Stable Manifold Theorem: Part 4

Lecture 34
Math 634
11/17/99

Stretching of C^1 Unstable Disks

The next lemma shows that if F is applied to a C^1 unstable disk (*i.e.*, an unstable disk that is the graph of a C^1 function), then part of the image gets stretched out of $\mathcal{B}(r)$, but the part that remains in is again a C^1 unstable disk.

Lemma (Unstable Disks) *Let $\mathcal{D}_0$ be a C^1 unstable disk, and recursively define*

$$\mathcal{D}_j = F(D_{j-1}) \cap \mathcal{B}(r)$$

for each $j \in \mathbb{N}$. Then each $\mathcal{D}_j$ is a C^1 unstable disk, and

$$\operatorname{diam}\left(\pi_u \bigcap_{i=0}^{j} F^{-i}(\mathcal{D}_i)\right) \leq 2(\lambda - \varepsilon/\alpha - \varepsilon)^{-j} r \tag{1}$$

for each $j \in \mathbb{N}$.

Proof. Because of induction, we only need to handle the case $j = 1$. The estimate on the diameter of the π_u projection of the preimage of $\mathcal{D}_1$ under F is a consequence of part **(b)** of the lemma on moving invariant cones. That $\mathcal{D}_1$ is the graph of an α^{-1}-Lipschitz function ψ_1 from a subset of $\mathcal{E}^u(r)$ to $\mathcal{E}^s(r)$ is a consequence of part **(c)** of that same lemma. Thus, all we need to show is that $\operatorname{dom}(\psi_1) = \mathcal{E}^u(r)$ and that ψ_1 is C^1.

Let $\psi_0 : \mathcal{E}^u(r) \to \mathcal{E}^s(r)$ be the C^1 function (with Lipschitz constant less than or equal to α^{-1}) such that

$$\mathcal{D}_0 = \left\{ v_u + \psi_0(v_u) \,\middle|\, v_u \in \mathcal{E}^u(r) \right\}.$$

Define $g : \mathcal{E}^u(r) \to \mathcal{E}^u$ by the formula $g(v_u) = \pi_u F(v_u + \psi_0(v_u))$. If we can show that for each $y \in \mathcal{E}^u(r)$ there exists $x \in \mathcal{E}^u(r)$ such that

$$g(x) = y, \tag{2}$$

then we will know that $\operatorname{dom}(\psi_1) = \mathcal{E}^u(r)$.

Let $y \in \mathcal{E}^u(r)$ be given. Let $L = A_{uu}(0)$. Since $m(L) > \lambda$, we know that $L^{-1} \in \mathcal{L}(\mathcal{E}^u, \mathcal{E}^u)$ exists and that $\|L^{-1}\| \leq 1/\lambda$. Define $G : \mathcal{E}^u(r) \to \mathcal{E}^u$ by the formula $G(x) = x - L^{-1}(g(x) - y)$, and note that fixed points of G are solutions of (2), and vice versa. We shall show that G is a contraction and takes the compact set $\mathcal{E}^u(r)$ into itself and that, therefore, (2) has a solution $x \in \mathcal{E}^u(r)$.

Note that

$$\begin{aligned} Dg(x) &= \pi_u DF(x + \psi_0(x))(I + D\psi_0(x)) \\ &= A_{uu}(x + \psi_0(x)) + A_{us}(x + \psi_0(x))D\psi_0(x), \end{aligned}$$

so

$$\begin{aligned} \|DG(x)\| = \|I - L^{-1}Dg(x)\| &\leq \|L^{-1}\|\|L - Dg(x)\| \\ &\leq \frac{1}{\lambda}(\|A_{uu}(x + \psi_0(x)) - A_{uu}(0)\| + \|A_{us}(x + \psi_0(x))\|\|D\psi_0(x)\|) \\ &\leq \frac{\varepsilon + \varepsilon/\alpha}{\lambda} < 1. \end{aligned}$$

The Mean Value Theorem then implies that G is a contraction.

Now, suppose that $x \in \mathcal{E}^u(r)$. Then

$$\begin{aligned} \|G(x)\| \leq \|G(0)\| + \|G(x) - G(0)\| &\leq \|L^{-1}\|(\|g(0)\| + \|y\|) + \frac{\varepsilon + \varepsilon/\alpha}{\lambda}\|x\| \\ &\leq \frac{1}{\lambda}(\|g(0)\| + r + (\varepsilon + \varepsilon/\alpha)r). \end{aligned}$$

Let $\rho : \mathcal{E}^s(r) \to \mathcal{E}^u(r)$ be defined by the formula $\rho(v_s) = \pi_u F(v_s)$. Since $\rho(0) = 0$ and, for any $v_s \in \mathcal{E}^s(r)$, $\|D\rho(v_s)\| = \|A_{us}(v_s)\| \leq \varepsilon$, the Mean Value Theorem tells us that

$$\|g(0)\| = \|\pi_u F(\psi_0(0))\| = \|\rho(\psi_0(0))\| \leq \varepsilon\|\psi_0(0)\| \leq \varepsilon r. \tag{3}$$

Plugging (3) into the previous estimate, we see that

$$\|G(x)\| \leq \frac{1}{\lambda}(\varepsilon r + r + (\varepsilon + \varepsilon/\alpha)r) = \frac{1 + \varepsilon/\alpha + 2\varepsilon}{\lambda} r < r,$$

so $G(x) \in \mathcal{E}^u(r)$.

That completes the verification that (2) has a solution for each $y \in \mathcal{E}^u(r)$ and, therefore, that $\operatorname{dom}(\psi_1) = \mathcal{E}^u(r)$. To finish the proof, we need to show that ψ_1 is C^1. Let $\tilde{g}$ be the restriction of g to $g^{-1}(\mathcal{D}_1)$, and observe that

$$\psi_1 \circ \tilde{g} = \pi_s \circ F \circ (I + \psi_0). \tag{4}$$

We have shown that $\tilde{g}$ is a bijection of $g^{-1}(\mathcal{D}_1)$ with $\mathcal{D}_1$ and, by the Inverse Function Theorem, $\tilde{g}^{-1}$ is C^1. Thus, if we rewrite (4) as

$$\psi_1 = \pi_s \circ F \circ (I + \psi_0) \circ \tilde{g}^{-1}$$

we can see that ψ_1, as the composition of C^1 functions, is indeed C^1. □

W_r^s is a Lipschitz Manifold

Recall that W_r^s was defined to be all points in the box $\mathcal{B}(r)$ that produced forward orbits that remain confined within $\mathcal{B}(r)$. The next lemma shows that this set is a manifold.

Lemma (Nature of W_r^s) *W_r^s is the graph of a function $h : \mathcal{E}^s(r) \to \mathcal{E}^u(r)$ that satisfies $h(0) = 0$ and that has a Lipschitz constant less than or equal to α.*

Proof. For each $v_s \in \mathcal{E}^u(r)$, consider the set

$$\mathcal{D} := \{v_s\} + \mathcal{E}^u(r).$$

$\mathcal{D}$ is a C^1 unstable disk, so by the lemma on unstable disks, the subset $\mathcal{S}_j$ of $\mathcal{D}$ that stays in $\mathcal{B}(r)$ for at least j iterations of F has a diameter less than or equal to $2(\lambda - \varepsilon/\alpha - \varepsilon)^{-j}r$. By the continuity of F, $\mathcal{S}_j$ is closed. Hence, the subset $\mathcal{S}_\infty$ of $\mathcal{D}$ that stays in $\mathcal{B}(r)$ for an unlimited number of iterations of F is the intersection of a nested collection of closed sets whose diameters approach 0. This means that $\mathcal{S}_\infty$ is a singleton. Call the single point in $\mathcal{S}_\infty$ $h(v_s)$.

It should be clear that W_r^s is the graph of h. That $h(0) = 0$ follows from the fact that $0 \in W_r^s$, since $F(0) = 0$. If h weren't α-Lipschitz, then there would be two points $p, q \in W_r^s$ such that $p \in \{q\} + C^u(\alpha)$. Repeated application of parts **(b)** and **(c)** of the lemma on moving unstable cones would imply that either $F^j(p)$ or $F^j(q)$ is outside of $\mathcal{B}(r)$ for some $j \in \mathbb{N}$, contrary to definition. □

$W_{\text{loc}}^s(0)$ is a Lipschitz Manifold

Our next lemma shows that $W_{\text{loc}}^s(0) = W_r^s$ and that, in fact, orbits in this set converge to 0 exponentially. (The constant $\tilde{\mu}$ in the statement of the theorem can be chosen to be $\mu + \varepsilon$ if $\alpha \leq 1$.)

Lemma (Exponential Decay) *If $\alpha \leq 1$, then for each $p \in W_r^s$,*

$$\|F^j(p)\| \leq (\mu + \varepsilon)^j \|p\|. \tag{5}$$

In particular, $W_r^s = W_{\text{loc}}^s(0)$.

Proof. Suppose that $\alpha \leq 1$ and $p \in W_r^s$. By mathematical induction (and the positive invariance of W_r^s), it suffices to verify (5) for $j = 1$. Estimating, we find that

$$\begin{aligned}
\|F(p)\| \leq \|\pi_s F(p)\| &= \left\| \int_0^1 \frac{d}{dt} \pi_s F(tp)\, dt \right\| = \left\| \int_0^1 \pi_s DF(tp) p\, dt \right\| \\
&= \left\| \int_0^1 A_{ss}(tp)\pi_s p + A_{su}(tp)\pi_u p\, dt \right\| \\
&\leq \int_0^1 [\|A_{ss}(tp)\| \|\pi_s p\| + \|A_{su}(tp)\| \|\pi_u p\|]\, dt \\
&\leq \mu \|\pi_s p\| + \varepsilon \|\pi_u p\| \leq (\mu + \varepsilon)\|p\|.
\end{aligned}$$

□