Stable Manifold Theorem: Part 5

Lecture 35
Math 634
11/19/99

$W^s_{\text{loc}}(0)$ is C^1

Lemma (Differentiability) *The function $h : \mathcal{E}^s(r) \to \mathcal{E}^u(r)$ for which*

$$W^s_{\text{loc}}(0) = \left\{ v_s + h(v_s) \;\middle|\; v_s \in \mathcal{E}^s(r) \right\}$$

is C^1, and $Dh(0) = 0$.

Proof. Let $q \in W^s_r$ be given. We will first come up with a candidate for a plane that is tangent to W^s_r at q, and then we will show that it really is.

For each $j \in \mathbb{N}$ and each $p \in W^s_r$, define

$$C^{s,j}(p) := [D(F^j)(p)]^{-1} C^s(\alpha),$$

and let

$$C^{s,0}(p) := C^s(\alpha).$$

By definition (and by the invertibility of $DF(v)$ for all $v \in \mathcal{B}(r)$), $C^{s,j}(p)$ is the image of the stable cone under an invertible linear transformation. Note that

$$C^{s,1}(p) = [DF(p)]^{-1} C^s(\alpha) \subset C^s(\alpha) = C^{s,0}(p)$$

by the (proof of the) lemma on linear invariance of the unstable cone. Similarly,

$$\begin{aligned} C^{s,2}(p) &= [D(F^2)(p)]^{-1} C^s(\alpha) = [DF(F(p))DF(p)]^{-1} C^s(\alpha) \\ &= [DF(p)]^{-1}[DF(F(p))]^{-1} C^s(\alpha) = [DF(p)]^{-1} C^{s,1}(F(p)) \\ &\subset [DF(p)]^{-1} C^s(\alpha) = C^{s,1}(p) \end{aligned}$$

and

$$\begin{aligned} C^{s,3}(p) &= [D(F^3)(p)]^{-1} C^s(\alpha) = [DF(F^2(p))DF(F(p))DF(p)]^{-1} C^s(\alpha) \\ &= [DF(p)]^{-1}[DF(F(p))]^{-1}[DF(F^2(p))]^{-1} C^s(\alpha) \\ &= [DF(p)]^{-1}[DF(F(p))]^{-1} C^{s,1}(F^2(p)) \\ &\subset [DF(p)]^{-1}[DF(F(p))]^{-1} C^s(\alpha) = C^{s,2}(p). \end{aligned}$$

Recursively, we find that, in particular,

$$C^{s,0}(q) \supset C^{s,1}(q) \supset C^{s,2}(q) \supset C^{s,3}(q) \supset \cdots .$$

The plane that we will show is the tangent plane to W_r^s at q is the intersection

$$C^{s,\infty}(q) := \bigcap_{j=0}^{\infty} C^{s,j}(q)$$

of this nested sequence of "cones".

First, we need to show that this intersection *is* a plane. Suppose that $x \in C^{s,j}(q)$. Then $x \in C^s(\alpha)$, so

$$\begin{aligned} \|\pi_s DF(q)x\| &= \|A_{ss}(q)\pi_s x + A_{su}(q)\pi_u x\| \leq \|A_{ss}(q)\|\|\pi_s x\| + \|A_{su}(q)\|\|\pi_u x\| \\ &\leq (\mu + \varepsilon\alpha)\|\pi_s x\|. \end{aligned}$$

Repeating this sort of estimate, we find that

$$\begin{aligned} \|\pi_s D(F^j)(q)x\| &= \|\pi_s DF(F^{j-1}(q))DF(F^{j-2}(q)) \cdots DF(q)x\| \\ &\leq (\mu + \varepsilon\alpha)^j \|\pi_s x\|. \end{aligned}$$

On the other hand, if y is also in $C^{s,j}(q)$ and $\pi_s x = \pi_s y$, then repeated applications of the estimates in the lemma on linear invariance of the unstable cone yield

$$\|\pi_u D(F^j)(q)x - \pi_u D(F^j)(q)y\| \geq (\lambda - \varepsilon/\alpha)^j \|\pi_u x - \pi_u y\|.$$

Since $D(F^j)(q)C^{s,j}(q) = C^s(\alpha)$, it must, therefore, be the case that

$$\frac{(\lambda - \varepsilon/\alpha)^j \|\pi_u x - \pi_u y\|}{(\mu + \varepsilon\alpha)^j \|\pi_s x\|} \leq 2\alpha.$$

This implies that

$$\|\pi_u x - \pi_u y\| \leq 2\alpha \left(\frac{\mu + \varepsilon\alpha}{\lambda - \varepsilon/\alpha} \right)^j \|\pi_s x\|. \tag{1}$$

Letting $j \uparrow \infty$ in (1), we see that for each $v_s \in \mathcal{E}^s$ there can be no more than 1 point x in $C^{s,\infty}(q)$ satisfying $\pi_s x = v_s$. On the other hand, each $C^{s,j}(q)$ contains a plane of dimension $\dim(\mathcal{E}^s)$ (namely, the preimage of $\mathcal{E}^s$ under $D(F^j)(q)$), so (since the set of planes of that dimension passing through the origin is a compact set in the natural topology), $C^{s,\infty}(q)$ contains a plane, as well. This means that $C^{s,\infty}(q)$ is a plane $\mathcal{P}_q$ that is the graph of a linear function $L_q : \mathcal{E}^s \to \mathcal{E}^u$.

Before we show that $L_q = Dh(q)$, we make a few remarks.

(a) Because $\mathcal{E}^s \subset C^{s,j}(0)$ for every $j \in \mathbb{N}$, $\mathcal{P}_0 = \mathcal{E}^s$ and $L_0 = 0$.

(b) The estimate (1) shows that the size of the largest angle between two vectors in $C^{s,j}(q)$ having the same projection onto $\mathcal{E}^s$ goes to zero as $j \uparrow \infty$.

(c) Also, the estimates in the proof of the lemma on linear invariance of the unstable cone show that the size of the minimal angle between a vector in $C^{s,1}(F^j(q))$ and a vector outside of $C^{s,0}(F^j(q))$ is bounded away from zero. Since

$$C^{s,j}(q) = [D(F^j)(q)]^{-1}C^s(\alpha) = [D(F^j)(q)]^{-1}C^{s,0}(F^j(q))$$

and

$$\begin{aligned} C^{s,j+1}(q) &= [D(F^{j+1})(q)]^{-1}C^s(\alpha) = [D(F^j)(q)]^{-1}[DF(F^j(q))]^{-1}C^s(\alpha) \\ &= [D(F^j)(q)]^{-1}C^{s,1}(F^j(q)), \end{aligned}$$

this means that the size of the minimal angle between a vector in $C^{s,j+1}(q)$ and a vector outside of $C^{s,j}(q)$ is also bounded away from zero.

(d) Thus, since $C^{s,j+1}(q)$ depends continuously on q,

$$\mathcal{P}_{q'} \in C^{s,j+1}(q') \subset C^{s,j}(q)$$

for a given j if q' is sufficiently close to q. This means that $\mathcal{P}_q$ depends continuously on q.

Now, we show that $DF(q) = L_q$. Let $\varepsilon > 0$ be given. By remark **(b)** above, we can choose $j \in \mathbb{N}$ such that

$$\|\pi_u v - L_q \pi_s v\| \leq \varepsilon \|\pi_s v\| \tag{2}$$

whenever $v \in C^{s,j}(q)$. By remark **(c)** above, we know that we can choose $\varepsilon' > 0$ such that if $w \in C^{s,j+1}(q)$ and $\|r\| \leq \varepsilon' \|w\|$, then $w + r \in C^{s,j}(q)$. Because of the differentiability of F^{-j-1}, we can choose $\eta > 0$ such that

$$\|F^{-j-1}(F^{j+1}(q) + v) - q - [D(F^{-j-1})(F^{j+1}(q))]v\| \leq \frac{\varepsilon'}{\|D(F^{j+1})(q)\|}\|v\| \tag{3}$$

whenever $\|v\| \leq \eta$. Define the truncated stable cone

$$C^s(\alpha, \eta) := C^s(\alpha) \cap \pi_s^{-1}\mathcal{E}^s(\eta).$$

From the continuity of F and the α-Lipschitz continuity of h, we know that we can pick $\delta > 0$ such that

$$F^{j+1}(v_s + h(v_s)) \in \{F^{j+1}(q)\} + C^s(\alpha, \eta). \tag{4}$$

whenever $\|v_s - \pi_s q\| < \delta$.

Now, suppose that $v \in C^s(\alpha, \eta)$. Then (assuming $\alpha \leq 1$) we know that $\|v\| \leq \eta$, so (3) tells us that

$$\begin{aligned} F^{-j-1}(F^{j+1}(q) + v) &= q + [D(F^{-j-1})(F^{j+1}(q))]v + r \\ &= q + [D(F^{j+1})(q)]^{-1}v + r \end{aligned} \tag{5}$$

for some r satisfying

$$\|r\| \leq \frac{\varepsilon'}{\|D(F^{j+1})(q)\|}\|v\|.$$

Let $w = [D(F^{j+1})(q)]^{-1}v$. Since $v \in C^s(\alpha)$, $w \in C^{s,j+1}(q)$. Also,

$$\|w\| = \|[D(F^{j+1})(q)]^{-1}v\| \geq m([D(F^{j+1})(q)]^{-1})\|v\| = \frac{\|v\|}{\|D(F^{j+1})(q)\|},$$

so $\|r\| \leq \varepsilon'\|w\|$. Thus, by the choice of ε', $w + r \in C^{s,j}(q)$. Consequently, (5) implies that

$$F^{-j-1}(F^{j+1}(q) + v) \in \{q\} + C^{s,j}(q).$$

Since v was an arbitrary element of $C^s(\alpha, \eta)$, we have

$$F^{-j-1}(\{F^{j+1}(q)\} + C^s(\alpha, \eta)) \subseteq \{q\} + C^{s,j}(q). \tag{6}$$

Set $q_s := \pi_s q$, and suppose that $v_s \in \mathcal{E}^s(r)$ satisfies $\|v_s - q_s\| \leq \delta$. By (4),

$$F^{j+1}(v_s + h(v_s)) \in \{F^{j+1}(q)\} + C^s(\alpha, \eta).$$

This, the invertibility of F, and (6) imply

$$v_s + h(v_s) \in \{q\} + C^{s,j}(q),$$

or, in other words,

$$v_s + h(v_s) - q_s - h(q_s) \in C^{s,j}(q).$$

The estimate (2) then tells us that

$$\|h(v_s) - h(q_s) - L_q(v_s - q_s)\| \leq \varepsilon \|v_s - q_s\|,$$

which proves that $Dh(q) = L_q$ (since ε was arbitrary).

Remark **(d)** above implies that $Dh(q)$ depends continuously on q, so $h \in C^1$. Remark **(a)** above implies that $Dh(0) = 0$. □