Smooth Conjugacies

Lecture 30
Math 634
11/8/99

The examples we looked at last time showing that topological conjugacies often cannot be chosen to be differentiable all involved two maps or vector fields with different linearizations at the origin. What about when, as in the Hartman-Grobman Theorem, we are looking for a conjugacy between a map (or flow) and its linearization? An example of Hartman shows that the conjugacy cannot always be chosen to be C^1.

Hartman's Example

Consider the system

$$\begin{cases} \dot{x} = \alpha x \\ \dot{y} = (\alpha - \gamma)y + \varepsilon xz \\ \dot{z} = -\gamma z, \end{cases}$$

where $\alpha > \gamma > 0$ and $\varepsilon \neq 0$. We will not cut off this vector field but will instead confine our attention to x, y, z small. A calculation shows that the time-1 map $F = \varphi(1, \cdot)$ of this system is given by

$$F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} ax \\ ac(y + \varepsilon xz) \\ cz \end{bmatrix},$$

where $a = e^{\alpha}$ and $c = e^{-\gamma}$. Note that $a > ac > 1 > c > 0$. The time-1 map B of the linearization of the differential equation is given by

$$B\begin{bmatrix} x \\ y \\ y \end{bmatrix} = \begin{bmatrix} ax \\ acy \\ cz \end{bmatrix}.$$

A local conjugacy $H = (f, g, h)$ of B with F must satisfy

$$\begin{aligned} af(x,y,z) &= f(ax, acy, cz) \\ ac[g(x,y,z) + \varepsilon f(x,y,z)h(x,y,z)] &= g(ax, acy, cz) \\ ch(x,y,z) &= h(ax, acy, cz) \end{aligned}$$

for every x, y, z near 0. Writing $k(x, z)$ for $k(x, 0, z)$, where $k \in \{f, g, h\}$, we have

$$\begin{aligned} af(x,z) &= f(ax, cz) && (1) \\ ac[g(x,z) + \varepsilon f(x,z)h(x,z)] &= g(ax, cz) && (2) \\ ch(x,z) &= h(ax, cz) && (3) \end{aligned}$$

for every x, z near 0.

Before proceeding further, we state and prove a lemma.

Lemma *Suppose that j is a continuous real-valued function of a real variable, defined on an open interval $\mathcal{U}$ centered at the origin. Suppose that there are constants $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha j(u) = j(\beta u) \tag{4}$$

whenever $u, \beta u \in \mathcal{U}$. Then if $|\beta| < 1 < |\alpha|$ or $|\alpha| < 1 < |\beta|$, $j(u) = 0$ for every $u \in \mathcal{U}$.

Proof. If $|\beta| < 1 < |\alpha|$, fix $u \in \mathcal{U}$ and apply (4) inductively to get

$$\alpha^n j(u) = j(\beta^n u) \tag{5}$$

for every $n \in \mathbb{N}$. Letting $n \uparrow \infty$ in (5), we see that $j(u)$ must be zero. If $|\alpha| < 1 < |\beta|$, substitute $v = \beta u$ into (4) to get

$$\alpha j(\beta^{-1} v) = j(v) \tag{6}$$

for every $v, \beta^{-1} v \in \mathcal{U}$. Fix $v \in \mathcal{U}$, and iterate (6) to get

$$\alpha^n j(\beta^{-n} v) = j(v) \tag{7}$$

for every $n \in \mathbb{N}$. Letting $n \uparrow \mathbb{N}$ in (7), we get $j(v) = 0$. □

Setting $x = 0$ in (1) and applying the Lemma gives

$$f(0, z) = 0 \tag{8}$$

for every z near zero. Setting $z = 0$ in (3) and applying the Lemma gives

$$h(x, 0) = 0 \tag{9}$$

for every x near zero. Setting $x = 0$ in (2), using (8), and applying the Lemma gives

$$g(0, z) = 0 \tag{10}$$

for every z near zero. If we set $z = 0$ in (2), use (9), and then differentiate both sides with respect to x, we get $cg_x(x, 0) = g_x(ax, 0)$; applying the Lemma yields

$$g_x(x, 0) = 0 \tag{11}$$

for every x near zero. Setting $z = 0$ in (10) and using (11), we get

$$g(x, 0) = 0 \tag{12}$$

for every x near zero.

Now, using (2) and mathematical induction, it can be verified that

$$a^n c^n [g(x, z) + n\varepsilon f(x, z)h(x, z)] = g(a^n x, c^n z) \tag{13}$$

for every $n \in \mathbb{N}$. Similarly, mathematical induction applied to (1) gives

$$f(x, z) = a^{-n} f(a^n x, c^n z) \tag{14}$$

for every $n \in \mathbb{N}$. If we substitute (14) into (13), divide through by c^{-n}, and replace x by $a^{-n}x$ we get

$$a^n g(a^{-n}x, z) + n\varepsilon f(x, c^n z)h(a^{-n}x, z) = c^{-n} g(x, c^n z) \tag{15}$$

for every $n \in \mathbb{N}$.

The existence of $g_x(0, z)$ and $g_z(0, x)$ along with equations (10) and (12) imply that $a^n g(a^{-n}x, z)$ and $c^{-n} g(x, c^n z)$ stay bounded as $n \uparrow \infty$. Using this fact, and letting $n \uparrow \infty$ in (15), we get

$$f(x, 0)h(0, z) = 0,$$

so $f(x, 0) = 0$ or $h(0, z) = 0$. If $f(x, 0) = 0$, then, in combination with (9) and (12), this tells us that H is not injective in a neighborhood of the origin. Similarly, if $h(0, z) = 0$ then, in combination with (8) and (10), this implies a violation of injectivity, as well.

Poincaré's Linearization Theorem

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is analytic and satisfies $f(0) = 0$. It is possible to establish conditions under which an *analytic* change of variables will turn the nonlinear equation

$$\dot{x} = f(x) \tag{16}$$

into its linearization

$$\dot{u} = Df(0)u. \tag{17}$$

Definition Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $Df(0)$, listed according to multiplicity. We say that $Df(0)$ is *resonant* if there are nonnegative integers $m_1, m_2, \dots, m_n$ and a number $s \in \{1, 2, \dots, n\}$ such that

$$\sum_{k=1}^{n} m_k \geq 2$$

and

$$\lambda_s = \sum_{k=1}^{n} m_k \lambda_k.$$

If $Df(0)$ is not resonant, we say that it is *nonresonant*.

Note that in Hartman's example there is resonance. As we will see in Math 635, nonresonance permits us to make changes of variable that remove nonlinear terms up to any specified order in the right-hand side of the differential equation. In order to be able to guarantee that *all* nonlinear terms may be removed, some extra condition beyond nonresonance is required.

Definition We say that $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{C}^n$ satisfy a *Siegel condition* if there are constants $C > 0$ and $\nu > 1$ such that

$$\left| \lambda_s - \sum_{k=1}^{n} m_k \lambda_k \right| \geq \frac{C}{\left(\sum_{k=1}^{n} m_k\right)^{\nu}}$$

for all nonnegative integers $m_1, m_2, \dots, m_n$ satisfying

$$\sum_{k=1}^{n} m_k \geq 2.$$

Theorem (Poincaré's Linearization Theorem) *Suppose that* f *is analytic, and that all the eigenvalues of* $Df(0)$ *are nonresonant and either all lie in the open left half-plane, all lie in the open right half-plane, or satisfy a Siegel condition. Then there is a change of variables* $u = g(x)$ *that is analytic near* 0 *and that turns* (16) *into* (17) *near* 0*.*