Hartman-Grobman Theorem: Part 5

Lecture 28
Math 634
11/3/99

Modifying the Vector Field

Consider the continuously differentiable autonomous differential equation

$$\dot{x} = f(x) \tag{1}$$

with an equilibrium point that, without loss of generality, is located at the origin. For x near 0, $f(x) \approx Bx$, where $B = Df(0)$. Our goal is to come up with a modification $\tilde{f}$ of f such that $\tilde{f}(x) = f(x)$ for x near 0 and $\tilde{f}(x) \approx Bx$ for *all* x. If we accomplish this goal, whatever information we obtain about the relationship between the equations

$$\dot{x} = \tilde{f}(x) \tag{2}$$

and

$$\dot{x} = Bx \tag{3}$$

will also hold between (1) and (3) for x small.

Pick $\beta : [0, \infty) \to [0, 1]$ to be a C^∞ function satisfying

$$\beta(s) = \begin{cases} 1 & \text{if } s \leq 1 \\ 0 & \text{if } s \geq 2, \end{cases}$$

and let $C = \sup_{s \in [0,\infty)} |\beta'(s)|$. Given $\varepsilon > 0$, pick $r > 0$ so small that

$$\|Df(x) - B\| < \frac{\varepsilon}{2C + 1}$$

whenever $\|x\| \leq 2r$. (We can do this since $Df(0) = B$ and Df is continuous.) Define $\tilde{f}$ by the formula

$$\tilde{f}(x) = Bx + \beta\left(\frac{\|x\|}{r}\right)(f(x) - Bx).$$

Note that $\tilde{f}$ is continuously differentiable, agrees with f for $\|x\| \leq r$, and agrees with B for $\|x\| \geq 2r$. We claim that $\tilde{f} - B$ has Lipschitz constant less than ε. Assuming, without loss of generality, that $\|x\|$ and $\|y\|$ are less than or equal to $2r$, we have (using the Mean Value Theorem)

$$\begin{aligned}
\|(\tilde{f}(x) - Bx) - (\tilde{f}(y) - By)\| \\
&= \left\|\beta\left(\frac{\|x\|}{r}\right)(f(x) - Bx) - \beta\left(\frac{\|y\|}{r}\right)(f(y) - By)\right\| \\
&\leq \beta\left(\frac{\|x\|}{r}\right)\|(f(x) - Bx) - (f(y) - By)\| \\
&\quad + \left|\beta\left(\frac{\|x\|}{r}\right) - \beta\left(\frac{\|y\|}{r}\right)\right| \|f(y) - By\| \\
&\leq \frac{\varepsilon}{2C+1}\|x - y\| + C\frac{|\|x\| - \|y\||}{r}\|y\|\frac{\varepsilon}{2C+1} \\
&\leq \varepsilon\|x - y\|.
\end{aligned}$$

Now, consider the difference between e^B and $\varphi(1, \cdot)$, where φ is the flow generated by $\tilde{f}$. Let $g(x) = \varphi(1, x) - e^B x$. Then, since $\tilde{f}(x) = B(x)$ for all large x, $g(x) = 0$ for all large x. Also, g is continuously differentiable, so $g \in C_b^1(\mathbb{R}^n)$. If we apply the variation of constants formula to (2) rewritten as

$$\dot{x} = Bx + (\tilde{f}(x) - Bx),$$

we find that

$$g(x) = \int_0^1 e^{(1-s)B}[\tilde{f}(\varphi(s, x)) - B\varphi(s, x)]\, ds,$$

so

$$\begin{aligned}
\|g(x) - g(y)\| \\
&\leq \int_0^1 \|e^{(1-s)B}\|\|(\tilde{f}(\varphi(s, x)) - B\varphi(s, x)) - (\tilde{f}(\varphi(s, y)) - B\varphi(s, y))\|\, ds \\
&\leq \varepsilon \int_0^1 \|e^{(1-s)B}\|\|\varphi(s, x) - \varphi(s, y)\|\, ds \\
&\leq \|x - y\|\varepsilon \int_0^1 \|e^{(1-s)B}\|\|e^{(\|B\|+\varepsilon)s} - 1\|\, ds,
\end{aligned}$$

by continuous dependence on initial conditions. Since

$$\varepsilon \int_0^1 \|e^{(1-s)B}\|\|e^{(\|B\|+\varepsilon)s} - 1\|\, ds \to 0$$

as $\varepsilon \downarrow 0$, we can make the Lipschitz constant of g as small as we want by making ε small (through shrinking the neighborhood of the origin on which $\tilde{f}$ and f agree).

Conjugacy for $t = 1$

If 0 is a hyperbolic equilibrium point of (1) (and therefore of (2)) then none of the eigenvalues of B are imaginary. Setting $A = e^B$, it is not hard to show that the eigenvalues of A are the exponentials of the eigenvalues of B, so none of the eigenvalues of A have modulus 1; *i.e.*, A is hyperbolic. Also, A is invertible (since $A^{-1} = e^{-B}$), so we can apply the global Hartman-Grobman Theorem for maps and conclude that there is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\varphi(1, h(x)) = h(e^B x) \tag{4}$$

for every $x \in \mathbb{R}^n$.

Conjugacy for $t \neq 1$

For the Hartman-Grobman Theorem for flows, we need

$$\varphi(t, h(x)) = h(e^{tB} x)$$

for every $x \in \mathbb{R}^n$ and *every* $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$, and consider the function $\tilde{h}$ defined by the formula

$$\tilde{h}(x) = \varphi(t, h(e^{-tB} x)). \tag{5}$$

As the composition of homeomorphisms, $\tilde{h}$ is a homeomorphism. Also, the fact that h satisfies (4) implies that

$$\begin{aligned}\varphi(1, \tilde{h}(x)) &= \varphi(1, \varphi(t, h(e^{-tB} x))) = \varphi(t, \varphi(1, h(e^{-tB} x))) = \varphi(t, h(e^B e^{-tB} x)) \\ &= \varphi(t, h(e^{-tB} e^B x))) = \tilde{h}(e^B x),\end{aligned}$$

so (4) holds if h is replaced by $\tilde{h}$.

Now,

$$\begin{aligned}\tilde{h} - I &= \varphi(t, \cdot) \circ h \circ e^{-tB} - I \\ &= (\varphi(t, \cdot) - e^{tB}) \circ h \circ e^{-tB} + e^{tB} \circ (h - I) \circ e^{-tB} =: v_1 + v_2.\end{aligned}$$

The fact that $\varphi(t, x)$ and $e^{tB} x$ agree for large x implies that $\varphi(t, \cdot) - e^{tB}$ is bounded, so v_1 is bounded, as well. The fact that $h - I$ is bounded implies that v_2 is bounded. Hence, $\tilde{h} - I$ is bounded.

The uniqueness part of the global Hartman-Grobman Theorem for maps now implies that h and $\tilde{h}$ must be the same function. Using this fact and substituting $y = e^{-tB}x$ in (5) yields

$$h(e^{tB}y) = \varphi(t, h(y))$$

for every $y \in \mathbb{R}^n$ and every $t \in \mathbb{R}^n$. This means that the flows generated by (3) and (2) are globally topologically conjugate, and the flows generated by (3) and (1) are locally topologically conjugate.