

Stable Manifold Theorem: Part 3

Lecture 33

Math 634

11/15/99

The Action of $DF(p)$ on the Unstable Cone

The first lemma shows that if the derivative of the map is applied to a point in the unstable cone, the image is also in the unstable cone.

Lemma (Linear Invariance of the Unstable Cone) *If $p \in \mathcal{B}(r)$, then*

$$DF(p)C^u(\alpha) \subseteq C^u(\alpha).$$

Proof. Let $p \in \mathcal{B}(r)$ and $v \in C^u(\alpha)$. Then, if we let $v_s = \pi_s v$ and $v_u = \pi_u v$, we have $\|v_u\| \geq \alpha \|v_s\|$, so

$$\begin{aligned} \|\pi_u DF(p)v\| &= \|A_{us}(p)v_s + A_{uu}(p)v_u\| \geq \|A_{uu}(p)v_u\| - \|A_{us}(p)v_s\| \\ &\geq m(A_{uu}(p))\|v_u\| - \|A_{us}(p)\|\|v_s\| \geq \lambda\|v_u\| - \varepsilon\|v_s\| \\ &\geq (\lambda - \varepsilon/\alpha)\|v_u\|, \end{aligned}$$

and

$$\begin{aligned} \|\pi_s DF(p)v\| &= \|A_{ss}(p)v_s + A_{su}(p)v_u\| \leq \|A_{ss}(p)v_s\| + \|A_{su}(p)v_u\| \\ &\leq \|A_{ss}(p)\|\|v_s\| + \|A_{su}(p)\|\|v_u\| \leq \mu\|v_s\| + \varepsilon\|v_u\| \\ &\leq (\mu/\alpha + \varepsilon)\|v_u\|. \end{aligned}$$

Since $\lambda - \varepsilon/\alpha \geq \alpha(\mu/\alpha + \varepsilon)$,

$$\|\pi_u DF(p)v\| \geq \alpha \|\pi_s DF(p)v\|,$$

so $DF(p)v \in C^u(\alpha)$. □

The Action of F on Moving Unstable Cones

The main part of the second lemma is that moving unstable cones are positively invariant. More precisely, if two points are in $\mathcal{B}(r)$ and one of the two points is in a translate of the unstable cone that is centered at the second point, then their images under F satisfy the same relationship. The lemma also provides estimates on the rates at which the stable and unstable parts of the difference between the two points contract or expand, respectively.

In this lemma (and later) we use the convention that if \mathcal{X} and \mathcal{Y} are subsets of a vector space, then

$$\mathcal{X} + \mathcal{Y} := \{x + y \mid x \in \mathcal{X} \text{ and } y \in \mathcal{Y}\}.$$

Lemma (Moving Unstable Cones) *If $p, q \in \mathcal{B}(r)$ and $q \in \{p\} + C^u(\alpha)$, then:*

- (a) $\|\pi_s(F(q) - F(p))\| \leq (\mu/\alpha + \varepsilon)\|\pi_u(q - p)\|;$
- (b) $\|\pi_u(F(q) - F(p))\| \geq (\lambda - \varepsilon/\alpha - \varepsilon)\|\pi_u(q - p)\|;$
- (c) $F(q) \in \{F(p)\} + C^u(\alpha).$

Proof. We will write differences as integrals (using the Fundamental Theorem of Calculus) and use our estimates on $DF(v)$, for $v \in \mathcal{B}(r)$, to estimate these integrals.

Since $\mathcal{B}(r)$ is convex,

$$\begin{aligned} \|\pi_s(F(q) - F(p))\| &= \left\| \int_0^1 \frac{d}{dt} \pi_s F(tq + (1-t)p) dt \right\| \\ &= \left\| \int_0^1 \pi_s DF(tq + (1-t)p)(q - p) dt \right\| \\ &= \left\| \int_0^1 [A_{ss}(tq + (1-t)p)\pi_s(q - p) + A_{su}(tq + (1-t)p)\pi_u(q - p)] dt \right\| \\ &\leq \int_0^1 [\|A_{ss}(tq + (1-t)p)\|\|\pi_s(q - p)\| + \|A_{su}(tq + (1-t)p)\|\|\pi_u(q - p)\|\] dt \\ &\leq \int_0^1 [\mu\|\pi_s(q - p)\| + \varepsilon\|\pi_u(q - p)\|\] dt \leq (\mu/\alpha + \varepsilon)\|\pi_u(q - p)\|. \end{aligned}$$

This gives **(a)**.

Similarly,

$$\begin{aligned}
& \|\pi_u(F(q) - F(p))\| \\
&= \left\| \int_0^1 [A_{us}(tq + (1-t)p)\pi_s(q-p) + A_{uu}(tq + (1-t)p)\pi_u(q-p)] dt \right\| \\
&\geq \left\| \int_0^1 A_{uu}(0)\pi_u(q-p) dt \right\| - \left\| \int_0^1 [A_{us}(tq + (1-t)p)\pi_s(q-p) dt] \right\| \\
&\quad - \left\| \int_0^1 (A_{uu}(tq + (1-t)p) - A_{uu}(0))\pi_u(q-p) dt \right\| \\
&\geq m(A_{uu}(0))\|\pi_u(q-p)\| - \int_0^1 \|A_{us}(tq + (1-t)p)\| \|\pi_s(q-p)\| dt \\
&\quad - \int_0^1 \|A_{uu}(tq + (1-t)p) - A_{uu}(0)\| \|\pi_u(q-p)\| dt \\
&\geq \lambda\|\pi_u(q-p)\| - \varepsilon\|\pi_s(q-p)\| - \varepsilon\|\pi_u(q-p)\| \geq (\lambda - \varepsilon/\alpha - \varepsilon)\|\pi_u(q-p)\|.
\end{aligned}$$

This gives **(b)**.

From **(a)**, **(b)**, and the choice of ε , we have

$$\begin{aligned}
\|\pi_u(F(q) - F(p))\| &\geq (\lambda - \varepsilon/\alpha - \varepsilon)\|\pi_u(q-p)\| \geq (\mu + \varepsilon\alpha)\|\pi_u(q-p)\| \\
&\geq \alpha\|\pi_s(F(q) - F(p))\|,
\end{aligned}$$

so $F(q) - F(p) \in C^u(\alpha)$, which means that **(c)** holds. □