Constructing Conjugacies

Lecture 29
Math 634
11/5/99

The Hartman-Grobman Theorem gives us conditions under which a conjugacy between certain maps or between certain flows may exist. Some limitations of the theorem are:

- The conditions it gives are sufficient, but certainly not necessary, for a conjugacy to exist.
- It doesn't give a simple way to construct a conjugacy (in closed form, at least).
- It doesn't indicate how smooth the conjugacy might be.

These shortcomings can be addressed in a number of different ways, but we won't really go into those here. We will, however, consider some aspects of conjugacies.

Differentiable Conjugacies of Flows

Consider the autonomous differential equations

$$\dot{x} = f(x) \tag{1}$$

and

$$\dot{x} = g(x), \tag{2}$$

generating, respectively, the flows φ and ψ. Recall that the conjugacy equation for φ and ψ is

$$\varphi(t, h(x)) = h(\psi(t, x)) \tag{3}$$

for every x and t. Not only is (3) somewhat complicated, it appears to require you to solve (1) and (2) before you can look for a conjugacy h. Suppose, however, that h is a differentiable conjugacy. Then, we can differentiate both sides of (3) with respect to t to get

$$f(\varphi(t, h(x))) = Dh(\psi(t, x))g(\psi(t, x)). \tag{4}$$

Substituting (3) into the right-hand side of (4) and replacing $\psi(t,x)$ by x, we get the equivalent equation

$$f(h(x)) = Dh(x)g(x). \tag{5}$$

Note that (5) involves the functions appearing in the differential equations, rather than the formulas for the solutions of those equations. Note, also, that (5) is the same equation you would get if you took a solution x of (2) and required the function $h \circ x$ to satisfy (1).

An Example for Flows

As the simplest nontrivial example, let $a, b \in \mathbb{R}$ be *distinct* constants and consider the equations

$$\dot{x} = ax \tag{6}$$

and

$$\dot{x} = bx \tag{7}$$

for $x \in \mathbb{R}$. Under what conditions on a and b does their exist a topological conjugacy h taking solutions of (7) to solutions of (6)? Equation (5) tells us that if h is differentiable then

$$ah(x) = h'(x)bx. \tag{8}$$

If $b \neq 0$, then separating variables in (8) implies that on intervals avoiding the origin h must be given by the formula

$$h(x) = C|x|^{a/b} \tag{9}$$

for some constant C. Clearly, (9) does not define a topological conjugacy for a single constant C, because it fails to be injective on $\mathbb{R}$; however, the formula

$$h(x) = \begin{cases} x|x|^{a/b-1} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases} \tag{10}$$

which is obtained from (9) by taking $C = 1$ for positive x and $C = -1$ for negative x, defines a homeomorphism if $ab > 0$. Even though the function defined in (10) may fail to be differentiable at 0, substitution of it into

$$e^{ta}h(x) = h(e^{tb}x), \tag{11}$$

which is (3) for this example, shows that it does, in fact, define a topological conjugacy when $ab > 0$. (Note that in no case is this a C^1-conjugacy, since either $h'(0)$ or $(h^{-1})'(0)$ does not exist.)

Now, suppose that $ab \leq 0$. Does a topological (possibly nondifferentiable) conjugacy exist? If $ab = 0$, then (11) implies that h is constant, which violates injectivity, so suppose that $ab < 0$. In this case, substituting $x = 0$ and $t = 1$ into (11) implies that $h(0) = 0$. Fixing $x \neq 0$ and letting $t \operatorname{sgn} b \downarrow -\infty$ in (11), we see that the continuity of h implies that $h(x) = 0$, also, which again violates injectivity.

Summarizing, for $a \neq b$ there is a topological conjugacy of (6) and (7) if and only if $ab > 0$, and these are not C^1-conjugacies.

An Example for Maps

Let's try a similar analysis for maps. Let $a, b \in \mathbb{R}$ be distinct constants, and consider the maps $F(x) = ax$ and $G(x) = bx$ (for $x \in \mathbb{R}$). For what (a, b)-combinations does there exist a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that

$$F(h(x)) = h(G(x)) \tag{12}$$

for every $x \in \mathbb{R}$? Can h and h^{-1} be chosen to be differentiable?

For these specific maps, the general equation (12) becomes

$$ah(x) = h(bx). \tag{13}$$

If $a = 0$ or $b = 0$ or $a = 1$ or $b = 1$, then injectivity is immediately violated. Note that, by induction, (13) gives

$$a^n h(x) = h(b^n x) \tag{14}$$

for every $n \in \mathbb{Z}$. In particular, $a^2 h(x) = h(b^2 x)$, so the cases when $a = -1$ or $b = -1$ cause the same problems as when $a = 1$ or $b = 1$.

So, from now on, assume that $a, b \notin \{-1, 0, 1\}$. Observe that:

- Setting $x = 0$ in (13) yields $h(0) = 0$.
- If $|b| < 1$, then fixing $x \neq 0$ in (14) and letting $n \uparrow \infty$, we have $|a| < 1$.
- If $|b| > 1$, we can, similarly, let $n \downarrow -\infty$ to conclude that $|a| > 1$.

- If $b > 0$ and $a < 0$, then (13) implies that $h(1)$ and $h(b)$ have opposite signs even though 1 and b have the same sign; consequently, the Intermediate Value Theorem yields a contradiction to injectivity.
- If $b < 0$ and $a > 0$, then (13) gives a similar contradiction.

Thus, the only cases where we could possibly have conjugacy is if a and b are both in the same component of

$$(-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty).$$

When this condition is met, experimentation (or experience) suggests trying h of the form $h(x) = x|x|^{p-1}$ for some constant $p > 0$ (with $h(0) = 0$). This is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$, and plugging it into (13) shows that it provides a conjugacy if $a = b|b|^{p-1}$ or, in other words, if

$$p = \frac{\log|a|}{\log|b|}.$$

Since $a \neq b$, either h or h^{-1} fails to be differentiable at 0. Is there some other formula that provides a C^1-conjugacy? No, because if there were we could differentiate both sides of (13) with respect to x and evaluate at $x = 0$ to get $h'(0) = 0$, which would mean that $(h^{-1})'(0)$ is undefined.

Exercise 16 Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x/2 \\ 2y + x^2 \end{bmatrix},$$

and let $A = DF(0)$.

(a) Show that the maps F and A are topologically conjugate.

(b) Show that the flows generated by the differential equations

$$\dot{z} = F(z)$$

and

$$\dot{z} = Az$$

are topologically conjugate.

(Hint: Try quadratic conjugacy functions.)