Stable Manifold Theorem: Part 2

Lecture 32
Math 634
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Statements

Given a normed vector space $\mathcal{X}$ and a positive number r, we let $\mathcal{X}(r)$ stand for the closed ball of radius r centered at 0 in $\mathcal{X}$.

The first theorem refers to the differential equation

$$\dot{x} = f(x). \tag{1}$$

Theorem (Stable Manifold Theorem) *Suppose that Ω is an open neighborhood of the origin in $\mathbb{R}^n$, and $f : \Omega \to \mathbb{R}^n$ is a C^k function ($k \geq 1$) such that 0 is a hyperbolic equilibrium point of (1). Let $\mathcal{E}^s \oplus \mathcal{E}^u$ be the decomposition of $\mathbb{R}^n$ corresponding to the matrix $Df(0)$. Then there is a norm $\|\cdot\|$ on $\mathbb{R}^n$, a number $r > 0$, and a C^kfunction $h : \mathcal{E}^s(r) \to \mathcal{E}^u(r)$ such that $h(0) = Dh(0) = 0$ and such that the local stable manifold $W^s_{\text{loc}}(0)$ of 0 relative to $\mathcal{B}(r) := \mathcal{E}^s(r) \oplus \mathcal{E}^u(r)$ is the set*

$$\left\{v_s + h(v_s) \;\middle|\; v_s \in \mathcal{E}^s(r)\right\}.$$

Moreover, there is a constant $c > 0$ such that

$$W^s_{\text{loc}}(0) = \left\{v \in \mathcal{B}(r) \;\middle|\; \gamma^+(v) \subset \mathcal{B}(r) \text{ and } \lim_{t\uparrow\infty} e^{ct}\varphi(t,v) = 0\right\}.$$

Two immediate and obvious corollaries, which we will not state explicitly, describe the stable manifolds of other equilibrium points (via translation) and describe unstable manifolds (by time reversal).

We will actually prove this theorem by first proving an analogous theorem for maps (much as we did with the Hartman-Grobman Theorem). Given a neighborhood $\mathcal{U}$ of a fixed point p of a map F, we can define the local stable manifold of p (relative to $\mathcal{U}$) as

$$W^s_{\text{loc}}(p) := \left\{x \in \mathcal{U} \;\middle|\; F^j(x) \in \mathcal{U} \text{ for every } j \in \mathbb{N} \text{ and } \lim_{j\uparrow\infty} F^j(x) = p\right\}.$$

Theorem (Stable Manifold Theorem for Maps) *Suppose that Ω is an open neighborhood of the origin in $\mathbb{R}^n$, and $F : \Omega \to \Omega$ is an invertible C^k function*

($k \geq 1$) for which $F(0) = 0$ and the matrix $DF(0)$ is hyperbolic and invertible. Let $\mathcal{E}^s \oplus \mathcal{E}^u (= \mathcal{E}^- \oplus \mathcal{E}^+)$ be the decomposition of $\mathbb{R}^n$ corresponding to the matrix $DF(0)$. Then there is a norm $\|\cdot\|$ on $\mathbb{R}^n$, a number $r > 0$, a number $\tilde{\mu} \in (0,1)$, and a C^kfunction $h : \mathcal{E}^s(r) \to \mathcal{E}^u(r)$ such that $h(0) = Dh(0) = 0$ and such that the local stable manifold $W^s_{\mathrm{loc}}(0)$ of 0 relative to $\mathcal{B}(r) := \mathcal{E}^s(r) \oplus \mathcal{E}^u(r)$ satisfies

$$\begin{aligned} W^s_{\mathrm{loc}}(0) &= \left\{ v_s + h(v_s) \;\middle|\; v_s \in \mathcal{E}^s(r) \right\} \\ &= \left\{ v \in \mathcal{B}(r) \;\middle|\; F^j(v) \in \mathcal{B}(r) \text{ for every } j \in \mathbb{N} \right\} \\ &= \left\{ v \in \mathcal{B}(r) \;\middle|\; F^j(v) \in \mathcal{B}(r) \text{ and } \|F^j(v)\| \leq \tilde{\mu}^j \|v\| \text{ for every } j \in \mathbb{N} \right\}. \end{aligned}$$

Preliminaries

The proof of the Stable Manifold Theorem for Maps will be broken up into a series of lemmas. Before stating and proving those lemmas, we need to lay a foundation by introducing some terminology and notation and by choosing some constants.

We know that $F(0) = 0$ and $DF(0)$ is hyperbolic. Then $\mathbb{R}^n = \mathcal{E}^s \oplus \mathcal{E}^u$, π_s and π_u are the corresponding projection operators, $\mathcal{E}^s$ and $\mathcal{E}^u$ are invariant under $DF(0)$, and there are constants $\mu < 1$ and $\lambda > 1$ such that all of the eigenvalues of $DF(0)|_{\mathcal{E}^s}$ have magnitude less than μ and all of the eigenvalues of $DF(0)|_{\mathcal{E}^u}$ have magnitude greater than λ.

When we deal with a matrix representation of $DF(q)$, it will be with respect to a basis that consists of a basis for $\mathcal{E}^s$ followed by a basis for $\mathcal{E}^u$. Thus,

$$DF(q) = \left[\begin{array}{c|c} A_{ss}(q) & A_{su}(q) \\ \hline A_{us}(q) & A_{uu}(q) \end{array} \right],$$

where, for example, $A_{su}(q)$ is a matrix representation of $\pi_s DF(q)|_{\mathcal{E}^u}$ in terms of the basis for $\mathcal{E}^u$ and the basis for $\mathcal{E}^s$. Note that, by invariance, $A_{su}(0) = A_{us}(0) = 0$. Furthermore, we can pick our basis vectors so that, with $\|\cdot\|$ being the corresponding Euclidean norm of a vector in $\mathcal{E}^s$ or in $\mathcal{E}^u$,

$$\|A_{ss}(0)\| := \sup_{v_s \neq 0} \frac{\|A_{ss}(0)v_s\|}{\|v_s\|} < \mu$$

and

$$m(A_{uu}(0)) := \inf_{v_u \neq 0} \frac{\|A_{uu}(0)v_u\|}{\|v_u\|} > \lambda.$$

(The functional $m(\cdot)$ defined implicitly in the last formula is sometimes called the *minimum norm* even though it is not a norm.) For a vector in $v \in \mathbb{R}^n$, let $\|v\| = \max\{\|\pi_s v\|, \|\pi_u v\|\}$. This will be the norm on $\mathbb{R}^n$ that will be used throughout the proof. Note that $\mathcal{B}(r) := \mathcal{E}^s(r) \oplus \mathcal{E}^u(r)$ is the closed ball of radius r in $\mathbb{R}^n$ by this norm.

Next, we choose r. Fix $\alpha > 0$. Pick $\varepsilon > 0$ small enough that

$$\mu + \varepsilon\alpha + \varepsilon < 1 < \lambda - \varepsilon/\alpha - 2\varepsilon.$$

Pick $r > 0$ small enough that if $q \in \mathcal{B}(r)$ then

$$\begin{aligned} \|A_{ss}(q)\| &< \mu, \\ m(A_{uu}(q)) &> \lambda, \\ \|A_{su}(q)\| &< \varepsilon, \\ \|A_{us}(q)\| &< \varepsilon, \\ \|DF(q) - DF(0)\| &< \varepsilon, \end{aligned}$$

and $DF(q)$ is invertible. (We can do this since F is C^1, so $DF(\cdot)$ is continuous.)

Now, define

$$W_r^s := \bigcap_{j=0}^{\infty} F^{-j}(\mathcal{B}(r)),$$

and note that W_r^s is the set of all points in $\mathcal{B}(r)$ that produce forward semiorbits (under the discrete dynamical system generated by F) that stay in $\mathcal{B}(r)$ for all forward iterates. By definition, $W_{\text{loc}}^s(0) \subseteq W_r^s$; we will show that these two sets are, in fact, equal.

Two other types of geometric sets play vital roles in the proof: *cones* and *disks*. The cones are of two types: *stable* and *unstable*. The stable cone (of "slope" α) is

$$C^s(\alpha) := \left\{ v \in \mathbb{R}^n \;\middle|\; \|\pi_u v\| \le \alpha \|\pi_s v\| \right\},$$

and the unstable cone (of "slope" α) is

$$C^u(\alpha) := \left\{ v \in \mathcal{R}^n \;\middle|\; \|\pi_u v\| \ge \alpha \|\pi_s v\| \right\}.$$

An *unstable disk* is a set of the form

$$\left\{ v_u + \psi(v_u) \;\middle|\; v_u \in \mathcal{E}^u(r) \right\}$$

for some Lipschitz continuous function $\psi : \mathcal{E}^u(r) \to \mathcal{E}^s(r)$ with Lipschitz constant (less than or equal to) α^{-1}.