R}^n)$; furthermore, if $v_1, v_2 \in C_b^0(\mathbb{R}^n)$, then

$$\begin{aligned}
\|\Theta(v_1) - \Theta(v_2)\|_0 &= \|\mathcal{L}^{-1}(\Psi(v_1) - \Psi(v_2))\|_0 \le \frac{1}{1-a}\|\Psi(v_1) - \Psi(v_2)\|_0 \\
&= \frac{1}{1-a}\|g \circ (I + v_1) \circ A^{-1} - g \circ (I + v_2) \circ A^{-1}\|_0 \\
&= \frac{1}{1-a} \sup_{x \in \mathbb{R}^n} \|g(A^{-1}x + v_1(A^{-1}x)) - g(A^{-1}x + v_2(A^{-1}x))\| \\
&\le \frac{\varepsilon}{1-a} \sup_{x \in \mathbb{R}^n} \|(A^{-1}x + v_1(A^{-1}x)) - (A^{-1}x + v_2(A^{-1}x))\| \\
&= \frac{\varepsilon}{1-a} \sup_{x \in \mathbb{R}^n} \|v_1(A^{-1}x) - v_2(A^{-1}x)\| \\
&= \frac{\varepsilon}{1-a} \sup_{y \in \mathbb{R}^n} \|v_1(y) - v_2(y)\| = \frac{\varepsilon}{1-a}\|v_1 - v_2\|_0.
\end{aligned}$$

This shows that Θ is a contraction, since ε was chosen to be less than $1 - a$. By the contraction mapping theorem, we know that Θ has a unique fixed point $v \in C_b^0(\mathbb{R}^n)$; the function $h := I + v$ satisfies $F \circ h = h \circ A$, where $F := A + g$. It remains to show that h is a homeomorphism.

Injectivity

First, we show that F is injective. Suppose it weren't. Then we could choose $x_1, x_2 \in \mathbb{R}^n$ such that $x_1 \neq x_2$ but $F(x_1) = F(x_2)$. This would mean that $Ax_1 + g(x_1) = Ax_2 + g(x_2)$, so

$$\frac{\|A(x_1 - x_2)\|}{\|x_1 - x_2\|} = \frac{\|Ax_1 - Ax_2\|}{\|x_1 - x_2\|} = \frac{\|g(x_1) - g(x_2)\|}{\|x_1 - x_2\|} \le \operatorname{Lip}(g) < \varepsilon < \inf_{x \neq 0} \frac{\|Ax\|}{\|x\|},$$

which would be a contradiction.

Next, we show that h is injective. Let $x_1, x_2 \in \mathbb{R}^n$ satisfying $h(x_1) = h(x_2)$ be given. Then

$$h(Ax_1) = F(h(x_1)) = F(h(x_2)) = h(Ax_2),$$

and, by induction, we have $h(A^n x_1) = h(A^n x_2)$ for every $n \in \mathbb{N}$. Also,

$$F(h(A^{-1}x_1)) = h(AA^{-1}x_1) = h(x_1) = h(x_2) = h(AA^{-1}x_2) = F(h(A^{-1}x_2)),$$

so the injectivity of F implies that $h(A^{-1}x_1) = h(A^{-1}x_2)$; by induction, $h(A^{-n}x_1) = h(A^{-n}x_2)$ for every $n \in \mathbb{N}$. Set $z = x_1 - x_2$. Since $I = h - v$, we know that for any $n \in \mathbb{Z}$

$$\begin{aligned}\|A^n z\| &= \|A^n x_1 - A^n x_2\| = \|(h(A^n x_1) - v(A^n x_1)) - (h(A^n x_2) - v(A^n x_2))\| \\ &= \|v(A^n x_1) - v(A^n x_2)\| \leq 2\|v\|_0.\end{aligned}$$

Because of the way the norm was chosen, we then know that for $n \geq 0$

$$\|P^+ z\| \leq a^n \|A^n P^+ z\| \leq a^n \|A^n z\| \leq 2a^n \|v\|_0 \to 0,$$

as $n \uparrow \infty$, and we know that for $n \leq 0$

$$\|P^- z\| \leq a^{-n} \|A^n P^- z\| \leq a^{-n} \|A^n z\| \leq 2a^{-n} \|v\|_0 \to 0,$$

as $n \downarrow -\infty$. Hence, $z = P^- z + P^+ z = 0$, so $x_1 = x_2$.

Surjectivity

It may seem intuitive that a map like h that is a bounded perturbation of the identity is surjective. Unfortunately, there does not appear to be a way of proving this that is simultaneously elementary, short, and complete. We will therefore rely on the following topological theorem without proving it.

Theorem (Brouwer Invariance of Domain) *Every continuous injective map from $\mathbb{R}^n$ to $\mathbb{R}^n$ maps open sets to open sets.*

In particular, this theorem implies that $h(\mathbb{R}^n)$ is open. If we can show that $h(\mathbb{R}^n)$ is closed, then (since $h(\mathbb{R}^n)$ is clearly nonempty) this will mean that $h(\mathbb{R}^n) = \mathbb{R}^n$, *i.e.*, h is surjective.

So, suppose we have a sequence $(h(x_k))$ of points in $h(\mathbb{R}^n)$ that converges to a point $y \in \mathbb{R}^n$. Without loss of generality, assume that

$$\|h(x_k) - y\| \leq 1$$

for every k. This implies that $\|h(x_k)\| \leq \|y\| + 1$, which in turn implies that $\|x_k\| \leq \|y\| + \|v\|_0 + 1$. Thus, the sequence (x_k) is bounded and therefore has a subsequence (x_{k_ℓ}) converging to some point $x_0 \in \mathbb{R}^n$. By continuity of h, $(h(x_{k_\ell}))$ converges to $h(x_0)$, which means that $h(x_0) = y$. Hence, $h(\mathbb{R}^n)$ is closed.

Continuity of the Inverse

The bijectivity of h implies that h^{-1} is defined. We now show that it is continuous (which will complete the verification that h is a homeomorphism). The proof will be very similar to the proof that $h(\mathbb{R}^n)$ is closed.

Let (y_k) be a sequence in $\mathbb{R}^n$ that converges to some point $y \in \mathbb{R}^n$. Without loss of generality, assume that

$$\|y_k - y\| \leq 1$$

for every k. This implies that $\|y_k\| \leq \|y\| + 1$, which in turn implies that $\|h^{-1}(y_k)\| \leq \|y\| + \|v\|_0 + 1$. Suppose that $(h^{-1}(y_k))$ does not converge to $h^{-1}(y)$. Then the boundedness of $(h^{-1}(y_k))$ implies that some subsequence $(h^{-1}(y_{k_\ell}))$ converges to some point $x_0 \neq h^{-1}(y)$. By the continuity of h, $h(h^{-1}(y_{k_\ell})) \to h(x_0)$ as $\ell \uparrow \infty$. But $h(h^{-1}(y_{k_\ell})) = y_{k_\ell} \to y$ as $\ell \uparrow \infty$. This means that $y = h(x_0)$ or, equivalently, $x_0 = h^{-1}(y)$, contrary to assumption.