Uniqueness of Solutions

Lecture 3
Math 634
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Uniqueness

If more than continuity of f is assumed, it may be possible to prove that

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = a, \end{cases} \tag{1}$$

has a *unique* solution. In particular Lipschitz continuity of $f(t, \cdot)$ is sufficient. (Recall that $g : \mathrm{dom}(g) \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is *Lipschitz continuous* if there exists a constant $L > 0$ such that $|g(x_1) - g(x_2)| \leq L|x_1 - x_2|$ for every $x_1, x_2 \in \mathrm{dom}(g)$; L is called a *Lipschitz constant* for g.)

One approach to uniqueness is developed in the following exercise, which uses what are know as *Gronwall inequalities*.

Exercise 3 Assume that the conditions of the Cauchy-Peano Theorem hold and that, in addition, $f(t,\cdot)$ is Lipschitz continuous with Lipschitz constant L for every t. Show that the solution of (1) is unique on $[t_0, t_0+b]$ by completing the following steps. (The solution can similarly be shown to be unique on $[t_0-b, t_0]$, but we won't bother doing that here.)

(a) If x_1 and x_2 are each solutions of (1) on $[t_0, t_0+b]$ and $U : [t_0, t_0+b] \to \mathbb{R}$ is defined by $U(t) := |x_1(t) - x_2(t)|$, show that

$$U(t) \le L \int_{t_0}^{t} U(s)\, ds \tag{2}$$

for every $t \in [t_0, t_0+b]$.

(b) Pick $\varepsilon > 0$ and let

$$V(t) := \varepsilon + L \int_{t_0}^{t} U(s)\, ds.$$

Show that

$$V'(t) \le LV(t) \tag{3}$$

for every $t \in [t_0, t_0+b]$, and that $V(t_0) = \varepsilon$.

(c) Dividing both sides of (3) by $V(t)$ and integrating from $t = t_0$ to $t = T$, show that $V(T) \le \varepsilon \exp[L(T - t_0)]$.

(d) By using (2) and letting $\varepsilon \downarrow 0$, show that $U(T) = 0$ for all $T \in [t_0, t_0+b]$, so $x_1 = x_2$.

We will prove an existence-uniqueness theorem that combines the results of the Cauchy-Peano Theorem and Exercise 3. The reason for this apparently redundant effort is that the concepts and techniques introduced in this proof will be useful throughout this course.

First, we review some definitions and results pertaining to metric spaces.

Definition A *metric space* is a set $\mathcal{X}$ together with a function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ satisfying:

1. $d(x, y) \geq 0$ for every $x, y \in \mathcal{X}$, with equality if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for every $x, y \in \mathcal{X}$;
3. $d(x, y) + d(y, z) \geq d(x, z)$ for every $x, y, z \in \mathcal{X}$.

Definition A *normed vector space* is a vector space together with a function $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ satisfying:

1. $\|x\| \geq 0$ for every $x \in \mathcal{X}$, with equality if and only if $x = 0$;
2. $\|\lambda x\| = |\lambda|\|x\|$ for every $x \in \mathcal{X}$ and every scalar λ;
3. $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in \mathcal{X}$.

Every normed vector space is a metric space with metric $d(x, y) = \|x - y\|$.

Definition An *inner product space* is a vector space together with a function $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ satisfying:

1. $\langle x, x \rangle \geq 0$ for every $x \in \mathcal{X}$, with equality if and only if $x = 0$;
2. $\langle x, y \rangle = \langle y, x \rangle$ for every $x, y \in \mathcal{X}$;
3. $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$ for every $x, y, z \in \mathcal{X}$ and all scalars μ, λ.

Every inner product space is a normed vector space with norm $\|x\| = \sqrt{\langle x, x \rangle}$.

Definition A sequence (x_n) in a metric space $\mathcal{X}$ is said to be (a) *Cauchy* (sequence) if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ whenever $m, n \geq N$.

Definition A sequence (x_n) in a metric space $\mathcal{X}$ *converges* to x if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq N$.

Definition A metric space is said to be *complete* if every Cauchy sequence in $\mathcal{X}$ converges (in $\mathcal{X}$). A complete normed vector space is called a *Banach space*. A complete inner product space is called a *Hilbert space*.

Definition A function $f : \mathcal{X} \to \mathcal{Y}$ from a metric space to a metric space is said to be *Lipschitz continuous* if there exists $L \in \mathbb{R}$ such that $d(f(u), f(v)) \leq Ld(u, v)$ for every $u, v \in \mathcal{X}$. We call L a *Lipschitz constant*, and write $\operatorname{Lip}(f)$ for the smallest Lipschitz constant that works.

Definition A *contraction* is a Lipschitz continuous function from a metric space to itself that has Lipschitz constant less than 1.

Definition A *fixed point* of a function $T : \mathcal{X} \to \mathcal{X}$ is a point $x \in \mathcal{X}$ such that $T(x) = x$.

Theorem (Contraction Mapping Principle) *If $\mathcal{X}$ is a complete metric space and $T : \mathcal{X} \to \mathcal{X}$ is a contraction, then T has a unique fixed point in $\mathcal{X}$.*

Proof. Pick $\lambda < 1$ such that $d(T(x), T(y)) \leq \lambda d(x, y)$ for every $x, y \in \mathcal{X}$. Pick any point $x_0 \in \mathcal{X}$. Define a sequence (x_k) by the recursive formula

$$x_{k+1} = T(x_k). \tag{4}$$

If $k \geq \ell \geq N$, then

$$\begin{aligned} d(x_k, x_\ell) &\leq d(x_k, x_{k-1}) + d(x_{k-1}, x_{k-2}) + \cdots + d(x_{\ell+1}, x_\ell) \\ &\leq \lambda d(x_{k-1}, x_{k-2}) + \lambda d(x_{k-2}, x_{k-3}) + \cdots + \lambda d(x_\ell, x_{\ell-1}) \\ &\vdots \\ &\leq \lambda^{k-1} d(x_1, x_0) + \lambda^{k-2} d(x_1, x_0) + \cdots + \lambda^\ell d(x_1, x_0) \\ &\leq \frac{\lambda^N}{1-\lambda} d(x_1, x_0). \end{aligned}$$

Hence, (x_k) is a Cauchy sequence. Since $\mathcal{X}$ is complete, (x_k) converges to some $x \in \mathcal{X}$. By letting $k \uparrow \infty$ in (4) and using the continuity of T, we see that $x = T(x)$, so x is a fixed point of T.

If there were another fixed point y of T, then $d(x, y) = d(T(x), T(y)) \leq \lambda d(x, y)$, so $d(x, y) = 0$, which means $x = y$. This shows uniqueness of the fixed point. □